STATIC AND DYNAMIC ELECTRICITY

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WILLIAM R. SMYTHE

Professor Emeritus of Physics
California Institute of Technology

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CHAPTER V

THREE-DIMENSIONAL POTENTIAL DISTRIBUTIONS

5.00. When Can a Set of Surfaces Be Equipotentials?—At first glance, one might think that the class of three-dimensional potential distributions in which there is symmetry about an axis could be obtained by rotation of a section of a two-dimensional distribution provided that, in so doing, the boundaries in the latter case generated the boundaries in the former. This is not true in general. We shall now find the condition that a set of nonintersecting surfaces in space must satisfy in order to be a possible set of equipotential surfaces. Let the equation of the surfaces be

$$F(x, y, z) = C \tag{1}$$

Since one member of the family corresponds to each value of C, if it is to be an equipotential, we must have one value of V for each value of C, so that

$$V = f(C)$$

must satisfy Laplace's equation. Differentiating results in

$$\frac{\partial V}{\partial x} = f'(C)\frac{\partial C}{\partial x}, \text{ etc.,} \qquad \frac{\partial^2 V}{\partial x^2} = f''(C)\left(\frac{\partial C}{\partial x}\right)^2 + f'(C)\frac{\partial^2 C}{\partial x^2}, \text{ etc.}$$

Substituting in Laplace's equation gives

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = f''(C)(\nabla C)^2 + f'(C)\nabla^2 C = 0$$

giving

$$\frac{\nabla^2 C}{(\nabla C)^2} = -\frac{f''(C)}{f'(C)} = \Phi(C) \tag{2}$$

The condition then that the surface $F(x, y, z) = C$ can be an equipotential is that $\nabla^2 C/(\nabla C)^2$ can be a function of C only.

By integration of (2), we can now obtain the actual potential. Since $f''(C)/f'(C) = d[\ln f'(C)]/dC$, we have

$$\int \Phi(C)\, dC = -\ln [f'(C)] + A'$$

or

$$f'(C) = Ae^{-\int \Phi(C)\, dC}$$

Integrating again gives

$$V = f(C) = A\int e^{-\int \Phi(C)\, dC}\, dC + B \tag{3}$$

The constants A and B can be determined by specifying the values of the potential on any two of the surfaces given by (1).

5.01. Potentials for Confocal Conicoids.—As an application of the formula just derived, we shall now show that any one of the three sets of nonintersecting confocal conicoids, given by the equation

$$\frac{x^2}{a^2 + \theta} + \frac{y^2}{b^2 + \theta} + \frac{z^2}{c^2 + \theta} = 1 \tag{1}$$

where $c > b > a$ and $-c^2 < \theta < \infty$, is a possible set of equipotential surfaces. To get a picture of these surfaces, let us vary θ over the given range. For the range $-a^2 < \theta < \infty$, every term in (1) is positive so that it represents an ellipsoid. When $\theta = \infty$, we have a sphere of infinite radius, and when $\theta = -a^2$, the ellipsoid is flattened to an elliptical disk lying in the yz-plane. When θ passes from $-a^2 + \delta$ to $-a^2 - \delta$, we pass from the region of the yz-plane inside the disk to that outside. The latter is one limiting case of the hyperboloid of one sheet which (1) represents when $-b^2 < \theta < -a^2$. When $\theta = -b^2$, the hyperboloid of one sheet is flattened into that region of the xz-plane which includes the x-axis and lies between the hyperbolas cutting the z-axis at

$$z = \pm(c^2 - b^2)^{\frac{1}{2}}.$$

When θ passes from $-b^2 + \delta$ to $-b^2 - \delta$, we pass to the region of the xz-plane on the other side of these hyperbolas which is the limiting case in which the hyperboloid of two sheets, represented by (1) when $-c^2 < \theta < -b^2$, is flattened into the xz-plane. When $\theta = -c^2$, we have the other limiting case in which the two sheets of this hyperboloid coalesce in the xy-plane. Thus one curve of each set passes through each point in space; and, since it can be shown that the three sets are orthogonal, we can apply to them the theory developed in 3.03 for orthogonal curvilinear coordinates which leads to ellipsoidal harmonics. The latter are too complicated to be treated here, although later we shall treat the special cases of spheroidal harmonics.

To return to our problem: Let

$$M_n = \frac{x^2}{(a^2 + \theta)^n} + \frac{y^2}{(b^2 + \theta)^n} + \frac{z^2}{(c^2 + \theta)^n}$$

and

$$N = \frac{1}{a^2 + \theta} + \frac{1}{b^2 + \theta} + \frac{1}{c^2 + \theta}$$

With this notation, (1) becomes $M_1 = 1$, and differentiating this we have

$$\frac{2x}{a^2 + \theta} - M_2\frac{\partial \theta}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial \theta}{\partial x} = \frac{2x}{M_2(a^2 + \theta)}, \text{ etc.}$$

so that

$$(\nabla\theta)^2 = \left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 = \frac{4M_2}{M_2^2} = \frac{4}{M_2} \tag{1.1}$$

Differentiating again gives

$$\frac{\partial^2\theta}{\partial x^2} = \frac{2}{M_2(a^2+\theta)} - \frac{2x}{M_2(a^2+\theta)^2}\frac{\partial\theta}{\partial x} - \frac{2x}{a^2+\theta}\frac{1}{M_2^2}\left[\frac{2x}{(a^2+\theta)^2} - 2M_3\frac{\partial\theta}{\partial x}\right]$$

$$= \frac{2}{M_2(a^2+\theta)} - \frac{4x^2}{M_2^2(a^2+\theta)^3} - \frac{4x^2}{M_2^2(a^2+\theta)^3} + \frac{8x^2M_3}{(a^2+\theta)^2M_2^3}, \text{ etc.}$$

Adding similar expressions for y and z gives

$$\nabla^2\theta = \frac{2N}{M_2} - \frac{8M_3}{M_2^2} + \frac{8M_2M_3}{M_2^3} = \frac{2N}{M_2}$$

Substituting in 5.00 (2), we have

$$\frac{\nabla^2\theta}{(\nabla\theta)^2} = \frac{2N}{M_2}\cdot\frac{M_2}{4} = \frac{N}{2} \quad \text{so} \quad \Phi(\theta) = \frac{1}{2}\left(\frac{1}{a^2+\theta} + \frac{1}{b^2+\theta} + \frac{1}{c^2+\theta}\right) \tag{2}$$

This proves that such a set of equipotentials is possible. We now find the potential by 5.00 (3) to be

$$V = A\int^{\theta}[(a^2+\theta)(b^2+\theta)(c^2+\theta)]^{-\frac{1}{2}}\,d\theta + B \tag{3}$$

This is an elliptic integral given by Peirce 542 to 549 with $x = -\theta$. The constants A and B may be taken real or imaginary, whichever makes V real.

5.02. Charged Conducting Ellipsoid.—If we choose $V = 0$ when $\theta = \infty$, 5.01 (3) takes the form

$$V = -A\int_{\theta}^{\infty}[(a^2+\theta)(b^2+\theta)(c^2+\theta)]^{-\frac{1}{2}}\,d\theta \tag{1}$$

If we choose $V = V_0$ when $\theta = 0$ then, substituting in (1) gives

$$-A = V_0\left\{\int_0^{\infty}[(a^2+\theta)(b^2+\theta)(c^2+\theta)]^{-\frac{1}{2}}\,d\theta\right\}^{-1} \tag{2}$$

The field at infinity due to this ellipsoid, if its total charge is Q, will be $Q/(4\pi\epsilon r^2)$. We see from 5.01 (1) that as $\theta \to \infty$, $x^2 + y^2 + z^2 = r^2 \to \theta$, and so $\partial\theta/\partial r \to 2r$ giving

$$\frac{\partial V}{\partial r} \underset{r\to\infty}{\longrightarrow} \frac{\partial V}{\partial\theta}\frac{\partial\theta}{\partial r} = \frac{A}{r^3}\cdot 2r = \frac{2A}{r^2} = -\frac{Q}{4\pi\epsilon r^2} \tag{3}$$

Hence

The capacitance of the ellipsoid is, from (2),

$$C = \frac{Q}{V_0} = -\frac{8\pi\epsilon A}{V_0} = 8\pi\epsilon\left\{\int_0^{\infty}[(a^2+\theta)(b^2+\theta)(c^2+\theta)]^{-\frac{1}{2}}\,d\theta\right\}^{-1}$$

$$= 4\pi\epsilon(a^2-b^2)^{\frac{1}{2}}F[(a^2-b^2)^{\frac{1}{2}}(a^2-c^2)^{\frac{1}{2}}, \sin^{-1}(1-c^2a^{-2})^{\frac{1}{2}}] \tag{4}$$

The surface density is given by

$$\sigma = -\epsilon(\nabla V)_{\theta=0} = -\epsilon\left(\frac{\partial V}{\partial\theta}|\nabla\theta|\right)_{\theta=0}$$

From (1), $(\partial V/\partial\theta)_{\theta=0} = A(abc)^{-1}$ and, from 5.01 (1.1), $|\nabla\theta| = 2M_2^{-\frac{1}{2}}$ so that

$$\sigma = \frac{Q}{4\pi abc}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-\frac{1}{2}} \tag{5}$$

5.03. Elliptic and Circular Disks.—The capacitance of an elliptic disk, obtained by putting $a = 0$ in 5.02 (4), is still an elliptic integral. To get the surface density, we write 5.02 (5) in the form

$$\sigma = \frac{Q}{4\pi bc}\left(\frac{x^2}{a^2} + \frac{a^2y^2}{b^4} + \frac{a^2z^2}{c^4}\right)^{-\frac{1}{2}}.$$

Now let $a \to 0$, and the terms involving y and z can be neglected. Since both x and a are zero, the first term must be evaluated from 5.01 (1) where θ is put equal to zero, giving

$$\sigma = \frac{Q}{4\pi bc}\left(1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)^{-\frac{1}{2}} \tag{1}$$

The capacitance of a circular disk is obtained by putting $a = 0$ and $b = c$ in 5.02 (4), giving by *Pc* 114 or *Dw* 186.11

$$C = 8\pi\epsilon\left[\int_0^{\infty}\theta^{-\frac{1}{2}}(b^2+\theta)^{-1}d\theta\right]^{-1} = 8\pi\epsilon\left(\frac{2}{b}\left|\tan^{-1}\frac{\theta^{\frac{1}{2}}}{b}\right|_0^{\infty}\right)^{-1} = 8\epsilon b \tag{2}$$

Letting $\rho^2 = y^2 + z^2$ and $b = c$ in (1), the surface density on each side is

$$\sigma = \frac{Q}{4\pi b(b^2-\rho^2)^{\frac{1}{2}}} \tag{3}$$

The potential due to such a disk given by 5.02 (1) with $a = 0$ and $b = c$ is

$$V = \frac{2V_0}{\pi}\left(\frac{1}{2}\pi - \tan^{-1}\frac{\theta^{\frac{1}{2}}}{b}\right) = \frac{2V_0}{\pi}\tan^{-1}\frac{b}{\theta^{\frac{1}{2}}}$$

Putting in the value of θ obtained by letting $r^2 = x^2 + y^2 + z^2$, $a = 0$, and $b = c$ in 5.01 (1) gives

$$V = \frac{2V_0}{\pi}\tan^{-1}\left(2^{\frac{1}{2}}b\{r^2 - b^2 + [(r^2-b^2)^2 + 4b^2x^2]^{\frac{1}{2}}\}^{-\frac{1}{2}}\right) \tag{4}$$

This problem can also be solved by oblate spheroidal harmonics (5.271).

5.04. Method of Images. Conducting Planes.—An application of the test of 5.00 shows that in no case involving more than one point charge can we obtain the potential from the analogous two-dimensional case. Nevertheless, two of the methods used in such cases can also be applied to three-dimensional problems. One of these is the method of images. Any case in which the equation of a closed conducting surface under the influence of a point charge can be expressed in the form

$$0 = \frac{q}{r} + \sum_{s=1}^{n}\frac{q_s}{r_s}$$

Differentiating again gives

$$\frac{\partial^2\theta}{\partial x^2} = \frac{2}{M_2(a^2+\theta)} - \frac{2x}{M_2(a^2+\theta)^2}\frac{\partial\theta}{\partial x} - \frac{2x}{a^2+\theta}\frac{1}{M_2^2}\left[\frac{2x}{(a^2+\theta)^2} - 2M_3\frac{\partial\theta}{\partial x}\right]$$

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