4 Lecture 10-2

4.1 Chapter 2 Projectiles and Charged Particles (con)

4.1.1 Quadratic Air Resistance

While we can find examples for which the drag of an object is linear with respect to its velocity, notably very small objects or for very small velocities, e.g. the Millikan oil drops, more obvious examples such as baseballs etc. are subject to quadratic drag. For this case the x and y components of the equation of motion are not in general separable. Additionally the equations are nonlinear which are often significantly more complicated than linear differential equations. For these reasons we shall consider purely horizontal or vertical motion.

In the case of purely horizontal motion, Newton's equation of motion is given by

$$m\frac{dv_x}{dt} = -cv_x^2. \tag{1}$$

This equation is easily separated which allows us to obtain the integrals

$$m\int_{v_{x0}}^{v_x} \frac{dv_x}{v_x^2} = -c\int_0^t dt. \tag{2}$$

These integrals are well known and we find

$$m\left(\frac{1}{v_{x0}} - \frac{1}{v_x}\right) = -ct.$$

Solving for v_x yields

$$\begin{aligned} \frac{1}{v_x} &= \frac{1}{m}ct + \frac{1}{v_{x0}} = \frac{1}{v_{x0}}\left(1 + cv_{x0}t/m\right) \\ v_x &= \frac{v_{x0}}{1 + t/\tau}, \end{aligned} \tag{3}$$

where

$$\tau = m/cv_{x0}. \tag{4}$$

This is a different time constant that the one we obtained for the case of linear drag. Here when $t = \tau$ the velocity is reduced by a factor of 2 versus e^{-1}. NTL, both time constants give us a measure of the time required for wind resistance to slow the motion of the object appreciably.

To find the position of the object as a function of time we merely integrate this solution via

$$\begin{aligned} x &= x_0 + \int_0^t \frac{v_{x0}}{1 + t/\tau} dt \\ x &= x_0 + v_{x0}\tau \ln\left(1 + t/\tau\right). \end{aligned} \tag{5}$$

The velocity still goes to zero as $t \to \infty$, but in this case it does so much more slowly. So slow in fact that x increases without limit. Remember however that when the velocity of the particle becomes small enough the drag becomes linear and the velocity will begin to fall off exponentially. Thus no real body can coast to infinity.

For vertical motion Newton's equation of motion is

$$m\frac{dv_y}{dt} = mg - cv_y^2, \tag{6}$$

where we are measuring positive y to be vertically down. Again as in the linear case it is useful to find the terminal velocity. In this situation $dv_y/dt = 0$ (the same as in the linear case) and

$$v_{\text{ter}} = \sqrt{mg/c}. \tag{7}$$

Rewriting the equation of motion in terms of the terminal velocity yields

$$\frac{dv_y}{dt} = \frac{g}{v_{\text{ter}}^2}\left(v_{\text{ter}}^2 - v_y^2\right) \tag{8}$$

We will assume that the object (ball) is dropped from rest. Then using the technique of separation of variables we find

$$\int_0^{v_y} \frac{dv_y}{v_{\text{ter}}^2 - v_y^2} = \frac{g}{v_{\text{ter}}^2}\int_0^t dt. \tag{9}$$

This integrand can be expanded into partial fractions,

$$\frac{1}{v_{\text{ter}}^2 - v_y^2} = \frac{1}{2v_{\text{ter}}}\left(\frac{1}{v_{\text{ter}} - v_y} + \frac{1}{v_{\text{ter}} + v_y}\right), \tag{10}$$

which enables us to perform the integral in a straightforward fashion. The result is

$$\ln\frac{v_{\text{ter}} + v_y}{v_{\text{ter}} - v_y} = \frac{2gt}{v_{\text{ter}}}. \tag{11}$$

Solving for v_y leads to

$$\begin{aligned} v_{\text{ter}} + v_y &= (v_{\text{ter}} - v_y)\exp 2gt/v_{\text{ter}} \\ v_y &= v_{\text{ter}}\frac{e^{2gt/v_{\text{ter}}} - 1}{e^{2gt/v_{\text{ter}}} + 1} = v_{\text{ter}}\tanh gt/v_{\text{ter}}. \end{aligned} \tag{12}$$

For $gt/v_{\text{ter}} << 1$ this expression reduces to

$$v_y = gt, \tag{13}$$

which is what you would expect for a falling object. However, the hyperbolic tangent rapidly approaches 1 as gt/v_{ter} increases beyond 1, so that the velocity of the object quickly reaches its terminal velocity. To find the distance the object has fallen we simply integrate the vertical velocity to find

$$y = \frac{v_{\text{ter}}^2}{g}\ln\cosh gt/v_{\text{ter}} \tag{14}$$

If the particle has both horizontal and vertical velocity components then the equations of motion are

$$m\dot{v}_x = -c\left(v_x^2 + v_y^2\right)\frac{v_x}{v} = -c\sqrt{v_x^2 + v_y^2}v_x \quad (15a)$$

$$m\dot{v}_y = mg - c\left(v_x^2 + v_y^2\right)\frac{v_y}{v} = mg - c\sqrt{v_x^2 + v_y^2}v_y. \quad (15b)$$

These equations are both nonlinear and coupled and cannot be solved analytically. Our only choice is to choose a set of initial conditions for v_x and v_y and numerically integrate these equations. Since $v_x(t+\delta t) = v_x(t) + \dot{v}_x\delta t$, and v_y satisfies a similar expression, the numerical integration of these equations is straightforward. An example for a baseball thrown off of a cliff is shown in figure 2-3.

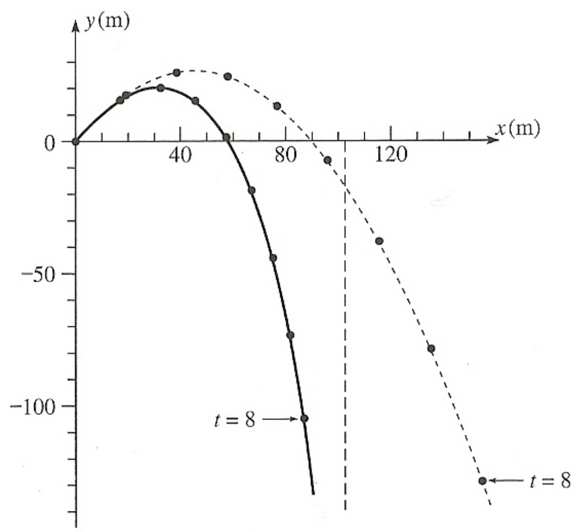

Figure 2-3. Trajectory of a baseball thrown off of a cliff subject to quadratic air resistance. The initial velocity is $30m/s$ at $50°$ above the horizontal. The terminal speed is $35m/s$ with a vertical asymptote just beyond $x \simeq 100m$. The dashed line the trajectory in a vacuum. The dots show the position of the ball at one-second intervals.

Note that, as with the case for linear resistance, there is a vertical asymptote for the horizontal range. This can be seen by examining the EOM that as $v_y \to v_{ter} >> v_x$, the horizontal velocity approaches $v_x \propto \exp(-kt)$, where $k = cv_{ter}/m$. Integrating this expression easily demonstrates that there is an asymptotic limit to the range.

4.1.2 Motion of Charge in a Uniform Magnetic Field

Again to introduce some important mathematical methods, we will consider the motion of a charged particle in a magnetic field. The net force on a particle moving in a magnetic field is

$$\overrightarrow{F} = q\left(\overrightarrow{v} \times \overrightarrow{B}\right). \tag{16}$$

This leads to an equation of motion given by

$$m\dot{\overrightarrow{v}} = q\left(\overrightarrow{v} \times \overrightarrow{B}\right) \tag{17}$$

Since the magnetic field is uniform (spatially and temporally) we will let it define the z direction. With this definition the three components of the equation of motion are

$$m\dot{v}_x = qv_yB \tag{18a}$$

$$m\dot{v}_y = -qv_xB \tag{18b}$$

$$m\dot{v}_z = 0. \tag{18c}$$

Clearly the z component of the velocity is a constant of the motion and will be ignored for the moment in this problem. Alternately we could choose an inertial frame in which $v_z = 0$, however in general, that will not be the laboratory frame from which we observe the orbit defining the motion of the charge. NTL, we will now only concern ourselves with v_x and v_y. Before we proceed it is convenient to define the *cyclotron frequency* as

$$\omega = \frac{qB}{m}. \tag{19}$$

With this definition the equations of motion become

$$\dot{v}_x = \omega v_y \tag{20a}$$

$$\dot{v}_y = -\omega v_x. \tag{20b}$$

These two relatively simple first order differential equation can be solved in several different ways. One of the more obvious methods might be to differentiate the equation for $\dot{v}_x$ and then substitute for $\dot{v}_y$. While this would clearly work, the solution to the resulting second order differential equation has two unknown constants of the motion. A similar approach used to solve for v_y introduces once again two undetermined constants of integration. The point here is that differentiating these equations introduces new unknown constants of integration. To resolve this issue requires that you substitute the solution for v_y back into the original first order differential equation for v_x. That is, you must ensure that your solutions solve the original coupled first order differential equations. This eliminates all but two of the unknown constants. The other two are determined by the initial conditions.

An interesting approach that does not introduce new unknown constants is to make use of complex numbers. Here we define the complex variable

$$\eta = v_x + iv_y, \tag{21}$$

where i denotes the square root of -1. The two coupled differential equations can then be reduced to a single first order differential equation as

$$\dot{\eta} = \dot{v}_x + i\dot{v}_y = \omega v_y - i\omega v_x = -i\omega\left(v_x + iv_y\right) = -i\omega\eta \tag{22}$$

This equation can be immediately integrated with the solution

$$\eta = Ae^{-i\omega t}. \tag{23}$$

It now appears that there is only one unknown constant, whereas we should expect two (one for each variable). The answer to this question is that in general A is also complex. We could assume that $A = a + ib$. However, it is more useful to take advantage of *Euler's formula,* $e^{i\theta} = \cos\theta + i\sin\theta$. We can now define a phase angle so that

$$A = ve^{i\delta} = v\left(\cos\delta + i\sin\delta\right), \tag{24}$$

where v is the magnitude of the transverse velocity, $v^2 = v_x^2 + v_y^2$, which as we see for this problem is a constant. With this definition the solution for our variable η becomes

$$\eta = ve^{i(\delta-\omega t)} = v\left(\cos\left(\delta-\omega t\right) + i\sin\left(\delta-\omega t\right)\right), \text{ and} \tag{25}$$

$$v_x = v\cos\left(\delta-\omega t\right), \quad v_y = v\sin\left(\delta-\omega t\right) \tag{26}$$

It is useful to see that Euler's formula implies the complex number $e^{i\theta}$ lies on

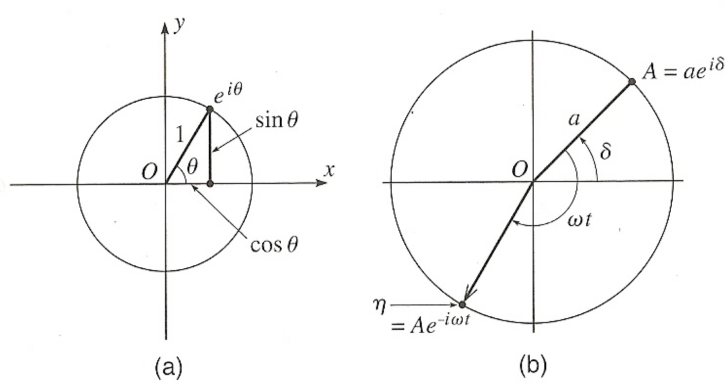

Figure 2-4. (a) Euler's formula implies that the complex number $e^{i\theta}$ lies on the unit circle center at the origin. (b) The complex function $A = ae^{i\delta}$ lies on a circle of radius a with polar angle δ. The function $\eta\left(t\right) = Ae^{-i\omega t}$ lies on the same circle with with polar angle $\left(\delta - \omega t\right)$.

the unit circle in the complex plane and the solution for η lies on a circle of radius v where the polar angle $\left(\delta - \omega t\right)$ rotates in a clockwise direction as t advances. Both of these descriptions are shown in Figure 2-4.

Integrating the solutions for v_z and η are straightforward with the results

$$\begin{aligned} z &= z_o + v_{zo}t \quad (27) \\ \xi &= x + iy = \int \eta dt = \frac{iv}{\omega} e^{i(\delta - \omega t)} + x_o + iy_o. \quad (28) \end{aligned}$$

Defining the origin so that $x_o = y_o = 0$, the solutions become

$$\begin{aligned} z &= z_o + v_{zo}t \quad (29) \\ \xi &= x + iy = \frac{v}{\omega} e^{i(\delta + \pi/2 - \omega t)}. \quad (30) \end{aligned}$$

Hence the orbit for the charged particle is also defined by its x and y positions lying on a circle, this time with a radius given by

$$r = \frac{v}{\omega} = \frac{mv}{qB}. \quad (31)$$

In general the motion is helical as the particle can propagate in the $\pm z$ direction depending on the sign of v_{zo}.

For the special case of $v_{zo} = 0$, the motion of the particle lies on the circle defined above. In a cyclotron, the particles are slowly accelerated by the timed application of an electric field. Ignoring relativistic effects, the frequency of the particle orbiting in a uniform magnetic field remains fixed and as the particle gains energy and momentum it proceeds to the outer radius of the field where it emerges and can be used in scattering experiments.

4.2 Chapter 3 Momentum and Angular Momentum

4.2.1 Conservation of Momentum

In chapter 1 we found that as long as all the internal forces in a system of N particles obeyed Newton's third law, the rate of change of the system's total linear momentum, $\vec{P} = \vec{p}_1 + \cdots + \vec{p}_N$, is determined only by the *external* forces on the system, i.e.

$$\dot{\vec{P}} = \vec{F}^{\text{ext}}. \quad (32)$$

In particular if the system is isolated (no external forces), then we have

Principle of Conservation of Momentum

> If the net external force $\vec{F}^{\text{ext}}$ on an N-particle system is zero, the system's total mechanical momentum $\vec{P} = \sum m_\alpha \vec{v}_\alpha$ is constant.

As an example consider two masses with mass m_1 and m_2 with velocities $\vec{v}_1$ and $\vec{v}_2$ respectively. Their initial total momentum is then

$$\vec{P}_{\text{i}} = m_1 \vec{v}_1 + m_2 \vec{v}_2. \quad (33)$$

If the collide and stick together (a perfectly inelastic collision) the final momentum is given by $\overrightarrow{P}_f = (m_1 + m_1)\,\overrightarrow{v}_f$ as both particles are moving with the same velocity $\overrightarrow{v}_f$. Now from the conservation of momentum we can find this velocity $\overrightarrow{v}$ by noting that in the absence of any external forces $\overrightarrow{P}_i = \overrightarrow{P}_f$, and

$$\overrightarrow{v}_f = \frac{m_1\overrightarrow{v}_1 + m_2\overrightarrow{v}_2}{m_1 + m_2}. \tag{34}$$

An important special case is when one of the bodies is initially at rest, as when a speeding car rear ends a stationary car at a stop light. Conservation of momentum then tells us

$$\overrightarrow{v}_f = \frac{m_1}{m_1 + m_2}\overrightarrow{v}_1. \tag{35}$$

We see that in this case the final velocity is in the same direction as $\overrightarrow{v}_1$ but at a reduced speed determined ratio of the masses. (This final velocity can be found from the skid marks of the combined wreck.) This sort of analysis of collisions, using the conservation of momentum, is an important tool in solving many problems ranging from nuclear reactions to the collisions of galaxies.

4.2.2 Rockets

A nice example that makes use of the conservation of momentum is the propulsion of a rocket. For a rocket there is nothing to push against, so how does it get itself moving? By ejecting the propellant out through the thrusters, it is accelerating the mass of the propellant in the opposite direction. Then by Newton's third law, the fuel pushes the rocket forward.

To analyze this problem we must examine the total momentum. Since the rocket is ejecting mass, the rocket's mass is steadily decreasing. At time t the momentum of the rocket is $P(t) = mv$ (We will ignore vector notation as the only direction of interest is that along which the rocket is traveling.) A short time later, $t + dt$, the rocket's mass is $m + dm$, where dm is negative, and its momentum is $(m + dm)(v + dv)$. If the fuel is ejected at a velocity u relative to the rocket, then the momentum of the ejected fuel (which has mass $-dm$) is $(-dm)(v - u)$. The total momentum at time $t + dt$ is

$$P(t + dt) = (m + dm)(v + dv) + dm(u - v) = mv + mdv + udm, \tag{36}$$

where we have neglected the second order term $dmdv$. The change in momentum during the time dt is

$$dP = P(t + dt) - P(t) = mdv + udm. \tag{37}$$

Now if there is an external force, e.g. gravity, then the change in momentum during the time dt is $F^{\text{ext}}dt$ and dividing by dt we find

$$F^{\text{ext}} = m\frac{dv}{dt} + u\frac{dm}{dt} \rightarrow m\frac{dv}{dt} = -u\frac{dm}{dt} + F^{\text{ext}}, \tag{38}$$

Here we shall assume no external forces and leave it as an exercise for the student to consider the effects of gravity. Hence $dP = 0$ and $mdv = -udm$. Dividing this by dt and we can rewrite this expression as

$$m\frac{dv}{dt} = -u\frac{dm}{dt}. \tag{39}$$

Here $-dm/dt$ is the rate at which the rocket's engine is ejecting mass. This equation looks just like Newton's second law except that the product $-udm/dt$ plays the role of the force. For this reason this product is often called the *thrust*.

Using the technique of separation of variable that we employed successfully when we examined projectiles in the presence of drag allows us to solve this equation as well. We find

$$\begin{aligned} dv &= -u\frac{dm}{m} \\ v - v_o &= u \ln \frac{m_o}{m}, \end{aligned} \tag{40}$$

where v_o and m_o are the initial velocity and mass respectively. This result puts a significant restriction on the maximum speed of the rocket. For example, even if the original mass is 90% fuel and all of this fuel is exhausted the quantity $\ln m_o/m = \ln 10 = 2.3$. So the speed gained cannot exceed $2.3u$. This means rocket engineers try to make u as large as possible and also design multistage rockets which can jettison the heavy fuel tanks of the early stages to reduce the total mass of the later stages.