Lecture 6

Conservation Laws and

Planetary Orbits

Outline of Lecture 6

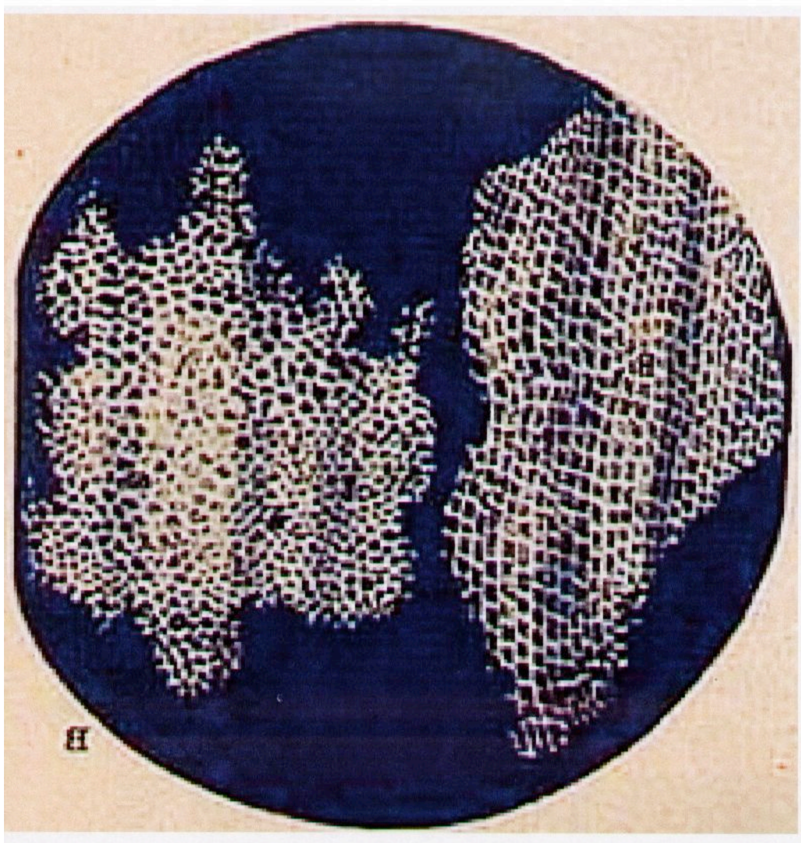
- Conservation of Energy
- Conservation of Angular Momentum
- Newton's explanation of Kepler's laws
  - Not detailed theory, which requires calculus
  - Emphasis is on role of conservation principles (need for future developments in astronomy, physics, and chemistry)
  - Orbits of comets as crowning triumph of Newtonian theory -- ultimate discriminant is predictive value.

Wren, Hooke, Halley, Newton

St. Paul’s Cathedral in London
Designed By Christopher Wren (1632-1723)

- Story of Christopher Wren, Robert Hooke, Edmund Halley, and Isaac Newton: What is the general shape of the orbit of a body moving in an attractive $1/r^2$ force field?
  - Hooke: an ellipse, but gave no proof.
  - Wren offers prize of book for mathematical demonstration.
- Halley approaches Newton
  - Newton claims to have solved problem years ago, but mislaid papers.
  - A few weeks later, Halley receives exposition in the mail.
  - Halley persuades Newton to publish *Principia* -- at Halley’s expense.

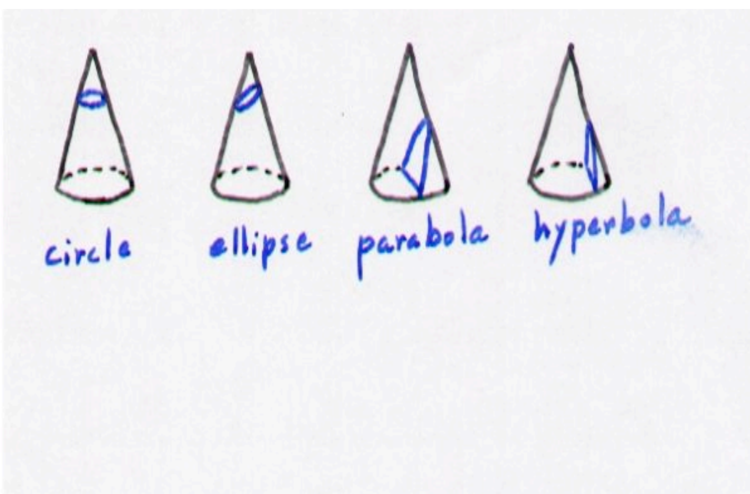
Drawing of Microscopic Cork Cell By Robert Hooke (1635-1703)

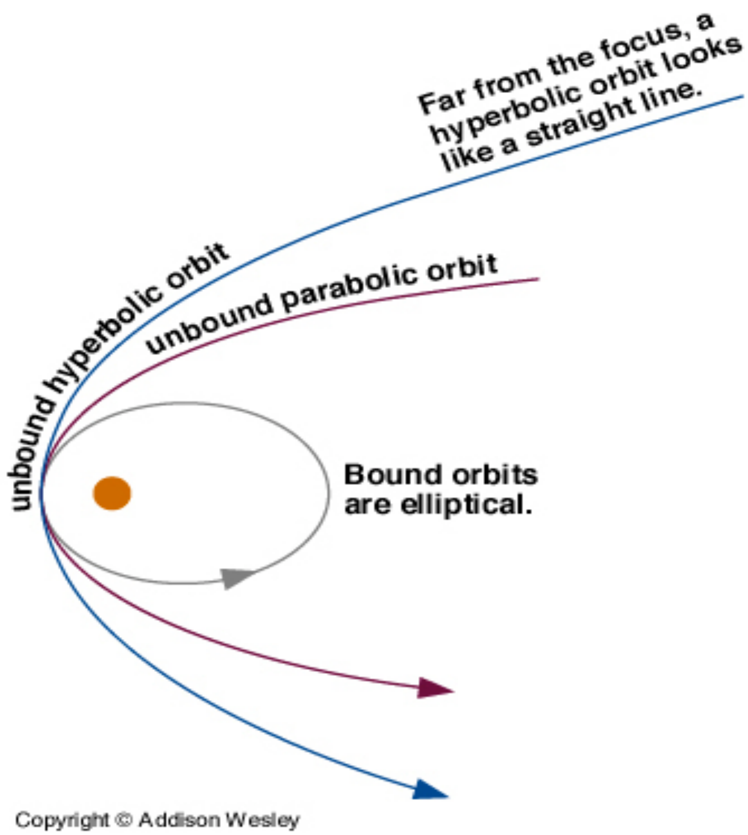


- Invented terminology “cell” meaning “little room” to describe the basic unit of life.
- Otherwise known today only for his force law of the spring (Hooke’s law).
- In 17th century, he was considered the greatest living experimentalist of his day.
- However, after his death, his reputation was systematically destroyed by Newton, who was not a nice man.

Newton in *Principia*: General Orbit under $1/r^2$ Attraction is a Conic Section

Conic Sections

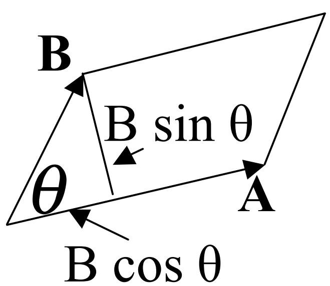




To understand results, need to discuss notion of conserved quantities: energy and angular momentum.

Product of Two Vectors (extra material)

- With a scalar times a vector, there is only one way to take a product. When one has two vectors, there are three ways to take products: dot, cross, and direct. We need be concerned only with dot and cross.
- The dot product of two vectors **A** and **B** yields a scalar, D = $\mathbf{A}\cdot\mathbf{B}$, whose magnitude is given by AB cos θ, where θis the angle between **A** and **B** when their tails are put at the same point.
- The cross product of two vectors **A** and **B** yields another vector, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, whose magnitude is given by AB sinθ, and whose direction is given by the “right-hand rule” (out of plane of page).



$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ = projection of **B** times A = AB cos θ.
$|\mathbf{A} \times \mathbf{B}|$ = area of parallelogram = AB sin θ. Note $\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$. If **A** and **B** are parallel, i.e., if θ= 0, then $\mathbf{A}\cdot\mathbf{B}$ = AB and $\mathbf{A} \times \mathbf{B} = 0$.

Orbital Potential Energy

- In the case where force is constant, we defined last lecture a quantity called potential energy which equals (minus) the force times the displacement of a body in the direction of the force.
- Even when the force is not constant, but depends on the position of the body, it is still possible to define a useful concept of potential energy W. But now, we need to sum up a lot of small displacements, each segment of which we may approximate the force to be almost constant (the method of integral calculus). When we do this for an attractive force which varies with radius r as $F = GMm/r^2$, it turns out (coincidentally) that W is given by F times r (dot product of **F** and **r**), with a minus sign attached in front:

$$W = -\frac{GMm}{r}.$$

Conservation of Energy

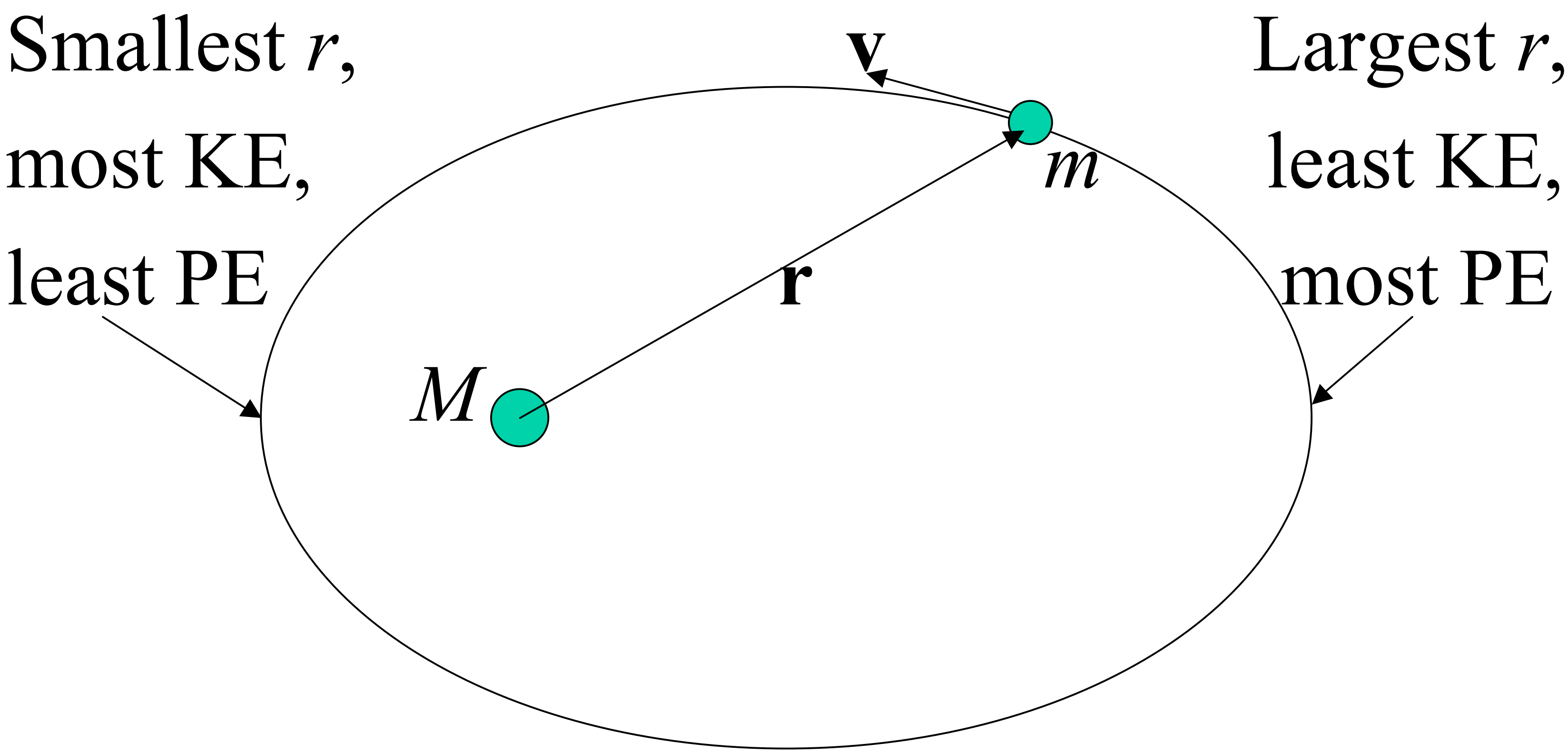
- The kinetic energy of mass m and mass M with speeds v and V are still given, respectively, by $m\mathrm{v}^2/2$ and $MV^2/2$. Thus, the total energy of the combined system is given by the expression:

$$E = \frac{1}{2}m\mathrm{v}^2 + \frac{1}{2}MV^2 - \frac{GMm}{r}.$$

- When $M >> m$, the large body hardly moves, and we can ignore its kinetic energy. In this approximation, we have the simplified statement of conservation of energy:

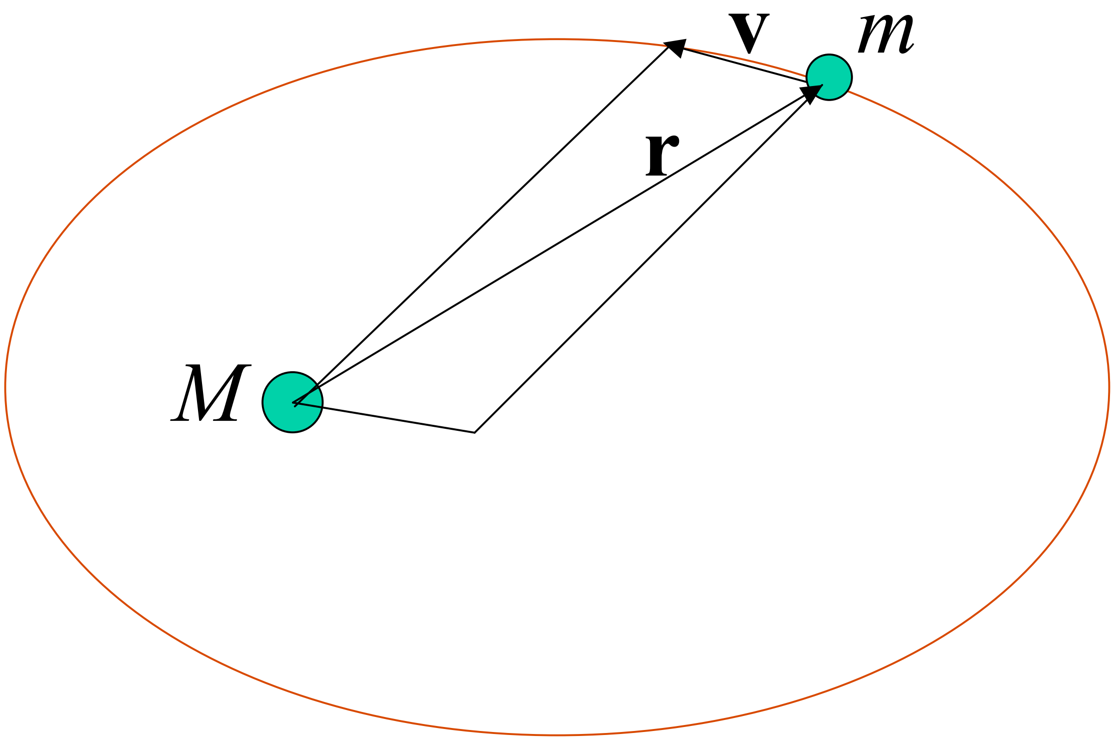
$$E = \frac{1}{2}m\mathrm{v}^2 - \frac{GMm}{r} = \text{const.}$$

Pictorial Representation



Conversion of PE to KE as m drops to small r; reconversion of KE to PE as r increases again.

Orbital Angular Momentum = Lever Arm Times Linear Momentum



Orbital angular momentum $\boldsymbol{L}$ = $\mathbf{r}$ x $\mathbf{p}$ = $m\mathbf{r}$ x $\mathbf{v}$ where $\mathbf{p}$ = $m\mathbf{v}$ is vector linear momentum. Cross product $\mathbf{r}$ x $\mathbf{v}$ is given in magnitude by area of black parallelogram, in direction by the right-hand rule (out of plane of diagram). A constancy of the direction of $\mathbf{r}$ x $\mathbf{v}$ implies that the orbit remains in the plane of the diagram. The magnitude of $\mathbf{r}$ x $\mathbf{v}$ is twice the rate of sweeping out area by the radius vector. Therefore, a conservation of $\boldsymbol{L}$ implies that the orbit takes place in a plane (part of Kepler's first law) and that equal areas are swept out in equal times (Kepler's second law).

Proof by Calculus (extra material)

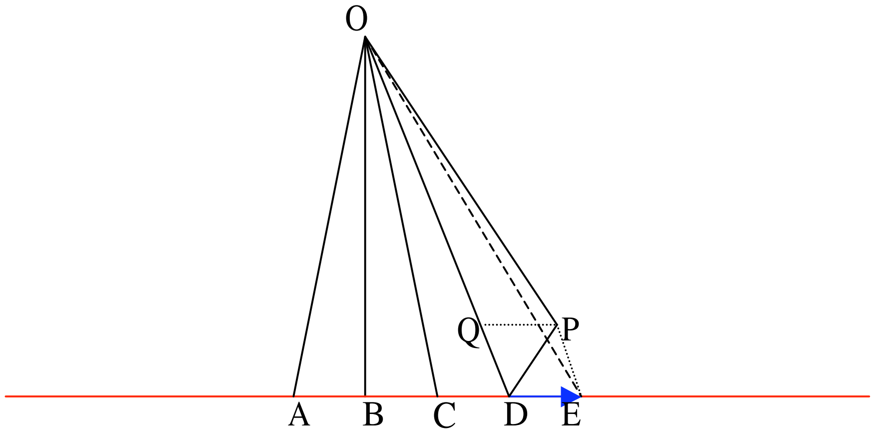
Orbital angular momentum: $\vec{L} = m\vec{r} \times \vec{\mathrm{v}}.$

$$\frac{d\vec{L}}{dt} = m\frac{d\vec{r}}{dt} \times \vec{\mathrm{v}} + m\vec{r} \times \frac{d\vec{\mathrm{v}}}{dt} = m\vec{\mathrm{v}} \times \vec{\mathrm{v}} + m\vec{r} \times \vec{a},$$

which equals zero because $\vec{\mathrm{v}} \times \vec{\mathrm{v}} = 0$ and $m\vec{a} = \vec{F}$ is in the direction of $-\vec{r}$. With $d\vec{L}/dt = 0$, we have

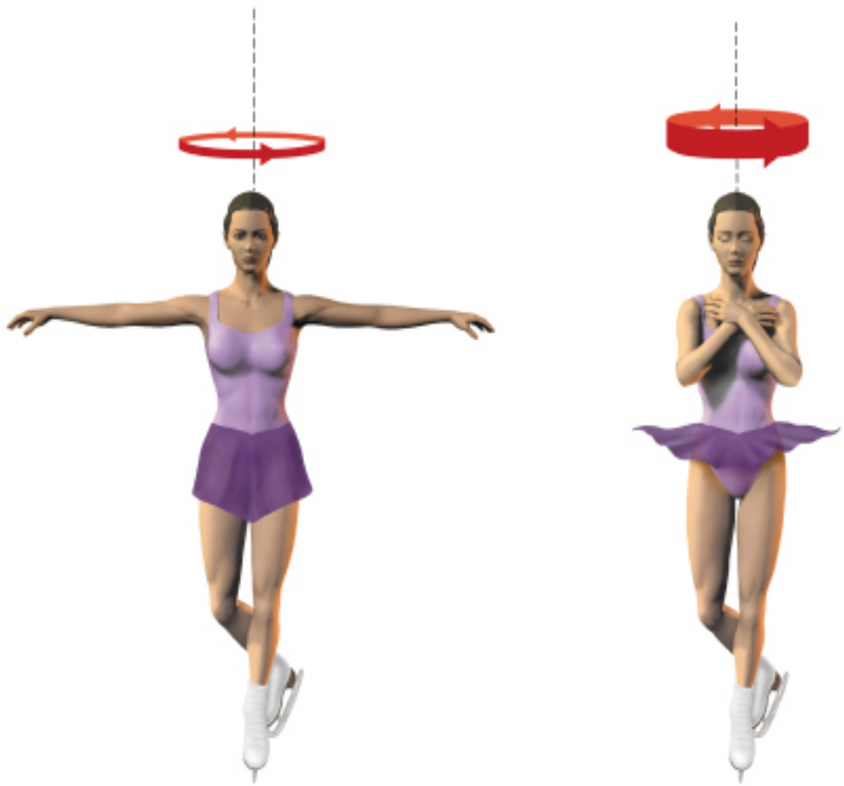
$$\vec{L} = \text{const.}$$

Newton’s Geometric Proof (extra material)



- If there were no force at all, and a planet were merely to go past the Sun at origin O at uniform speed along a straight line ABCD, equal areas would be swept out in equal times by a radius vector from the origin O to the planet. (Equal times implies that that the line segments AB, BC, and CD are all equal. On the other hand, the triangles OAB, OBC, and OCD all have the same height. With the heights and bases equal, these triangles all have the same areas, which demonstrates the desired proposition.)
- Imagine inserting the Sun into the origin when the planet is at position D. During the small interval of time that the planet would have gone from D to E, which we take for simplicity to have the same length DE = AB = BC = CD, the Sun impulsively pulls the planet to a position Q that would have been along the radius vector OD had the planet been stationary at D. But because the planet was moving, the actual position reached by the planet under the joint influence of the impulse and inertia is resolved by the parallelogram law for adding vectors so that the planet actually ends up at the position P. The triangles ODE and ODP have the same base OD. They also have the same height, a perpendicular dropped to the base OD, since EDQP is a parallelogram. Thus, the triangles ODE and ODP have the same areas, and the impulse supplied by a central force field has not prevented the radius vector from sweeping out equal areas in equal times. Extending the argument to many consecutive impulses demonstrates that a central force field associated with the Sun, independent of its specific character, has no effect whatsoever on the area theorem that applies to a planet moving in a straight line.

Conservation of Spin Angular Momentum in Spin-up of Ice Skater



Angular momentum represents inertia of rotating things to keep its "momentum" of rotation.

Role of *E* and *L* in Orbital Shape

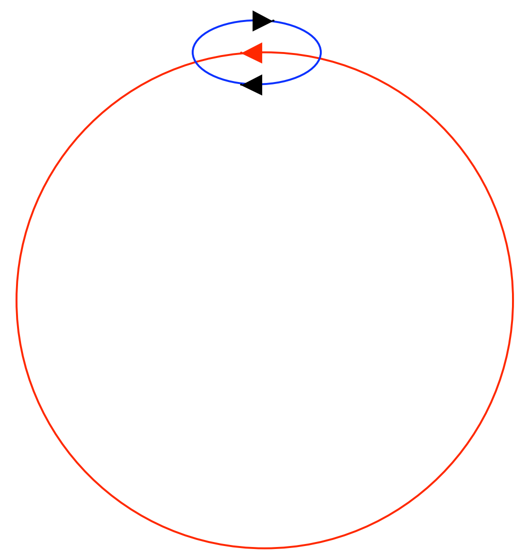
- Conservation of energy: $$E = \frac{1}{2} m \mathrm{v}^2 - \frac{GMm}{r} = \text{const.}$$
- If $E < 0$, orbit is *bound,* i.e., r cannot reach infinity because PE would then equal 0 while KE ≥ 0, contradicting assumption that $E < 0$. Orbit is ellipse.
- If $E > 0$, orbit is *unbound,* i.e., r can reach infinity with KE positive. Mass m can escape from system. Orbit is hyperbola.
- If $E = 0$, orbit is marginally unbound or *critical*. Orbit is parabola.
- Extra material: Of all orbits of a given angular momentum L, the circular orbit has least (most negative) energy E. Call this energy of the circular orbit $E_c(L)$, which is < 0.
- The eccentricity e of an orbit then turns out to be given by

$$e = \sqrt{1 - \frac{E}{E_c(L)}}.$$

- Ellipse: $0 \leq e < 1$; Parabola: $e = 1$ ($E = 0$); Hyperbola: $e > 1$ ($E > 0$).

Why Epicycles Worked Approximately for Copernicus (extra material)

- Reprise: of all orbits of a given angular momentum $\boldsymbol{L}$, the circular orbit at constant r has least energy E.
- If we keep $\boldsymbol{L}$ fixed, but increase E, then the planet m gains enough energy to oscillate inwards and outwards, varying its radial distance r from the Sun.
- To conserve angular momentum in the presence of a changing lever arm r, as the planet oscillates inwards, it has to speed up in the circular direction; as it oscillates outwards, it has to slow down.
- These motions therefore decompose into a guiding center that goes at a uniform angular speed around the Sun in a large circle (red curve), atop of which the planet travels during the same period in a small retrograde epicycle (blue curve).
- To conserve energy, the epicycle turns out to have a *centered elliptical shape with a 2:1 axial ratio*. The combination of a circular guiding center plus such a retrograde elliptical epicycle yields the equivalent of a Kepler ellipse with the Sun at a *focus*.

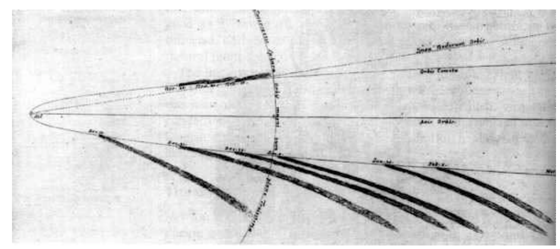


If Copernicus had used centered retrograde ellipses with a 2:1 axial ratio as epicycles rather than circles, he could have reduced the number of epicycles for the 6 planets that go around the Sun to 6 (the Moon going around the Earth is more complicated). But then he would have possessed the mathematical skills of a Kepler.

The Comets of 1680 and 1682

- Since ancient times, the appearance of comets had been sudden and unpredictable, and thus frightening.
- Comets became associated with harbingers of evil, the fall of royal houses and kingdoms.
- Tycho Brahe determined comets to have wild orbits that threatened to smash the crystalline spheres of Aristotle’s celestial imagination.
- Robert Hooke showed that comet tails always point away from the Sun, independent of the direction of travel of the comet head.
- In 1680 and 1682, two bright comets appeared that awed professional astronomers and public alike.
- After much confusion, Newton computed that the 1680 comet had a parabolic orbit, and thus was new to the solar system; while the 1682 comet had a retrograde elliptical orbit, and therefore must have made previous appearances as well as will make future ones.

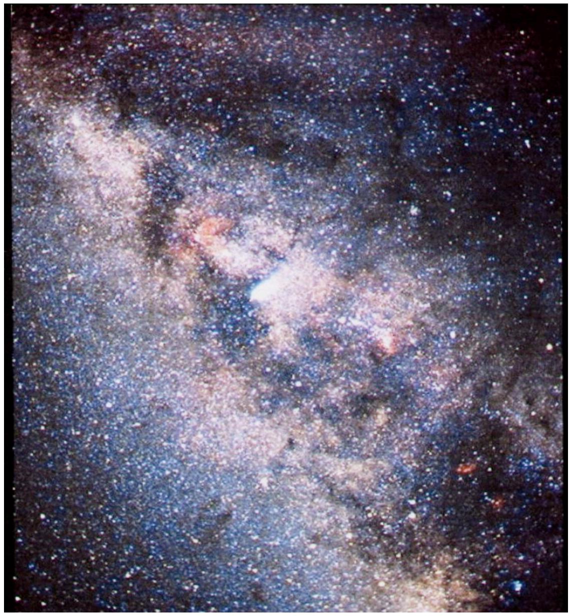
Marshall Space Flight Center/ NASA



Newton’s drawing of orbit of great comet of 1680.

The Odyssey of Halley's Comet

- Halley (1656-1742) guessed that the comet of 1682 may have been same as those found in 1531 and 1607 as well as perhaps 1378.
- Inexact repetition period of 75 to 77 (now known to be 74 to 79) years implies that the comet's orbit about Sun might be perturbed by the giant planets, in particular, by Jupiter and Saturn.
- Halley proceeded to use Newton's theory to correct for the effect of planetary perturbations.
- He predicted that the comet should next make perihelion passage, becoming brightest, on April 13, 1759$\pm$ a month.
- In 1758, 76 years since it had last been seen and long after Newton and Halley had both died, astronomers turned their telescopes to the night sky in order to have the honor of first finding Halley's comet. The whole year almost passed with no success. Then on Christmas eve 1758, Newton's birthday, Halley's comet was recovered! It made perihelion passage on March 12, 1759, within the error allowed by Halley. What had been previously regarded as chaotic and unknowable proved to be orderly and predictable. The triumph ushered in a new Age of Enlightenment.



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Halley's comet seen against the Milky Way in 1986. Look for it again in 2061!

Summary

- Kepler's kinematical description and Newton's dynamical explanation of planetary orbits have a clarity, simplicity, and beauty about them that speak to their essential truth.
- Alexander Pope: "Nature, and Nature's laws lay hid in night,
  God said, let Newton be, and all was light."
- Central to our physical understanding of planetary orbits are the principles of conservation of orbital energy E and angular momentum $\boldsymbol{L}$ in the approximation that the gravitational interactions of planets on each other are negligible in comparison with their corresponding interaction with the Sun.
- Conserved quantities are especially powerful because they often have a generalized validity that goes beyond their original derivations. Thus, the conservations of energy and angular momentum prove to be important guides in our exploration of systems that are both much smaller than planetary systems (atoms & nuclei of atoms, stars & black holes) and much larger (galaxies, the universe as a whole).

Postscript (extra material)

- For over 300 years, Newton's solution of the problem of planetary orbits in the two-body approximation (Sun plus planet) represented the ultimate triumph of deterministic solutions over chance occurrences, order over disorder, regularity (cosmos) over irregularity (chaos).
- During the last 40 years, however, scientists in many fields, including celestial mechanics, have come to realize that the Newtonian "clockwork universe" may be an illusion. Many systems lack enough "constants of motion" to make their behavior regular and predictable. Instead they are characterized by chaos and unpredictability.
- Thus, regularity and chaos both have important roles in the natural world and in the human experience. Our lives are both richer and more complex as a consequence. Challenge and opportunity abound in learning how to live with this dichotomy.