21 Lecture 11-16

21.1 Chapter 8 Two Body Central Force Problem (con)

Conservation of Energy To learn more details about the orbit we must examine the first integral of the radial equation of motion. Since we have reduced the problem to one generalized coordinate, that being the relative radial coordinate, the Lagrangian for this system is simply

$$\mathcal{L} = \frac{1}{2}\mu\dot{r}^2 - U_{eff}. \tag{1}$$

Since the Lagrangian is independent of t, $\mathcal{L} \neq \mathcal{L}(t)$, the Hamiltonian,

$$\mathcal{H} = p_r\dot{r} - \mathcal{L} \tag{2}$$

is a constant of the motion. Since the conjugate momentum to the radial coordinate is

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = \mu\dot{r}, \tag{3}$$

the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\mu\dot{r}^2 + U_{eff}.$$

Additionally the transformation from Cartesian coordinates to our generalized coordinate, the relative radial coordinate, was time independent as well. This means that the Hamiltonian is equal to the energy E. Hence we are lead to

$$\frac{\mu}{2}\dot{r}^2 + U_{eff}(r) = \frac{\mu}{2}\dot{r}^2 + \frac{L^2}{2\mu r^2} + U(r) = E, \tag{4}$$

where the energy E is a constant of the motion.

The total energy, E, can be thought of as the one-dimensional kinetic energy of the radial motion, plus the effective one-dimensional potential energy $U_{eff}(r)$. This effective potential energy is the actual potential energy U and the kinetic energy $\mu r^2\dot{\phi}^2/2$ of the angular motion. This means that all of our experience with one dimensional problems, both in terms of energy and forces, can be immediately applied to the two-body central force problem.

Energy Considerations for a Comet or Planet We will examine again an energy diagram for a comet (or planet) with a given energy E. This includes finding the equation that determines the maximum and minimum distances of the comet from the Sun, if $E > 0$ and again if $E < 0$.

In the energy equation (4) the radial kinetic energy $\mu\dot{r}^2/2$ is positive definite, therefore the comet's motion is confined to those regions where $E > U_{eff}$. To see what this implies consider figure 8.4

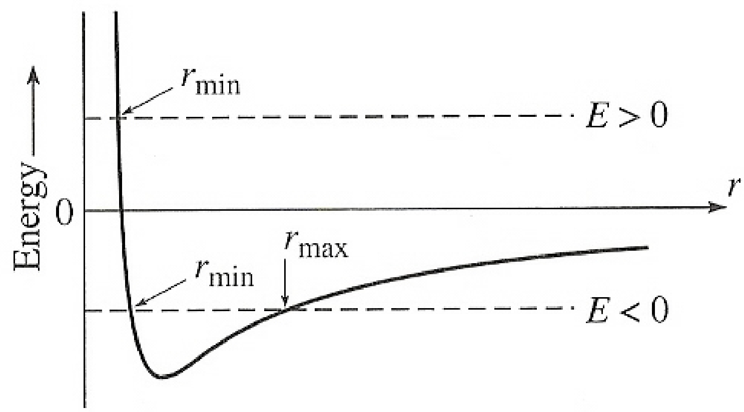

Figure 8.4. $U_{eff}(r)$ for a comet. For a given E the comet can only travel where $E > U_{eff}(r)$. For $E > 0$ there is a turning point at $r_{\min}$. For $E < 0$ there are two turning points. One at $r_{\min}$ and another at $r_{\max}$.

where we have plotted $U_{eff}(r)$ along with two different energy levels (shown with dashed lines). First we will consider the case when the comet's energy is greater than zero. A comet with this energy can move anywhere that its energy exceeds U_{eff}. From figure 8.4 this means it is allowed to be anywhere for $r > r_{\min}$. From our earlier discussions of one-dimensional energy diagrams we know that this is a turning point determined by the condition

$$U_{eff}(r_{\min}) = E. \tag{5}$$

If the comet is initially moving radially inward toward the Sun, then it will continue to do so until it reaches $r_{\min}$, where $\dot{r} = 0$. It then begins to move radially outward. Since there are no other turning points, points where $\dot{r} = 0$, it will continue moving radially outward toward infinity. We say that this orbit is *unbounded*.

If instead, $E < 0$, then the energy level at this height has two turning points labeled $r_{\min}$ and $r_{\max}$. A comet (or planet) with $E < 0$ is trapped between these two values of r. If it is moving away from the Sun it continues to do so until it reaches the turning point $r_{\max}$ at which point $\dot{r} = 0$. It then begins to move toward the Sun until it reaches the turning point at $r_{\min}$ only to repeat this process. For obvious reasons, this type of orbit is called a *bounded* orbit.

Finally if E is equal to the minimum value of U_{eff}, which depends on the value of its angular momentum, the radial coordinate is fixed. This means that the comet moves in a circular orbit of radius r_o. This radius is found from finding the minimum in U_{eff}. Taking the derivative of U_{eff} and setting it to zero we find

$$\begin{aligned} \frac{dU_{eff}}{dr} &= \frac{d}{dr}\left(\frac{L^2}{2\mu r^2} - \frac{G\mu M}{r}\right)_{r=r_o} = -\frac{L^2}{\mu r_o^3} + \frac{G\mu M}{r_o^2} = \\ r_o &= \frac{L^2}{\mu^2 GM} = \frac{\ell^2}{GM}, \end{aligned} \tag{6}$$

where we have defined the angular momentum per unit reduced mass as, $\ell = L/\mu$. The radius of this circular orbit only depends on the angular momentum (per unit mass). As we shall soon see increasing the energy of the orbit while holding the angular momentum fixed changes a bounded orbit from circular to elliptical.

In thinking about the radial motion of the two-body problem, you must not forget about the angular motion. Although the motion of the comet only depends on its radial coordinate, the path of the comet is not one dimensional. We need to remember that the comet possesses angular momentum, which means it has an angular component to its kinetic energy. Since its angular momentum is conserved its angular component is always changing at a rate given by

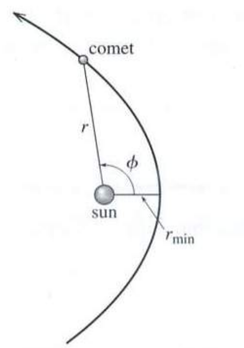

Figure 8.5 Typical unbounded orbit for a positive energy comet. As r decreases from infinity to $r_{\min}$ and the goes back out to infinity, ϕ is continually increasing.

$\dot{\phi} = L/\mu r^2 = \ell/r^2$ which never changes sign. For example as a comet with positive energy approaches the Sun, the angle ϕ changes, at a rate that increases as r decreases. As the comet moves away from the Sun, ϕ continues to change in the same direction but at a rate that decreases as r increases. The actual orbit of a positive energy comet looks something like that shown in figure 8.5. For the case of the gravitational inverse square law we shall see that the orbit is actually a hyperbola.

For the bounded orbits ($E < 0$), we have seen that r oscillates between the two values $r_{\min}$ and $r_{\max}$, all the while ϕ is continuing to increase (or decrease). For the case of the gravitational potential we shall see that the period of radial oscillations is exactly equal to the time for ϕ to make one complete revolution. Thus the orbits are closed and the motion repeats itself exactly once per revolution. We shall see that the orbit is an ellipse.

We have been considering the case of the inverse square law. But many of the two body problems have similar qualitative features. In chapter 4 we discussed the potential energy diagram for a diatomic molecule which had many of the same features as the effective gravitational potential. Thus all of our qualitative conclusions apply to the diatomic molecule and many other two-body problems. However many of the quantitative features do not apply. For instance for most other force laws the period of radial motion is different from the time to make one complete revolution and in most cases the orbit is not even closed. That is it never returns to its original position. A comparison of a closed elliptical orbit and one that is not closed is shown in figure 8.6. In figure 8.6 (b) we show an orbit for which r goes from $r_{\min}$ to $r_{\max}$ and back to $r_{\min}$ in the time that the

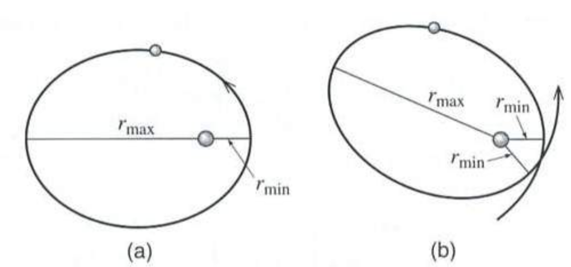

Figure 8.6. (a) The bounded orbit for a gravitational potential. (b) Typical orbit for a non inverse square law central force in which the orbit is not closed.

angle ϕ advances by about 330°. This orbit does not close on itself after one revolution.

21.1.1 The Orbit Equation

The radial equation of motion for r determines r as a function of t,

$$\mu\ddot{r} = \frac{L^2}{\mu r^3} - \frac{dU}{dr}, \tag{7}$$

but often we wish to know r as a function of ϕ. There are several ways to do this and one of the more straightforward ways starts with the expression for the conservation of energy,

$$E = \frac{\mu}{2}\dot{r}^2 + U(r) + \frac{L^2}{2\mu r^2}. \tag{8}$$

There are two tricks that we have to use. The first is the substitution

$$r = \frac{1}{u} \text{ or } u = \frac{1}{r}.$$

The second is to recognize that the time derivative d/dt can be written in terms of $d/d\phi$ by making use of the chain rule,

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \frac{L}{\mu r^2}\frac{d}{d\phi}. \tag{9}$$

Combining both of these "tricks" results in

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{dt} = -\frac{1}{u^2}\frac{d\phi}{dt}\frac{du}{d\phi} = -\frac{1}{u^2}\frac{Lu^2}{\mu}\frac{du}{d\phi} = -\frac{L}{\mu}\frac{du}{d\phi}. \tag{10}$$

Substituting this result into expression for the conservation of energy, equation (8), results in

$$\frac{L^2}{2\mu}\left[\left(\frac{du}{d\phi}\right)^2 + u^2\right] = E - U\left(u\right). \tag{11}$$

We now an equation for $du/dd\phi$ for any given potential U. We will call this the orbit equation. To solve this equation, it is a straightforward procedure to separate and integrate the resulting expression. Whether or not it is possible perform the integration depends on U, but in principle at least we have the solution and for a worst case the resulting integral can be performed numerically.

As our first example we will consider equation (11) for a free particle. This will also be the path of a photon (at least in classical mechanics) and we want to confirm that the resulting orbit is a straight line. Before we proceed we will define the energy per unit reduced mass in a way analogous to our definition of angular momentum per unit reduced mass as

$$\varepsilon = \frac{E}{\mu}, \tag{12}$$

for as we have seen the equation of motion is independent of the reduced mass μ. This means that the orbit equation must also be independent of μ. Substituting this into the expression for $du/d\phi$ when $U = 0$ yields

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{2\varepsilon}{\ell^2}, \tag{13}$$

where both ε and ℓ are conserved quantities. The quantity on the right hand side of this equation is a constant and corresponds to the value of u when $du/d\phi = 0$. For a free particle this is the distance of closest approach and our expression, equation (13), then becomes

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = u_o^2 = 1/\rho_o^2. \tag{14}$$

This equation is nonlinear, but a very simple nonlinear equation that is easily separated and integrated. However the solution is obvious from inspection and is given by $u = u_o \cos\left(\phi - \delta\right)$. Recognizing that $u = 1/r$ we are lead to the result

$$\rho_o = r\cos(\phi - \delta). \tag{15}$$

Expanding the cosine function we see that it becomes

$$\rho_o = x\cos\delta + y\sin\delta, \tag{16}$$

where as usual $x = r\cos\phi$ and $y = r\sin\phi$. This is the equation of the straight line with a slope of $-\cot\delta = \tan(\delta + \pi/2)$ and is shown in figure 8.7.

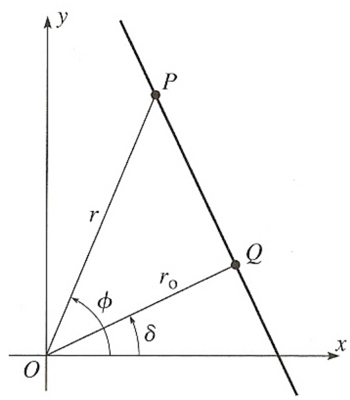

Figure 8.7 Path of a free particle in radial coordinates.

When $\phi = \delta$ the particle is at the point of closest approach, and as $\phi \to \delta \pm \pi/2$ we find that $r \to \infty$. A particularly simple solution occurs for the initial condition that requires $\delta = \pi/2$, for in that case the solution is $y = \rho_o$, a straight line that is parallel to the x axis. We will return to this solution when we solve the orbit equation of a photon in a gravitationally curved space. We will find that General Relativity induces an additional term which results in a photon being bent as it passes a large gravitational source.

21.1.2 Kepler Orbits

We will now return to the Kepler problem, that of finding the possible orbits of a comet or any other particle subject to an attractive inverse square law. The two important examples of this problem are the motion of comets or planets around the Sun or Earth satellites around the Earth. For this case we saw that we can express the gravitational potential as

$$U(r) = -\frac{G\mu M}{r}. \tag{17}$$

Returning to the equation for $du/d\phi$, i.e. the orbit problem equation (11), we find for the gravitational potential that

$$\frac{L^2}{2\mu}\left[\left(\frac{du}{d\phi}\right)^2 + u^2\right] = E + G\mu M u. \tag{18}$$

Again, as with the free particle problem, it is useful to define energy and angular momentum per unit reduced mass as $\varepsilon = E/\mu$ and $\ell = L/\mu$. Then dividing the above expression by μ results in

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2GM}{\ell^2}u = \left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2u}{r_o} = \frac{2\varepsilon}{\ell^2}, \tag{19}$$

where r_o is the radius of a circular orbit given an angular momentum ℓ. By completing the square this equation can be put in the form

$$\left(\frac{du}{d\phi}\right)^2 + \left(u - \frac{1}{r_o}\right)^2 = \frac{2\varepsilon}{\ell^2} + \frac{1}{r_o^2}. \tag{20}$$

Defining $y = u - 1/r_o$ this expression is reduced to the same form as that for a free particle

$$\left(\frac{dy}{d\phi}\right)^2 + y^2 = \frac{2\varepsilon}{\ell^2} + \frac{1}{r_o^2}. \tag{21}$$

From the free particle result we can now simply state the solution as

$$\frac{1}{r} = \sqrt{\frac{2\varepsilon}{\ell^2} + \frac{1}{r_o^2}}\cos\left(\phi - \delta\right) + \frac{1}{r_o}, \tag{22}$$

where the phase δ is a constant which we can take to be zero by a suitable choice for the direction $\phi = 0$. Defining the parameter ϵ as

$$\epsilon = \sqrt{1 + \frac{2\varepsilon r_o^2}{\ell^2}} = \sqrt{1 + \frac{2\varepsilon\ell^2}{G^2M^2}}, \tag{23}$$

allows us to write the radial solution more compactly as

$$\frac{1}{r\left(\phi\right)} = \frac{1}{r_o}\left(1 + \epsilon\cos\phi\right),$$

or

$$r\left(\phi\right) = \frac{r_o}{1 + \epsilon\cos\phi} \tag{24}$$

This is our solution for $r\left(\phi\right)$ in terms of the parameter ϵ. This parameter depends on the total energy (per unit μ) ε which may be positive (for an unbound particle) or negative (for a bound particle). Thus for bound orbits $\epsilon < 1$ while for unbound orbits $\epsilon > 1$. We shall now explore the properties of this solution, first for bounded orbits and then for unbounded orbits.

Bounded Orbits A glance at the radial solution shows that the behavior is very different depending on whether $\epsilon < 1$ or $\epsilon > 1$. If $\epsilon < 1$ we see that the denominator never vanishes and the radius is bounded for all ϕ. This is consistent with the energy being negative for a bound particle in the expression for ϵ, equation (23). It is this case that we wish to discuss first.

With $\epsilon < 1$ the radial coordinate oscillates between

$$r_{\min} = \frac{r_o}{1+\epsilon} \quad \text{and} \quad r_{\max} = \frac{r_o}{1-\epsilon}, \tag{25}$$

with $r = r_{\min}$ defined to be the *perihelion* which occurs at $\phi = 0$ and $r = r_{\max}$ defined to be the *aphelion* which occurs at $\phi = \pi$. Since $r(\phi)$ is periodic in ϕ with a period of 2π, the orbit closes on itself after one revolution.

It is a straightforward algebraic exercise to show that our solution for $r(\phi)$ can be written in Cartesian coordinates as

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{26}$$

where

$$a = \frac{r_o}{1-\epsilon^2}, \quad b = \frac{r_o}{\sqrt{1-\epsilon^2}}, \quad \text{and} \quad d = a\epsilon. \tag{27}$$

Equation (26) is the standard equation of an ellipse with a semimajor and semiminor axes a and b. This ellipse is centered at $x = -d$ which reflects that our origin the CM (which is essentially the center of the Sun) is not at the center of the ellipse as shown in figure 8.8.

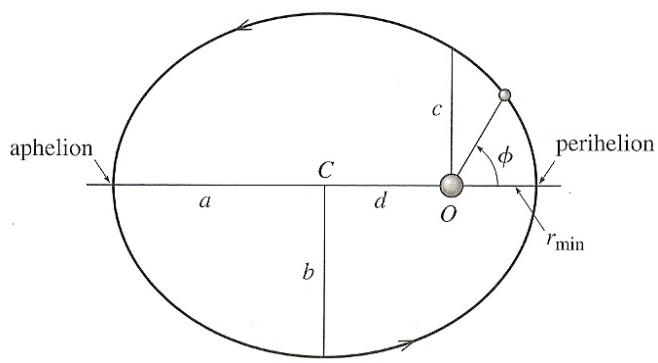

Figure 8.8. The bounded orbits described by equation (23) are ellipses. The points where a comet is closest and farthest from the Sun are called the perihelion and aphelion.

We can now identify the parameter ϵ in terms of the major and minor axes,

$$\frac{b}{a} = \sqrt{1-\epsilon^2}. \tag{28}$$

This is one definition for the eccentricity of an ellipse. We also could have found ϵ from the maximum values of u and or r as

$$\epsilon = \frac{u_{\max} - u_{\min}}{u_{\max} + u_{\min}} = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}, \tag{29}$$

which is also one definition for the eccentricity of an ellipse.

Having identified ϵ as the eccentricity, we identify the position of the Sun in relation to the ellipse. Its distance is $d = a\epsilon$ which is the distance from the center to either focus of the ellipse. Thus center of the Sun (which is very close to the CM) is one of the ellipse's two foci. This means that we have proved *Kepler's first law*, namely that the planets (or bound comets) follow elliptical orbits with the Sun at one focus.

Since all of the planets have close to circular orbits (Pluto is no longer considered to be a planet), it is of interest to examine the highly eccentric orbit of Halley's comet. It has an eccentricity of $\epsilon = .967$, and at closest approach or perihelion the comet is .59AU from the Sun. (The AU or astronomical unit is the mean distance of the Earth from the Sun and is approximately $150 \times 10^6 km$.) This means that the distance to the aphelion, which can be found from equation (29), is

$$r_{\max} = \frac{1+\epsilon}{1-\epsilon} r_{\min} \simeq 60 r_{\min} = 35\text{AU},$$

which is outside the orbit of Neptune.