

## 12 Lecture 10-21

### 12.1 Chapter 5 Oscillations (con)

**Complex Solutions for a Sinusoidal Driving Force** We shall consider the case when the driving force  $f(t)$  is a sinusoidal function of the form

$$f(t) = f_o \cos \omega t, \quad (1)$$

where  $f_o$  is the amplitude of the driving force (actually  $F(t)/m$ ) and  $\omega$  is the angular frequency of the driving force. It is important to distinguish between the frequency of the forcing function,  $\omega$ , from the natural frequency of the oscillator,  $\omega_o$ . These are entirely independent frequencies, although we shall see that the oscillator responds most when  $\omega = \omega_o$ . The driving force for many driven oscillators is approximately sinusoidal, a parent pushing a child on a swing, the EMF produced in the electronic components in your radio circuits etc. With this in mind we will assume that the equation of motion (??) takes the form

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_o^2 x = f_o \cos \omega t. \quad (2)$$

Solving this equation is simplified with the following trick: For any solution of this equation, there must also be a solution of the same equation but with the cosine forcing function replaced by a sine function. After all these two only differ by a shift in the origin in time. Accordingly there must be a solution to

$$\frac{d^2y}{dt^2} + 2\beta \frac{dy}{dt} + \omega_o^2 y = f_o \sin \omega t. \quad (3)$$

Suppose we now define the complex function

$$z(t) = x(t) + iy(t). \quad (4)$$

If we now multiply equation (3) by  $i$  and add it to equation (2) we obtain

$$\frac{d^2z}{dt^2} + 2\beta \frac{dz}{dt} + \omega_o^2 z = f_o e^{i\omega t}. \quad (5)$$

Although at first glance this may not appear to be much of an improvement, but it is. A sine or cosine function changes from one to the other during the differential process, but that is not the case with  $\exp(i\omega t)$ . This makes it much easier to solve for  $z(t)$  and once we do, we only have to take the real part to find the solution of equation (2).

The most obvious trial solution for  $z(t)$  is

$$z(t) = C e^{i\omega t}, \quad (6)$$

where  $C$  is an as yet undetermined constant. Substituting this trial solution into equation (5) results in

$$(-\omega^2 + 2i\beta\omega + \omega_o^2) C e^{i\omega t} = f_o e^{i\omega t}. \quad (7)$$

Our trial is a solution if and only if

$$C = \frac{f_o}{-\omega^2 + 2i\beta\omega + \omega_o^2}, \quad (8)$$

and we have succeeded in finding a particular solution for the equation of motion.

Before we take the real part of  $z(t) = Ce^{i\omega t}$  it is convenient to write the complex coefficient in the form  $C = Ae^{-i\delta}$ , where both  $A$  and  $\delta$  are real. To find  $A^2$  we multiply  $C$  by its complex conjugate and

$$A^2 = CC^* = \frac{f_o^2}{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}. \quad (9)$$

We will see in a moment that  $A$  is the amplitude of the oscillations caused by the driving force  $f(t)$ . Thus this result is the most important of this discussion as it shows how the amplitude depends on the various parameters. In particular note how  $A$  is a maximum when  $\omega_o \simeq \omega$  as this minimizes the denominator. In other words the oscillator responds best to the driving force when its frequency matches the natural frequency of the oscillator.

Before we proceed we need to find the phase angle  $\delta$ . From the definition  $C = Ae^{-i\delta}$  we see that

$$f_o e^{i\delta} = A(\omega_o^2 - \omega^2 + 2i\beta\omega). \quad (10)$$

Now both  $f_o$  and  $A$  are real so that the phase angle is the same as the phase of the complex number  $\omega_o^2 - \omega^2 + 2i\beta\omega$  which is

$$\tan \delta = \frac{2\beta\omega}{\omega_o^2 - \omega^2} \rightarrow \delta = \tan^{-1} \left( \frac{2\beta\omega}{\omega_o^2 - \omega^2} \right). \quad (11)$$

We have now found the solution, it is the real part of

$$z(t) = Ce^{i\omega t} = Ae^{i(\omega t - \delta)}, \quad (12)$$

or

$$x(t) = A \cos(\omega t - \delta), \quad (13)$$

where  $A$  and  $\delta$  are given by equation (9) and equation (11) respectively.

The solution in equation (13) is just one particular solution for the equation of motion. As we have seen, for the general solution we must add the any solution to the homogeneous equation. So our general solution is

$$x(t) = A \cos(\omega t - \delta) + C_1 e^{rt} + C_2 e^{-rt}. \quad (14)$$

Because the solutions to the homogeneous equation die out exponentially, they are called *transients* or transient solutions. They depend on the initial conditions (or any set of boundary conditions) of the problem but are eventually irrelevant. The long term solution is dominated by the particular solution, i.e.

the  $\cos(\omega t - \delta)$  term. Thus the particular solution is the one of interest for our following discussions.

Before we discuss examples of the solution in equation (14), it is important that you understand as to what type of system that this applies, namely any oscillator with a linear restoring force and a linear resistive force. It is the solution to a second order *linear* differential equation. Because nonlinear differential equations are difficult to solve (often requiring numerical techniques), until recently most text books focused on linear equations to the point of ignoring nonlinear systems. As we shall see in chapter 12 on nonlinear mechanics and chaos, oscillators that are nonlinear can behave in ways that are astonishing different from equation (14). One important reason for studying linear oscillators is to give you some background against which to study nonlinear oscillators later on.

The details of the general solution (just as with the homogeneous solution) depend on the strength of the damping parameter  $\beta$ . Consider for example a weakly damped oscillator where  $\beta < \omega_o$ . For this case the general solution can be written as

$$x(t) = A \cos(\omega t - \delta) + A_{tr} e^{-\beta t} \cos(\omega_1 t - \delta_{tr}). \quad (15)$$

The term on the right is the transient term and to emphasize this fact we have added the subscript *tr* to distinguish  $A_{tr}$  and  $\delta_{tr}$  from  $A$  and  $\delta$ . These constants of the transient motion are arbitrary constants and determined from the initial conditions. The factor  $e^{-\beta t}$  makes clear that the transient term decays exponentially and is indeed irrelevant to the long term behavior - hence the term transient. The first term is the particular solution and the constants  $A$  and  $\delta$  are certainly not arbitrary, they were determined to satisfy the motion of the system in the presence of a particular forcing function. If you change the forcing function then  $A$  and  $\delta$  will be required to change accordingly. This term oscillates with the frequency  $\omega$  and amplitude  $A$  for as long as the driving force is maintained.

To better understand these effects examine the response of a linear damped oscillator in Figure 5.11. Here we plot the response of a driven underdamped oscillator whose natural frequency is five times that of the driving frequency,  $\omega_o = 5\omega$ , and the decay constant is given by  $\beta = \omega_o/20$ . We have chose the initial conditions to be  $x_o = v_o = 0$ . For the first three cycles or so the effects of the transients are clearly visible. However after that the motion is virtually indistinguishable from a pure cosine oscillating (albeit slightly out of phase) at the drive frequency. That is the transients have died out and only the long term motion remains.

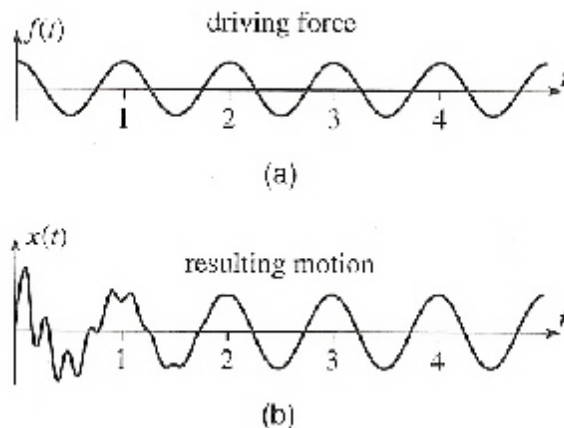


Figure 5.11 (a) The driving force for a damped linear oscillator is a pure cosine function. (b) The resulting motion for the initial conditions  $x_o = v_o = 0$ . The long term motion oscillates at the drive frequency. This sinusoidal motion is called an attractor.

Because the transient motion depends on the initial conditions, i.e.  $x_o$  and  $v_o$ , different initial conditions would result in different initial motion. After a short time however the motion settles down into the same sinusoidal motion of the particular solution, *irrespective of the initial conditions*. For this reason the motions of the particular solution are sometimes called an *attractor* - the motions of different initial conditions are “attracted” to the particular solution. We shall see that for nonlinear oscillators there can be several different attractors and that for some values of the parameters the motion of an attractor can be far more complicated than simple harmonic oscillation of the drive frequency.

### 12.1.1 Resonance

We have already commented that a driven oscillator responds best when driven at a frequency  $\omega$  that is close to its natural frequency  $\omega_o$ . We now have the solutions necessary to consider this effect in some detail. As we just discovered in equation (13), apart from the transient motions, the system’s response to the driving force is

$$x(t) = A \cos(\omega t - \delta),$$

with an amplitude given by

$$A^2 = \frac{f_o^2}{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}. \quad (16)$$

One obvious feature of this expression is that the amplitude of the response is proportional to the amplitude of the forcing function,  $A \propto f_o$ , a result that should be expected for a linear oscillator. More interesting however, is the dependence of the amplitude on the frequencies,  $\omega$  and  $\omega_o$ , as well as the damping

constant  $\beta$ . The most interesting case is when the damping constant is small. If the frequencies  $\omega$  and  $\omega_o$  are very different then the first term in the denominator is large and the amplitude of the oscillations is small. On the other hand if  $\omega \simeq \omega_o$ , then the amplitude of the oscillations can be very large. This means that we can vary either  $\omega$  or  $\omega_o$  and find dramatic changes in the amplitude of the oscillator's motion. This is shown in Figure 5.12, where we plot the amplitude as a function of the natural frequency of the system,  $\omega_o$ .

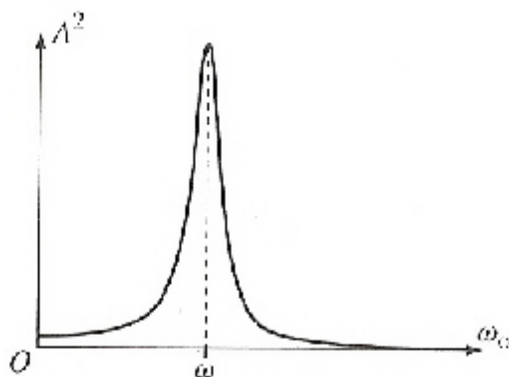


Figure 5.12 The amplitude squared of a driven oscillator as a function of the natural frequency,  $\omega_o$ , with the driving frequency,  $\omega$ , fixed.

Although the behavior is dramatic, the qualitative features are what you might have expected. In the absence of any forcing function the oscillator vibrates at its natural frequency  $\omega_o$  (actually a slightly lower frequency  $\omega_1$  when we allow for damping). If we drive the oscillator at its natural frequency then it responds very well, but if  $\omega$  is far from  $\omega_o$  then it hardly responds at all. This phenomena so clearly represented in Figure 2 is called *resonance*.

One of the simplest examples that we experience in our daily lives is the parent pushing on a swing. The parent always gives the child a push at exactly the same frequency that the swing oscillates, i.e. the parent plays the role of a resonant forcing function. Additional examples are numerous with another common occurrence being the tuning circuits inside your radio as you dial in to different radio stations.

The details of the resonance are a bit more complicated than they seem with just a cursory examination. For example the exact location of the maximum response depends on whether we vary  $\omega_o$  while holding  $\omega$  fixed or vice versa. The amplitude is a maximum when the denominator,

$$\text{denominator} = (\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2, \quad (17)$$

is a minimum. So if we vary  $\omega_o$  while holding  $\omega$  fixed, as we do when we are tuning our radio to a given carrier frequency, then as figure 2 showed, the denominator is a minimum when  $\omega_o = \omega$ . However if we vary  $\omega$  while holding  $\omega_o$

fixed then minimizing the denominator requires us to differentiate the expression in equation (17). We find that the minimum occurs when

$$\omega = \omega_2 = \sqrt{\omega_o^2 - 2\beta^2}. \quad (18)$$

However when  $\beta \ll \omega_o$  (which usually the most interesting case), the difference between  $\omega_2$  and  $\omega_o$  is negligible.

We have discussed so many frequencies that it is useful to review them. First there is the natural frequency of the oscillator (undamped)  $\omega_o = \sqrt{k/m}$ . Next we considered damping which reduced the frequency that the system oscillated to  $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$ . Then we added a forcing function which oscillated at  $\omega$ . In principle this frequency is independent of either of the other two. However due to the usual interest in resonance it is often close to  $\omega_o$ . In fact if we hold  $\omega$  fixed while varying  $\omega_o$  the maximum response occurs when  $\omega = \omega_o$ . Finally as we just saw, if we vary  $\omega$  while holding  $\omega_o$  fixed then the response is a maximum when  $\omega = \omega_2$ .

In any case, the maximum amplitude of the driven oscillator is found when  $\omega \simeq \omega_o$  and is then given by

$$A \simeq \frac{f_o}{2\beta\omega_o}. \quad (19)$$

In this expression it is clear that smaller values of the damping constant,  $\beta$ , lead to larger values of the maximum amplitude of oscillation.

**Width of the Resonance; Q factor** As we can see in figure 5.13, if we make the damping constant  $\beta$  smaller, not only does the resonant peak get higher, but it also gets narrower. This idea is expressed in the definition of the width

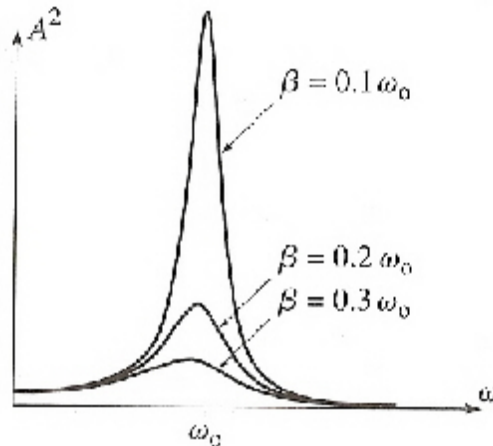


Figure 5.13 Amplitude as a function of the driving frequency  $\omega$  for different values of the damping parameter  $\beta$ .

or more precisely, *full width at half maximum/FWHM*. FWHM is the interval between two points on the resonance curve where  $A^2$  is equal to half its maximum height. It is a simple exercise for the student to show that the two half maximum points are at  $\omega \simeq \omega_o \pm \beta$ . Hence the full width at half maximum is

$$FWHM \simeq 2\beta. \quad (20)$$

The sharpness of the resonance peak is indicated by the ratio of its width,  $2\beta$ , to its position  $\omega_o$ . For many purposes, we want a very sharp resonance, to it is common practice to define a *quality factor*  $Q$  as a reciprocal of this ratio,

$$Q = \omega_o/2\beta. \quad (21)$$

A large  $Q$  indicates a narrow resonance, and vice versa. For example, clocks depend on the resonance in an oscillator (a pendulum or a quartz crystal) to regulate the mechanism to a well defined frequency. This requires that the width  $2\beta$  be very small compared to its natural frequency. In other words a good clock requires a high  $Q$ . The  $Q$  for a typical pendulum may be around 100 while that for a quartz crystal around 10,000. Thus a quartz crystal watches keep much better time than a typical grandfather clock.

**The Phase at Resonance** As we found in equation (11) the phase difference by which the oscillator's motion lags behind the driving force is

$$\delta = \tan^{-1} \left( \frac{2\beta\omega}{\omega_o^2 - \omega^2} \right).$$

It is a useful exercise to follow this phase as a function of  $\omega$ . For  $\omega \ll \omega_o$  the phase is very small meaning that the oscillations are almost perfectly in step with the driving force. However  $\delta$  increases as  $\omega$  is increased toward  $\omega_o$ . When  $\omega = \omega_o$  the phase satisfies the expression  $\tan \delta = \infty$ , which occurs when  $\delta = \pi/2$ . So at resonance the oscillations lag the forcing function by  $90^\circ$ . Once  $\omega > \omega_o$ ,  $\tan \delta$  becomes negative, but with a large amplitude as long as  $\delta$  is just slightly greater than  $\pi/2$ . As  $\omega$  continues to increase the magnitude of  $\tan \delta$  continues to decrease until it approaches 0 and  $\delta \rightarrow \pi$ . In particular, once  $\omega \gg \omega_o$  then the oscillations are almost perfectly out of step with the driving force. All of this is illustrated in figure 5.14. Additionally in figure 5.14, we see that the change in the phase is more abrupt for smaller  $\beta$ .

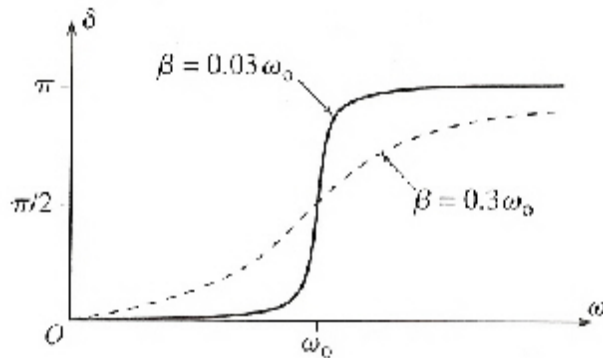


Figure 5.14 The phase shift  $\delta$  increases from 0 to  $\pi$  as the driving frequency  $\omega$  passes through resonance. If you drive a simple pendulum manually by moving your hand slowly from side to side the pendulum will eventually move in step with your hand. If you move your hand more quickly than the natural frequency of the pendulum it will move oppositely to your hand.

In the resonances of classical mechanics, the behavior of the phase is usually less important than that of the amplitude, figure 5.13. However in atomic and nuclear scattering experiments it is the phase shift that is often the quantity of primary interest. Such collisions are governed by quantum mechanics, but there is a corresponding phenomenon of resonance. When the incident beam of particles matches the energy difference between two different levels in an atom or nucleus then a resonance occurs and the phase shift increases rapidly from 0 to  $\pi$ .

**An Example** Consider a massless damped spring that is attached to the ceiling. If an attached mass is released then the equation of motion ( $x$  positive in the downward direction) is

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = mg \rightarrow \frac{d^2 x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_o^2 x = g. \quad (22)$$

Further assume that the spring is critically damped so that  $\omega_o^2 = \beta^2$ . By observation we note that the particular solution for this equation is simply  $x_p = g/\omega_o^2 = g/\beta^2$ . Adding the homogeneous solutions leads to the general solution

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t} + g/\beta^2, \quad (23)$$

subject to the initial conditions  $x(t=0) = \dot{x}(t=0) = 0$ . The condition  $x(t=0) = 0$  yields  $C_1 = -g/\beta^2$ . Whereas the condition  $\dot{x}(t=0) = 0$  leads to  $C_2 = -g/\beta$ . Hence the complete solution is

$$x(t) = \frac{g}{\beta^2} (1 - e^{-\beta t}) - \frac{g}{\beta} t e^{-\beta t}. \quad (24)$$



If the final resting place is  $.5m$  below the point of release then

$$\frac{g}{\beta^2} = .5 \rightarrow \beta = \sqrt{2g}, \quad (25)$$

and we have

$$x(t) = \frac{1}{2} \left( 1 - e^{-\sqrt{2gt}} \right) - \sqrt{g/2} t e^{-\sqrt{2gt}}. \quad (26)$$

In problem 5.28 they ask for the position of the mass 1 sec after it is released. From equation (26) and a value for  $g$  of  $g = 9.8m/sec^2$ , we have

$$x(t) = .4676m. \quad (27)$$

or just  $3.24cm$  above the final resting spot.

Now consider this problem from a slightly different point of view. Assume that the initial position is  $x_0 = -.5m$  above the equilibrium position of  $x(t \rightarrow \infty) = 0$ . Note that  $x$  is still positive in the downward direction. Basically we are defining  $x' = x - x_0$  (or  $x = x' + x_0$ ) where  $x_0 = -g/\beta^2 = -.5m$ . Ignoring the primes (for convenience) the equation of motion is now

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \beta^2 x = 0,$$

with the initial condition  $x_0 = g/\beta^2 = -.5m$ ,  $\dot{x}(t=0) = 0$ . The solution for a critically damped oscillator is

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}.$$

From the initial condition for  $x$ ,  $x = x_0 = -.5$  we find  $C_1 = -.5$ . The velocity is now given by

$$\dot{x}(t) = -\beta \left( -.5 e^{-\beta t} + C_2 t e^{-\beta t} \right) + C_2 e^{-\beta t}.$$

Evaluating this expression at  $t = 0$  yields

$$\dot{x}(t=0) = -\beta/2 + C_2 = 0 \rightarrow C_2 = \beta/2.$$

Our solution for the position of the mass on the spring is

$$x(t) = -\frac{1}{2} (1 + \beta t) e^{-\beta t} = -\frac{1}{2} \left( 1 + \sqrt{2gt} \right) e^{-\sqrt{2gt}}.$$

This is different from the earlier solution by exactly  $1/2$  which is as it had to be.