8 Lecture 10-12

8.1 Chapter 4 Energy (con)

8.1.1 Curvilinear One-Dimensional Systems

When an object is confined to travel along any one-dimensional curved path, we simply define the distance along curved path to be s. Then the kinetic energy still takes the simple form

$$T = \frac{1}{2}m\dot{s}^2, \tag{1}$$

as compared to $m\dot{x}^2/2$ for a straight track. As the object travels along this curved path there are normal forces required to keep the object on the path. However, normal forces do no work and do not impact the total mechanical energy of the system. Any tangential component of the force does do work and in an analogous way to that used for the linear one-dimensional problem we can show

$$F_{\text{tan}} = m\ddot{s}. \tag{2}$$

Again note the analogy with $F_x = m\ddot{x}$ along a straight track. Further if all the forces that have tangential components with our track are conservative then we can define a corresponding potential energy $U(s)$ such that $F_{\text{tan}} = -dU/ds$. The entire discussion that we have been involved in still applies and the total mechanical energy $E = T + U(s)$ is conserved.

Some of the examples that can be treated in this manner are more involved than an object traveling along a curved track. Consider a cube balanced on a cylinder in Figure 4.8. A hard rubber cylinder (no slip condition applies) of radius r is held fixed with its axis horizontal, and a wooden cube of side $2b$ is balanced on top of the cylinder. The center of the cube is directly above the axis of the cylinder and four of its sides are parallel to the axis as well (see Figure 4.8). The cube cannot slide on the cylinder but it can, of course, rock back and forth. The goal in this example will be to find the effective potential energy and determine the conditions required for stable equilibrium for the cube. As a single parameter we can define the distance s along the surface of the cylinder. However it is a bit more convenient to define the angle θ that determines the point of contact between the cylinder and the cube. The constraining forces, the normal force and frictional force, simply constrain the cube to rock on cylinder. Since they do no work we will not consider them explicitly. We will define the potential energy to be $U = mgh$, where h is the distance above the center of

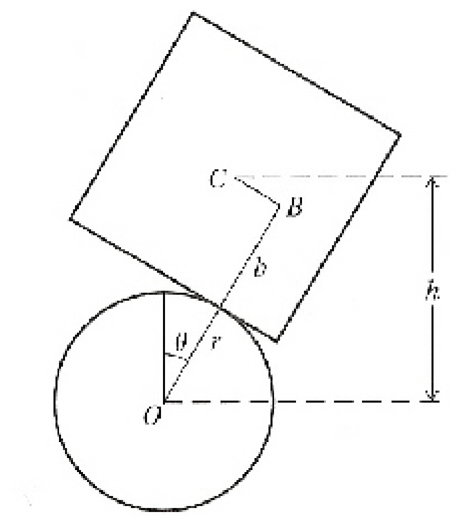

Figure 4.8. A cube of side $2b$ and center C is placed on a fixed horizontal cylinder of radius r and center O. It is originally placed so that C is directly above O. The cube can roll from side to side without slipping.

the cylinder. This problem can be analyzed using the geometry shown in Figure 4.8. The length of the line OB is $r + b$. In terms of the angle θ, the potential energy is given by

$$U(\theta) = mgh = mg\left((r+b)\cos\theta + CB\sin\theta\right). \tag{3}$$

Now imagine an additional point A that is the bisector of the bottom of the cube. This point is in contact with the cylinder when the cube is perfectly balanced on the cylinder, $\theta = 0$. As the cube rocks on the cylinder the distance from A to the point of contact between the cube and cylinder is equal to CB and given by $r\theta$. Thus the potential energy can be written as

$$U(\theta) = mgh = mg\left((r+b)\cos\theta + r\theta\sin\theta\right). \tag{4}$$

The derivative of $U(\theta)$ is found to be

$$\frac{dU}{d\theta} = mg\left(-b\sin\theta + r\theta\cos\theta\right). \tag{5}$$

This vanishes at $\theta = 0$, confirming our physical intuition that the system is in equilibrium at this point. To determine if this equilibrium is a position of stable equilibrium, we merely have to differentiate one more time to find

$$\left[\frac{d^2U}{d\theta^2}\right]_{\theta=0} = mg\left[-b\cos\theta + r\cos\theta - r\theta\sin\theta\right]_{\theta=0} = mg\,(r-b) \tag{6}$$

So we see that if the cube is smaller than the cylinder, $r > b$, so that $d^2U/dx^2 > 0$, the system is stable. Any small perturbation to the cube and it will rock back and forth atop the cylinder. Conversely if the cube is larger than the cylinder, $b > r$, so that $d^2U/dx^2 < 0$, the system is unstable. Any small perturbation will result in the cube rolling off the cube.

Further Generalizations There are many systems that can be described as one dimensional. All that is required is just one parameter to describe the system's position. As an additional example we will consider the Atwood machine in Figure 4.9 that involves two masses, an inextensible string, and a massless pulley (although it is straightforward to include the mass of the pulley). The two masses move up and down, but a nonslip condition between the inextensible string and the pulley ensures us that when one of the masses moves up the other moves down by exactly the same distance. Thus the description of the system can be specified by a single parameter, for example the distance x of m_1 below the pulley's center as shown

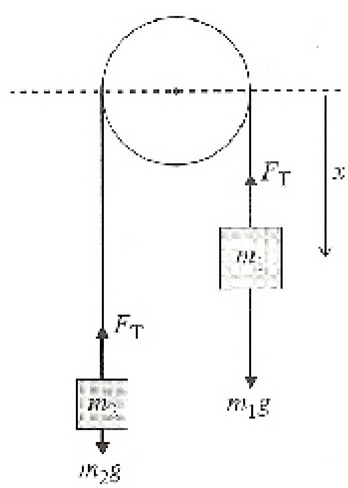

Figure 4.9. An Atwood machine consistine of two masses m_1 and m_2 with a massless pulley and string. Since the length of the string is fixed, the position of the whole system is specified by the distance x of m_1 below any convenient fixed level. The tension in the string F_T is the same all along the string.

in Figure 4.9. Thus this system is again one-dimensional.

Since the gravitational force is conservative, we can define potential energies U_1 and U_2 for the gravitational energies. The work done by the tension in the string on the two masses is W_1^{ten} and W_2^{ten}. This means that we can write the change in mechanical energy for the two masses as

$$\Delta T_1 + \Delta U_1 = W_1^{\text{ten}}, \tag{7}$$

and

$$\Delta T_2 + \Delta U_2 = W_2^{\text{ten}} \tag{8}$$

Now, with a massless pulley, the tension is the same all along the string. Additionally when one mass moves up the other moves down. Thus the total work done by the tension vanishes. Thus adding equations (5) and (6) yields

$$\Delta\left(T_1 + U_1 + T_2 + U_2\right) = \Delta E = 0, \tag{9}$$

where E is the total mechanical energy in the system. That is, the total mechanical energy

$$E = T_1 + U_1 + T_2 + U_2 \tag{10}$$

is conserved. The beauty of this result is that all reference to the constraining forces of the string and pulley has disappeared.

A careful discussion of constrained systems is far easier in the Lagrangian formulation which we will consider later in the course. Thus we shall postpone any further discussion of issues related to constraints until that time.

8.1.2 Central Forces

A three-dimensional situation that has some of the simplicity of one-dimensional problems is a particle that is subject to a central force. By definition that means that the force is everywhere directed toward a fixed "force center". Defining the force center to be the origin, a central force takes the form

$$\overrightarrow{F}\left(\overrightarrow{r}\right) = f\left(\overrightarrow{r}\right)\widehat{r}. \tag{11}$$

Two obvious examples of a central force are the Coulomb force and the gravitational force,

$$\overrightarrow{F}_C\left(\overrightarrow{r}\right) = k\frac{qQ}{r^2}\widehat{r} \text{ or } \overrightarrow{F}_G\left(\overrightarrow{r}\right) = -G\frac{mM}{r^2}\widehat{r}. \tag{12}$$

The negative sign for the gravitational force indicates that this force is always attractive, whereas the sign of the Coulomb force depends on the sign of the interacting charges, i.e. charges of opposite sign attract while those with the same sign repel. First, both the Coulomb force and the gravitational force are conservative (We have already shown that $\nabla\times\overrightarrow{F}_G = 0.$). Second these forces are spherically symmetric (rotationally invariant). This means that the magnitude of the force depends only on the magnitude of the distance from the origin. We can write this property as

$$f\left(\overrightarrow{r}\right) = f\left(r\right). \tag{13}$$

A feature of central forces is that if the force is conservative then it is automatically spherically symmetric, and, conversely, if a central force is spherically symmetric then it is conservative. Before we show this it is useful to review spherical polar coordinates.

Spherical Polar Coordinates In spherical polar coordinates the position P determined by the vector $\overrightarrow{r}$ is given by the coordinates (r, θ, ϕ) as defined in Figure 4.10. The magnitude of the distance from the origin is $r = |\overrightarrow{r}|$, θ is the angle measured from the z axis, and ϕ, often referred to as the azimuthal angle, is the angle from the x axis to the projection of $\overrightarrow{r}$ onto the $x - y$ plane.

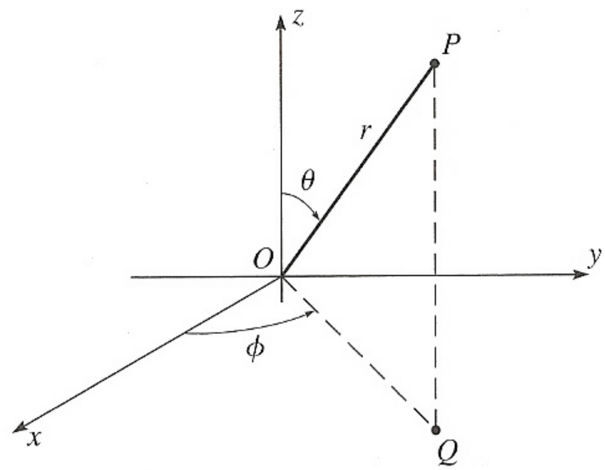

Figure 4.10. Spherical coordinates (r, θ, ϕ) of point P are defined so that r is the distance from from the origin to P, θ is the angle between OP and the z axis, and ϕ is the angle of between OQ from the x axis where Q is the projection of P onto the xy plane.

The Cartesian coordinates are found from the relations

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad \text{and} \quad z = \cos\theta. \tag{14}$$

It is often useful to note that on the Earth, θ is the angle of latitude, ϕ is the angle of longitude, and z coincides with the north polar axis. The spherical unit vectors are defined in the usual way. First $\widehat{r}$ is the unit vector pointing in the direction of increasing radial distance while θ and ϕ are held fixed. Similarly $\widehat{\theta}$ points in the direction of increasing θ while r and ϕ are held fixed. Finally, $\widehat{\phi}$ points in the direction of increasing ϕ while r and θ are held fixed. These unit vectors can be expanded in terms of the Cartesian unit vectors with the relations

$$\widehat{r} = \sin\theta\cos\phi\widehat{x} + \sin\theta\sin\phi\widehat{y} + \cos\theta\widehat{z}, \tag{15a}$$

$$\widehat{\theta} = \cos\theta\cos\phi\widehat{x} + \cos\theta\sin\phi\widehat{y} - \sin\theta\widehat{z}, \tag{15b}$$

$$\widehat{\phi} = -\sin\phi\widehat{x} + \cos\phi\widehat{y}. \tag{15c}$$

A little algebra also shows that these unit vectors are mutually orthogonal. Additionally they satisfy the cross product $\widehat{r} \times \widehat{\theta} = \widehat{\phi}$ with the obvious cyclic permutations being satisfied as well.

Now these unit vectors form a complete set and any vector can be expanded in this basis via

$$\overrightarrow{A} = A_r\widehat{r} + A_\theta\widehat{\theta} + A_\phi\widehat{\phi}. \tag{16}$$

Then the dot product of $\overrightarrow{A}$ and $\overrightarrow{B}$ is given by

$$\overrightarrow{A} \cdot \overrightarrow{B} = A_r B_r + A_\theta B_\theta + A_\phi B_\phi, \tag{17}$$

while the cross product is

$$\overrightarrow{A} \times \overrightarrow{B} = (A_\theta B_\phi - A_\phi B_\theta)\,\widehat{r} + (A_\phi B_r - A_r B_\phi)\,\widehat{\theta} + (A_r B_\theta - A_\theta B_r)\,\widehat{\phi}. \tag{18}$$

The Gradient in Spherical Polar Coordinates In Cartesian coordinates the gradient is defined as

$$\nabla f = \widehat{x}\frac{\partial f}{\partial x} + \widehat{y}\frac{\partial f}{\partial y} + \widehat{z}\frac{\partial f}{\partial z}. \tag{19}$$

Using the chain rule we could find the gradient in spherical coordinates but that would be cumbersome. There is a much easier way. We have already shown that the change in potential energy is found from

$$dU = -\overrightarrow{F} \cdot d\overrightarrow{r}, \tag{20}$$

where $\overrightarrow{F} = -\nabla U$. This means that we can write

$$dU = \nabla U \cdot d\overrightarrow{r}, \tag{21}$$

which is true for any arbitrary scaler function, f. To find $d\overrightarrow{r}$, we note that the position vector $\overrightarrow{r} = r\widehat{r} = r\left(\sin\theta\cos\phi\widehat{x} + \sin\theta\sin\phi\widehat{y} + \cos\theta\widehat{z}\right).$ Now the Cartesian unit vectors are constant so that from the chain rule

$$\begin{aligned}
d\overrightarrow{r} &= \frac{\partial\overrightarrow{r}}{\partial r}dr + \frac{\partial\overrightarrow{r}}{\partial\theta}d\theta + \frac{\partial\overrightarrow{r}}{\partial\phi}d\phi, \\
d\overrightarrow{r} &= (\sin\theta\cos\phi\widehat{x} + \sin\theta\sin\phi\widehat{y} + \cos\theta\widehat{z})\,dr \\
&\quad + r\,(\cos\theta\cos\phi\widehat{x} + \cos\theta\sin\phi\widehat{y} - \sin\theta\widehat{z})\,d\theta \\
&\quad + r\sin\theta\,(-\sin\phi\widehat{x} + \cos\phi\widehat{y})\,d\phi.
\end{aligned}$$

From the spherical unit vectors that we found in equations (29a,b,c) we find

$$d\overrightarrow{r} = \widehat{r}dr + r\widehat{\theta}d\theta + r\sin\theta\widehat{\phi}d\phi. \tag{22}$$

Note that we could have also written this down by decomposing any differential displacement from $\overrightarrow{r}$ into the $\widehat{r}$, $\widehat{\theta}$, and $\widehat{\phi}$ directions. Then with the appropriate radius for the θ and ϕ direction arrive at equation (36). Now we can evaluate the dot product in equation (35) for an arbitrary scalar function f, which yields

$$df = \nabla f \cdot d\overrightarrow{r} = (\nabla f)_r\,dr + (\nabla f)_\theta\,rd\theta + (\nabla f)_\phi\,r\sin\theta d\phi \tag{23}$$

Meanwhile our arbitrary scalar function f is simply a function of r, θ, and ϕ, so that from the chain rule

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi. \tag{24}$$

Comparing equations (37) and (38) allows us to identify the components of ∇f as

$$(\nabla f)_r = \frac{\partial f}{\partial r}, \quad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}, \quad \text{and} \quad (\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}, \tag{25}$$

or

$$\nabla f = \widehat{r}\frac{\partial f}{\partial r} + \widehat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \widehat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}. \tag{26}$$

In similar ways we can find the curl and other operators of vector calculus, all of which are more complicated in spherical coordinates than in Cartesian coordinates. They are listed in the back of Taylor's book.

Conservative and Spherically Symmetric, Central Forces To show that a central force is conservative if and only if it is spherically symmetric, we will make use of spherical polar coordinates. First we will assume that the central force $\overrightarrow{F}(\overrightarrow{r})$ is conservative and show that it must be spherically symmetric. Since it is conservative, it can be expressed as $-\nabla U$ which we just showed has the form

$$-\nabla U = -\widehat{r}\frac{\partial U}{\partial r} - \widehat{\theta}\frac{1}{r}\frac{\partial U}{\partial \theta} - \widehat{\phi}\frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}. \tag{27}$$

Since $\overrightarrow{F}(\overrightarrow{r})$ is a central force, only its radial component can be nonzero. This requires $\partial U/\partial\theta = \partial U/\partial\phi = 0$, or U is spherically symmetric. This leads directly to the result

$$\overrightarrow{F}(\overrightarrow{r}) = -\widehat{r}\frac{\partial U}{\partial r}. \tag{28}$$

Since U is spherically symmetric, the same must be true of $\partial U/\partial r$, and we see that $\overrightarrow{F}(\overrightarrow{r})$ is indeed spherically symmetric. The other half of the proof, that a spherically symmetric central force is necessarily conservative, is left as an exercise for the student. (Hint, remember that a conservative force satisfies $\nabla \times \overrightarrow{F} = 0$.)

Now because a central force $\overrightarrow{F}(\overrightarrow{r})$ is spherically symmetric, it has a magnitude that depends only on $r = |\overrightarrow{r}|$. Additionally, although $\overrightarrow{F}(\overrightarrow{r})$ is not actually one-dimensional as its direction depends on θ and ϕ, we shall see that any problem involving this kind of force is mathematically equivalent to a related one-dimensional problem.

As an application of these principles, consider a particle moving in a perfectly circular orbit in the field of a central attractive force with potential energy $U(r) = kr^n$. Since a central force conserves angular momentum, the motion of the particle must remain in a fixed plane which we define to be the equatorial plane in Figure 6. With this in mind, using spherical coordinates we can express the kinetic energy of a particle in the presence of a central force as

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2, \tag{29}$$

where ϕ is the azimuthal angle. We found that the angular momentum for a single particle (as measured from the origin of the force) is $\ell = mr^2\dot{\phi}^2$. Substituting for $\dot{\phi}$ in equation (43) we find

$$T = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2}. \tag{30}$$

The total energy is then

$$T + U = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + kr^n = E. \tag{31}$$

Here we have taken advantage of the conservation of angular momentum to replaced the kinetic energy term due to the angular velocity by a term that only depends on the radial coordinate, $\ell^2/\left(2mr^2\right)$. This expression for the total energy is now of the form

$$T + U_{eff} = E, \tag{32}$$

where

$$U_{eff} = \frac{\ell^2}{2mr^2} + kr^n. \tag{33}$$

For a circular orbit the radius is fixed, $\dot{r} = 0$, and the particle's radial location must occur at a minimum in the effective potential. Hence

$$\begin{aligned} \frac{dU_{eff}\left(r_{cir}\right)}{dr} &= -\frac{\ell^2}{mr_{cir}^3} + nkr_{cir}^{n-1} = 0, \\ r_{cir}^n &= \frac{\ell^2}{mnkr_{cir}^2}. \end{aligned} \tag{34}$$

In terms of this radius, the total energy for our particle in a circular orbit reduces to

$$E = \frac{\ell^2}{2mr_{cir}^2} + kr_{cir}^n = \frac{\ell^2}{2mr_{cir}^2} + \frac{\ell^2}{mnr_{cir}^2} = \frac{n}{2}U + U, \tag{35}$$

where the potential energy is now expressed as $U = \ell^2/mnr_{cir}^2$. Thus the kinetic energy in a circular orbit is related to the potential energy via

$$T = \frac{n}{2}U. \tag{36}$$

This is a statement of the *virial theorem* for circular orbits.