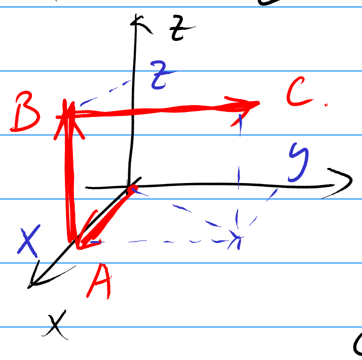
HW3

2.20. (a): $\nabla \times E = -k(3z, 2y, x) \neq 0.$

$\therefore$ Not an E field.

(b): $\nabla \times E = 0,$



choose path:

$A \to B \to C.$

Then at $C(x, y, z)$,

$$\varphi_C = -\left(\int_{OA} + \int_{AB} + \int_{BC}\right) \vec{E} \cdot d\vec{l}$$

$$\int_{OA} = \int_{AB} = 0$$

$$\Rightarrow \varphi_C = -\int_{BC} \vec{E} \cdot dl = -\int_0^y E_y(y')\,dy'$$

$$= -\int_0^y (2xy' + z^2)\,dy'$$

$$= -(xy^2 + z^2 y)$$

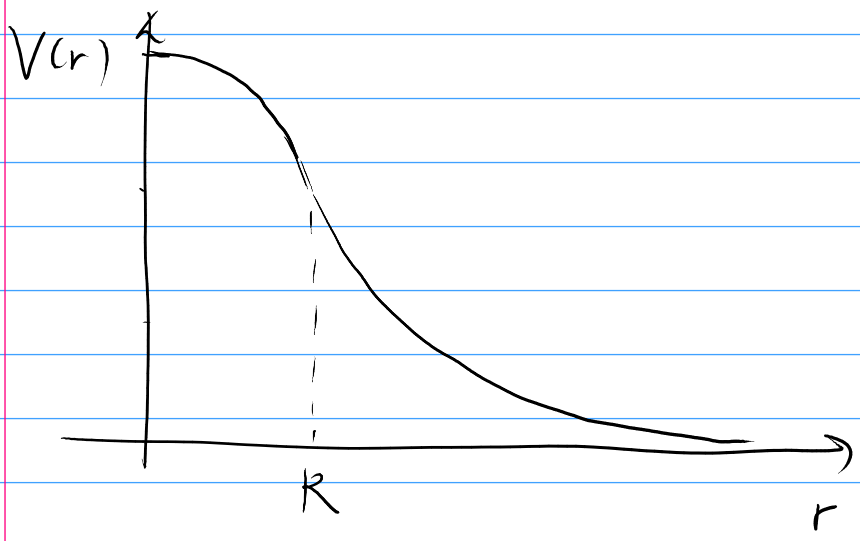
2.21.

$$\nabla V = -\vec{E} = \begin{cases} -\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} & r > R \\ -\frac{1}{4\pi\varepsilon_0} \underbrace{\left(\frac{r^3}{R^3} q\right)}_{\text{charge inside } r} \frac{1}{r^2} \hat{r} & r < R \end{cases}$$

$$\Rightarrow V(r) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{q}{r} & r \geq R \\ V(R) + \int_R^r \nabla V \cdot \hat{r}\, dr & \end{cases}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{R} + \frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}\frac{R^2 - r^2}{2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right) \qquad r < R$$



2.24. From prob. 2.16,

$$\vec{E} = \begin{cases} \frac{\rho}{2\varepsilon_0} r & r < a \\ \frac{\rho a^2}{2\varepsilon_0} \cdot \frac{1}{r} & r \in [a, b] \end{cases}$$

$$\Rightarrow V(b) - V(0) = \int \vec{E} \cdot \hat{r}\, dr$$

$$= -\frac{\rho}{2\varepsilon_0}\int_0^a r\, dr - \frac{\rho a^2}{2\varepsilon_0}\int_a^b \frac{1}{r}\, dr$$

$$= -\frac{\rho a^2}{4\varepsilon_0} \cdot \left(1 + 2\log\frac{b}{a}\right)$$

2.25

$$dV(r) = \frac{1}{4\pi\varepsilon_0}\frac{2\pi r\,dr\cdot\sigma}{\sqrt{r^2+z^2}}$$

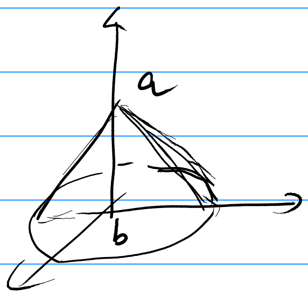
$$\Rightarrow V = \int dV$$

$$= \frac{\sigma}{4\varepsilon_0}\int_{r=0}^{R}\frac{d(r^2)}{\sqrt{r^2+z^2}}$$

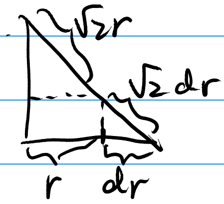
$$= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+z^2} - z\right)$$

$$\Rightarrow \vec{E} = -\hat{z}\frac{\partial V}{\partial z} = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)\hat{z}$$

2.26



a:

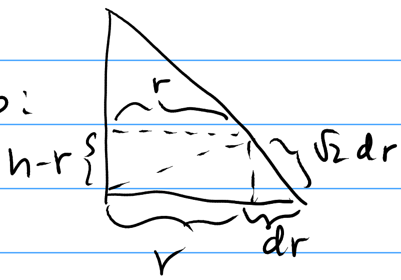


$$dV_a(r) = \frac{1}{4\pi\varepsilon_0}\frac{2\pi r\sqrt{2}\,dr\cdot\sigma}{\sqrt{2}\,r}$$

$$= \frac{\sigma}{2\varepsilon_0}dr$$

$$\Rightarrow V_a(r) = \int_{r=0}^{h} dV_a(r) = \frac{\sigma h}{2\varepsilon_0}$$

b:



$$dV_b(r) = \frac{1}{4\pi\varepsilon_0}\frac{2\pi r\cdot\sqrt{2}\,dr\cdot\sigma}{\sqrt{r^2+(h-r)^2}}$$

$$= \frac{\sigma h}{\sqrt{2}\,\varepsilon_0}\frac{t\,dt}{\sqrt{t^2+(1-t)^2}}, \quad t \equiv \frac{r}{h} \in [0,1]$$

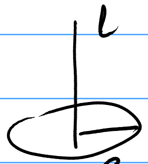
$$\Rightarrow V_b = \int_{t=0}^{1} dV_b(r) = \frac{\sigma h}{2\varepsilon_0} \sinh^{-1}(1)$$

see appendix for calculation

$$= \frac{\sigma h}{2\varepsilon_0} \log(1+\sqrt{2})$$

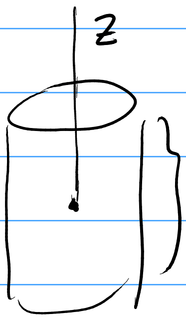
$$\Rightarrow V_a - V_b = \frac{\sigma h}{2\varepsilon_0}\left(1 - \log(1+\sqrt{2})\right)$$

2.27 Potential of a disk is:



$$dV(l) = \int_{r=0}^{R} \frac{2\pi r\, dr \cdot \sigma}{4\pi\varepsilon_0 \sqrt{l^2+r^2}} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+l^2} - l\right)$$

where $\sigma = \rho\, dl$



$$\Rightarrow V = \int_{l=z-\frac{L}{2}}^{z+\frac{L}{2}} dV(l)$$

$$= \frac{\rho}{2\varepsilon_0}\int \sqrt{R^2+l^2}\, dl - \frac{1}{2}\Big[\underbrace{\left(z+\tfrac{L}{2}\right)^2 - \left(z-\tfrac{L}{2}\right)^2}_{2zL}\Big]$$

Now, $\sqrt{R^2+l^2}\, dl = R^2 \cdot \sqrt{1+\left(\frac{l}{R}\right)^2} \cdot d\frac{l}{R}$

$$\left(\frac{l}{R} = sh(\theta) = \sinh(\theta)\right) = R^2\, ch(\theta)\, d\big(sh(\theta)\big) = R^2 ch^2(\theta)\, d\theta.$$

$$= R^2\, \frac{ch(2\theta)+1}{2}\, d\theta$$

$$= R^2 \cdot \left[\frac{1}{4} d(sh(2\theta)) + d\theta \right]$$

$$\Rightarrow \int \sqrt{R^2+l^2}\, dl = R^2 \left\{ \left[\frac{1}{4} sh(2\theta) \right|_{①}^{②} + \theta \Big|_{①}^{②} \right\}$$

$$\Rightarrow V = \frac{\rho}{2\varepsilon_0} \left\{ R^2 \left[\frac{1}{4} (sh\, 2\theta_+ - sh\, 2\theta_-) + \theta_+ - \theta_- \right] - L z \right\}$$

where $\theta_\pm = sh^{-1}\left(\frac{z \pm \frac{L}{2}}{R} \right)$.

$\left(\text{recall } sh^{-1}(x) = \log(x + \sqrt{1+x^2}) \right.$

$\left. \& \quad \frac{d}{dx} sh^{-1}(x) = \frac{1}{\sqrt{1+x^2}} \right)$

$$\Rightarrow \vec{E} = -\frac{\partial V}{\partial z} \hat{z}$$

$$= \left[\frac{\rho}{2\varepsilon_0} L - \frac{\partial V}{\partial \theta_+} \frac{\partial \theta_+}{\partial z} - \frac{\partial V}{\partial \theta_-} \frac{\partial \theta_-}{\partial z} \right] \hat{z}$$

Appendix:

$$I = \int \frac{\sigma h}{\sqrt{2}\,\varepsilon_0} \frac{t\,dt}{\sqrt{t^2+(1-t)^2}}, \quad t \in [0,1].$$

$$t^2+(1-t)^2 = 2t^2-2t+1 = 2\left(t-\tfrac{1}{2}\right)^2 + \tfrac{1}{2}$$

$$= \tfrac{1}{2}\left\{\left[2\left(t-\tfrac{1}{2}\right)\right]^2 + 1\right\}.$$

Let $2\left(t-\tfrac{1}{2}\right) = \sinh\theta \in [-1,1]$.

$$t\,dt = \tfrac{1}{2}(\sinh\theta+1)\tfrac{1}{2}\,d\sinh\theta$$

$$= \tfrac{1}{4}(\sinh\theta+1)\cdot\cosh\theta\,d\theta$$

$$\Rightarrow I = \frac{\sigma h}{\varepsilon_0}\int_{-\alpha}^{\alpha} \frac{\tfrac{1}{4}(\sinh\theta+1)\cdot\cosh\theta\,d\theta}{\sqrt{\sinh^2\theta+1}} \qquad \alpha = \sinh^{-1}(1)$$

$$= \frac{\sigma h}{4\varepsilon_0}\int_{-\alpha}^{\alpha}(\sinh\theta+1)\,d\theta.$$

$\sinh\theta$ is odd, $\therefore \int_{-\alpha}^{\alpha}\sinh\theta\,d\theta = 0$

$$\Rightarrow I = \frac{\sigma h}{4\varepsilon_0}\cdot 2\alpha = \frac{\sigma h}{2\varepsilon_0}\sinh^{-1}(1)$$

recall: $y = \sinh(x) = \dfrac{e^x - e^{-x}}{2}$

$\Rightarrow x = \log(x+\sqrt{1+x^2})$ (take positive root)