9 Lecture 10-14

9.1 Chapter 4 Energy (con)

9.1.1 Energy of Interaction of Two Particles

We would like to extend our discussion to a system of multiple particles. We will start by considering just two particles. We shall assume that the interaction between two particles is *translationally invariant*. This means that if we translate the system to a new position, without changing the relative positions of the two particles, the interaction between the two particles remains the same. This is consistent with our physical intuition, and we will assume that this property is always maintained. As an example we will examine the gravitational interaction between two particles with masses m_1 and m_2. The force on m_1 due to $\overrightarrow{F}_{12}$ is expressed as a function of $\overrightarrow{r}_1$ and $\overrightarrow{r}_2$ as

$$\overrightarrow{F}_{12} = -\frac{Gm_1m_2}{|\overrightarrow{r}|^2}\widehat{r} = -\frac{Gm_1m_2}{|\overrightarrow{r}|^3}\overrightarrow{r}, \tag{1}$$

where

$$\overrightarrow{r} = \overrightarrow{r}_1 - \overrightarrow{r}_2. \tag{2}$$

Rewriting this interaction as a function of $\overrightarrow{r}_1 - \overrightarrow{r}_2$ yields

$$\overrightarrow{F}_{12} = \overrightarrow{F}\left(\overrightarrow{r}_1 - \overrightarrow{r}_2\right) = -\frac{Gm_1m_2}{|\overrightarrow{r}_1 - \overrightarrow{r}_2|^3}\left(\overrightarrow{r}_1 - \overrightarrow{r}_2\right). \tag{3}$$

Here $\overrightarrow{r}_1$ and $\overrightarrow{r}_2$ are measured from the same arbitrary origin. In this form it is easily seen that translating the system by an amount $\overrightarrow{s}$ ($\overrightarrow{r}_1 \to \overrightarrow{r}_1 - \overrightarrow{s}$ and $\overrightarrow{r}_2 \to \overrightarrow{r}_2 - \overrightarrow{s}$) leaves the interaction unchanged. In this form it is also easy to see that Newton's third law is satisfied as $\overrightarrow{F}_{12} = -\overrightarrow{F}_{21}$. Before we leave this expression it is also useful to note that the vector $\overrightarrow{r}_1 - \overrightarrow{r}_2$ originates from the location of $\overrightarrow{r}_2$ and terminates at $\overrightarrow{r}_1$. This means that the gravitational interaction of m_2 on m_1 points toward m_2 meaning that this interaction is attractive.

The result in equation (3) simplifies our discussion. We can learn almost everything of interest about $\overrightarrow{F}_{12}$ by fixing $\overrightarrow{r}_2$ a convenient point, namely the origin. For example if the force $\overrightarrow{F}_{12}$ on particle 1 is conservative then it must satisfy

$$\nabla_1 \times \overrightarrow{F}_{12} = 0. \tag{4}$$

If the curl does vanish, then we can define a potential energy via

$$\overrightarrow{F}_{12} = -\nabla_1 U\left(\overrightarrow{r}_1\right), \tag{5}$$

where ∇_1 is the operator

$$\nabla_1 = \widehat{x}\frac{\partial}{\partial x_1} + \widehat{y}\frac{\partial}{\partial y_1} + \widehat{z}\frac{\partial}{\partial z_1} = \widehat{r}_1\frac{\partial}{\partial r_1} + \widehat{\theta}_1\frac{1}{r_1}\frac{\partial}{\partial \theta_1} + \widehat{\phi}_1\frac{1}{r_1\sin\theta_1}\frac{\partial}{\partial \phi_1}. \tag{6}$$

This gives the force when particle 2 is at the origin. To find it with particle 2 being at some arbitrary location we merely replace $\overrightarrow{r}_1$ with $\overrightarrow{r}_1 - \overrightarrow{r}_2$ and we have

$$\overrightarrow{F}_{12} = -\nabla_1 U\left(\overrightarrow{r}_1 - \overrightarrow{r}_2\right). \tag{7}$$

Notice that we didn't have to change the operator ∇_1 since partial derivatives are not impacted by (effectively) adding a constant to $\overrightarrow{r}_1$.

To find the reaction force we can merely change the sign of $\overrightarrow{F}_{12}$, as is clear in equation (3), or we can simply replace ∇_1 by ∇_2 which yields

$$\overrightarrow{F}_{21} = -\nabla_2 U\left(\overrightarrow{r}_1 - \overrightarrow{r}_2\right) = -\overrightarrow{F}_{12} \tag{8}$$

Equations (7) and (8) are an elegant result that generalizes to multiple particle systems. To emphasize this result we will rewrite them as

$$\begin{array}{rcl} \text{Force on particle 1} & = & -\nabla_1 U \\ \text{Force on particle 2} & = & -\nabla_2 U. \end{array} \tag{9}$$

So there is a *single* potential energy function U, from which we can derive *both* forces. To find the force on particle 1, we merely take the gradient of U with respect to the coordinates of particle 1, and to find the force on particle 2, we take the gradient of U with respect to the coordinates of particle 2.

Before we generalize this result to multiple particle systems, let us consider the conservation of energy for our two-particle system. From the work-KE theorem the work done on particle 1 during a short time period dt is

$$dT_1 = \overrightarrow{F}_{12} \cdot d\overrightarrow{r}_1, \tag{10}$$

while the work done on particle 2 is

$$dT_2 = \overrightarrow{F}_{21} \cdot d\overrightarrow{r}_2. \tag{11}$$

Adding these two expressions we find the change in the total kinetic energy as

$$dT = dT_1 + dT_2 = \overrightarrow{F}_{12} \cdot d\overrightarrow{r}_1 + \overrightarrow{F}_{21} \cdot d\overrightarrow{r}_2. \tag{12}$$

Since $\overrightarrow{F}_{21} = -\overrightarrow{F}_{12}$ we see that

$$dT = \overrightarrow{F}_{12} \cdot \left(d\overrightarrow{r}_1 - d\overrightarrow{r}_2\right) = \overrightarrow{F}_{12} \cdot d\left(\overrightarrow{r}_1 - \overrightarrow{r}_2\right). \tag{13}$$

Substituting $-\nabla_1 U$ for $\overrightarrow{F}_{12}$ yields

$$dT = -\nabla_1 U\left(\overrightarrow{r}_1 - \overrightarrow{r}_2\right) \cdot d\left(\overrightarrow{r}_1 - \overrightarrow{r}_2\right). \tag{14}$$

Now replacing $\overrightarrow{r}_1 - \overrightarrow{r}_2$ with the relative coordinate $\overrightarrow{r}$ we find

$$dT = -\nabla U \cdot d\overrightarrow{r} = -dU. \tag{15}$$

Moving $-dU$ to the other side of the equation allows us to conclude

$$d(T+U)=0. \tag{16}$$

That is the total energy,

$$E=T+U=T_1+T_2+U, \tag{17}$$

of our two particle system is conserved. It is important to note that the total energy of our two particle system contains the kinetic energy of both particles (of course), but only one potential energy, the potential energy of the interaction between the two particles. Here U accounts for the work done by both of the forces $\overrightarrow{F}_{12}$ and $\overrightarrow{F}_{21}$.

Elastic Collisions As an application of these ideas, consider an elastic collision. We will define an elastic collision as a collision between two particles that interact via a conservative force (so we can define a potential energy of interaction) that goes to zero as $|\overrightarrow{r}_1-\overrightarrow{r}_2|\to\infty$. Since the force goes to zero, the potential energy approaches a constant which we will usually define to be zero (remember our reference point $\overrightarrow{r}_0$). For example, the two particles could be an electron and a proton or two billiard balls. We can include billiard balls in this discussion because as it turns out billiard balls are designed so that they behave like almost perfect springs when they come into contact with each other. Clearly lumps of putty or other similar objects interact via nonconservative forces and the collisions between these types of objects (deform easily) are not elastic.

In an elastic collision between two particles the total energy is conserved, i.e. $T+U=T_1+T_2+U=E$, where E is a constant. Additionally $U\to 0$ as their separation becomes large. If we use the subscripts i and f to denote their initial and final conditions when they are far apart then we have

$$T_i=T_f\to(T_1+T_2)_i=(T_1+T_2)_f\,. \tag{18}$$

In other words an elastic collision is one in which two particles come together and reemerge with their total kinetic energy unchanged. Note, however, that the total kinetic energy is not conserved throughout the interaction. As they approach each other their potential energy is nonzero and consequently their kinetic energy is changing. It is only when they are well removed from each other (and therefore their potential energy is negligible) that their total kinetic energy is conserved.

Our discussion seems to imply that elastic collisions are very common as all that is required is two particles interacting with a conservative force. The problem is that we require two particles to enter and leave the collision. For example if we fire one of the billiard balls with enough energy, it may shatter the second billiard ball. Similarly, if we fire an electron with sufficient energy at an atom, the atom may break up or at least go into an excited state. Even the collision of two genuine particles such as an electron and a proton, we know that new particles may be created. Clearly at high enough energies the assumption that two particles come together and reemerge breaks down even if all the underlying forces are conservative. Never-the-less at reasonably low energies there are many situations where the collisions are perfectly elastic.

As an example of an elastic collision, consider the collision between two particles of equal mass (electrons or billiard balls). If one of the particles is at rest (particle 2) then from the conservation of momentum we have

$$\overrightarrow{v}_1 = \overrightarrow{v}'_1 + \overrightarrow{v}'_2, \tag{19}$$

where we denote the velocities after the collision with a prime. Since we have an elastic collision we can also state that

$$\overrightarrow{v}_1^2 = \overrightarrow{v}_1'^2 + \overrightarrow{v}_2'^2. \tag{20}$$

Squaring equation (19) yields

$$\overrightarrow{v}_1^2 = \overrightarrow{v}_1'^2 + \overrightarrow{v}_2'^2 + 2\overrightarrow{v}'_1 \cdot \overrightarrow{v}'_2. \tag{21}$$

Subtracting equation (20) from this results we find

$$\overrightarrow{v}'_1 \cdot \overrightarrow{v}'_2 = 0. \tag{22}$$

So unless one of the velocities is zero they exit the collision perpendicular to each other. This result is useful in interpreting scattering experiments. When an unknown projectile hits a stationary target particle, the fact that the two emerge at 90° is taken as evidence that the collision was elastic and the two particles had equal masses.

9.1.2 Energy of a Multiple Particle System

For a multiple particle system with N particles, the total kinetic energy is simply

$$T = T_1 + T_2 + \cdots = \sum_{\alpha=1}^{N} T_\alpha. \tag{23}$$

To define the potential energy, we must first examine the forces on the particles. First we consider the internal forces of the particles acting on each other. We shall take for granted that each of the interparticle forces $\overrightarrow{F}_{\alpha\beta}$ are unaffected by the presence of any of the other particles. Of course other particles can exert additional forces on the α particle, but we are claiming that only the force of the β particle on the α particle contributes to $\overrightarrow{F}_{\alpha\beta}$. One could imagine a world where this is not true (for example a three body interaction), but experiment seems to confirm our assumption. Thus we can treat the pair of forces $\overrightarrow{F}_{\alpha\beta}$ and $\overrightarrow{F}_{\beta\alpha}$ exactly as we did for the two body problem. Now provided the forces are conservative we can define a potential energy

$$U_{\alpha\beta} = U\left(\overrightarrow{r}_\alpha - \overrightarrow{r}_\beta\right). \tag{24}$$

The corresponding forces are then

$$\overrightarrow{F}_{\alpha\beta} = -\nabla_\alpha U\left(\overrightarrow{r}_\alpha - \overrightarrow{r}_\beta\right) \text{ and } \overrightarrow{F}_{\beta\alpha} = -\nabla_\beta U\left(\overrightarrow{r}_\alpha - \overrightarrow{r}_\beta\right). \tag{25}$$

Being careful to only include the potential energy between distinct pairs of particles, the total potential energy is then given by

$$U^{\text{int}} = \sum_\alpha \sum_{\beta>\alpha} U_{\alpha\beta} \tag{26}$$

where again

$$U_{\alpha\beta} = U\left(\overrightarrow{r}_\alpha - \overrightarrow{r}_\beta\right). \tag{27}$$

If the forces are central (as is usually the case) then, remembering our assumption about them being conservative, $U_{\alpha\beta}$ just depends on the magnitude of $\overrightarrow{r}_\alpha - \overrightarrow{r}_\beta$ and

$$U_{\alpha\beta} = U\left(|\overrightarrow{r}_\alpha - \overrightarrow{r}_\beta|\right). \tag{28}$$

Hence $U_{\alpha\beta} = U_{\beta\alpha}$ and the total energy potential becomes

$$U^{\text{int}} = \sum_\alpha \sum_{\beta>\alpha} U_{\alpha\beta} = \frac{1}{2} \sum_{\alpha,\beta\neq\alpha} U_{\alpha\beta} \tag{29}$$

If there is a conservative external field, e.g. N charged particles in an electric field, then each of the N particles has a potential energy associated with this field. The total potential energy is then

$$U = U^{\text{int}} + U^{\text{ext}} = \sum_\alpha U_\alpha^{\text{ext}} + \frac{1}{2} \sum_{\alpha,\beta\neq\alpha} U_{\alpha\beta} \tag{30}$$

Now, as a rigid body moves, the positions $\overrightarrow{r}_\alpha$ of the atoms can of course move but the distance $|\overrightarrow{r}_\alpha - \overrightarrow{r}_\beta|$ between any two atoms cannot change. Therefore, the internal potential energy does not change. Basically it is a constant and can be ignored. Thus in applying energy considerations to rigid bodies we totally ignore U^{int} and only consider the potential due to external fields, U^{ext}. This latter energy is often a very simple function and energy considerations as applied to a rigid body are often straightforward.

As an example of this, consider a cylinder of mass M and radius R rolling without slipping down an incline as shown in Figure 4.11. Its kinetic energy is a combination of translational and rotational terms,

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2, \tag{31}$$

where I is the momentum of inertia and as we have shown $I = mR^2/2$.

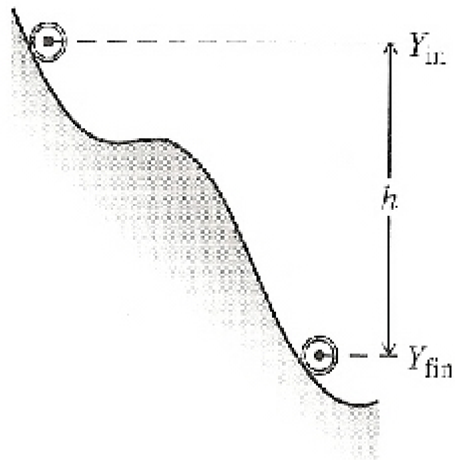

Figure 4.11. A uniform cylinder starts from rest and rolls without slipping through a total vertical height $h = Y_{fin} - Y_{in}$.

Since the cylinder rolls without slipping $v = R\omega$, and the total kinetic energy is

$$T = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\frac{v^2}{R^2} = \frac{3}{4}mv^2 \tag{32}$$

There are three external forces on the cylinder, the normal force of the incline, frictional forces, and gravity. The normal force does no work and as long as the cylinder does not slip neither does the force of friction. We are left with the gravitational force and it is conservative. If the particle starts from rest and descends a height h then from the conservation of energy

$$\Delta T + \Delta U = 0, \tag{33a}$$

$$\frac{3}{4}mv^2 - mgh = 0, \tag{33b}$$

so that

$$v = \sqrt{4gh/3}. \tag{34}$$

As a final example consider a hydrogen atom in which an electron, which we will label as electron 1, is in a circular orbit of radius r around a proton. A second electron approaches this atom from afar with kinetic energy T_2. After the collision, the original bound electron is knocked free and the second electron is captured into a circular orbit of radius r' and we wish to determine the kinetic energy of the first electron when it is far from the proton. Since the Coulomb force is of the form $U = kr^{-1}$, we know from the virial theorem for circular orbits that the kinetic energy is given by $T = nU/2 = -U/2$ and the total energy is $E = T + U = U/2$. Assuming that the proton is fixed, the total energy for this three particle system when the second electron is far away is

$$E = \frac{1}{2}mv_1^2 - \frac{ke^2}{r} + \frac{1}{2}mv_2^2 = \frac{1}{2}U_1 + T_2 = -\frac{ke^2}{2r} + T_2. \tag{35}$$

Long after the collision the energy is

$$E' = \frac{1}{2}mv_2'^2 - \frac{ke^2}{r'} + \frac{1}{2}mv_1'^2 = \frac{1}{2}U_2 + T_1' = -\frac{ke^2}{2r'} + T_1'. \tag{36}$$

By the conservation of energy we have

$$T_1' = T_2 - \frac{ke^2}{2}\left(\frac{1}{r} - \frac{1}{r'}\right) = T_2 + \frac{ke^2}{2}\left(\frac{1}{r'} - \frac{1}{r}\right) \tag{37}$$

9.1.3 Review for Chapter 4

The change in the kinetic energy of a particle as it moves from point 1 to point 2 is given by

$$\Delta T = \int_1^2 \overrightarrow{F} \cdot d\overrightarrow{r} \equiv W\left(\overrightarrow{r}_1 \rightarrow \overrightarrow{r}_2\right), \tag{38}$$

where W is the work done by the total force, $\overrightarrow{F}$, over the path from point 1 to point 2. The integral that defines the work done is a path integral and in general is path dependent. A force is conservative if (i) $\overrightarrow{F} = \overrightarrow{F}\left(\overrightarrow{r}\right)$, i.e. it is independent of time, velocity, or any other variable, and (ii) the work done is path independent. From Stokes theorem this implies that the curl of the force vanishes, $\nabla \times \overrightarrow{F} = 0$. If $\overrightarrow{F}$ is conservative then the work done only depends on the endpoints and we can define a potential energy as

$$U\left(\overrightarrow{r}\right) = W\left(\overrightarrow{r}_o \rightarrow \overrightarrow{r}\right) = -\int_{\overrightarrow{r}_o}^{\overrightarrow{r}} \overrightarrow{F} \cdot d\overrightarrow{r}, \tag{39}$$

so that $U\left(\overrightarrow{r}_o\right) = 0$.

If all of the forces are conservative with corresponding potential energies then the total mechanical energy

$$E = T + U_1 + \cdots + U_n \tag{40}$$

is conserved, hence the name conservative force. More generally if there are nonconservative forces then $\Delta E = W_{nc}$, the work done by the nonconservative forces.

A force $\overrightarrow{F}\left(\overrightarrow{r}\right)$ is a central force if the force originates (or terminates) from a force center. If this center is taken to be the origin, usually convenient, then

$$\overrightarrow{F}\left(\overrightarrow{r}\right) = f\left(\overrightarrow{r}\right)\widehat{r}. \tag{41}$$

A central force is spherically symmetric, $f\left(\overrightarrow{r}\right) = f\left(r\right)$, if and only if it is conservative.

For one-dimensional systems energy diagrams that are plots of the potential energy can be very useful in qualitatively understanding the motion of a particle. Points where the energy of the particle equals the potential energy, $E = U$, are turning points, and since the force is given by $-dU/dx$, the particle always experiences a force "in the downward direction". Points where $dU/dx = 0$ are locations of equilibrium. If $d^2U/dx^2 > 0$ then it is a position of stable equilibrium as opposed to the condition $d^2U/dx^2 < 0$ which denotes a position of unstable equilibrium.

For a multiparticle system in the presence of conservative forces the total potential energy is

$$U = U^{int} + U^{ext} = \frac{1}{2} \sum_{\alpha, \beta \neq \alpha} U_{\alpha\beta} + \sum U_{\alpha}^{ext}. \tag{42}$$

The net force on particle α is

$$\overrightarrow{F}_{\alpha} = -\nabla_{\alpha} U, \tag{43}$$

and the total energy is conserved.