Lecture 5

The Newtonian Triumph

Outline of Lecture 5

- Laws of Mechanics:
  - First law: In the absence of external forces, a body in motion remains in motion in a straight line at constant speed.
  - Second law: When a force ***F*** with magnitude and direction is present, a body of mass *m* will experience an acceleration ***a*** given by ***a*** = ***F***/*m*. Often, this is written as ***F*** = *m**a***, the most famous equation perhaps in all of science.
  - Third law: For every action, there is an equal and opposite reaction.
- Universal law of gravitation: Two point masses *m* and *M*, separated by a distance *r*, exert a force of gravitational attraction on each other along the line joining their centers given in magnitude by $F = GMm/r^2$.
- The laws of nature are the same everywhere in the universe.

Some Comments (extra material)

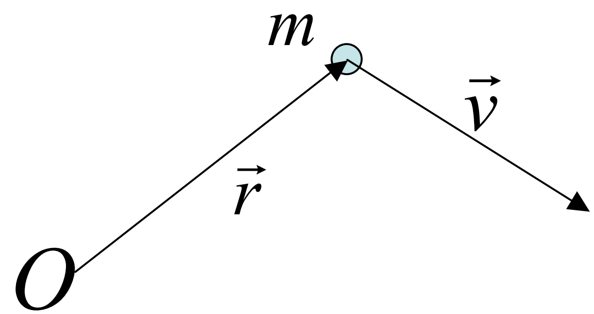
- For those students who desire more mathematical details, I have supplied additional material in the form of "proofs" of certain assertions that are made in class. They are included for the interested student with enough mathematical background to appreciate such derivations.
- This additional material (with headings in magenta and subheading "extra material") is not an essential part of the course and can be skipped by the general student. They will not be covered in the lectures or exams.

Some Preliminaries: Vectors (extra material)

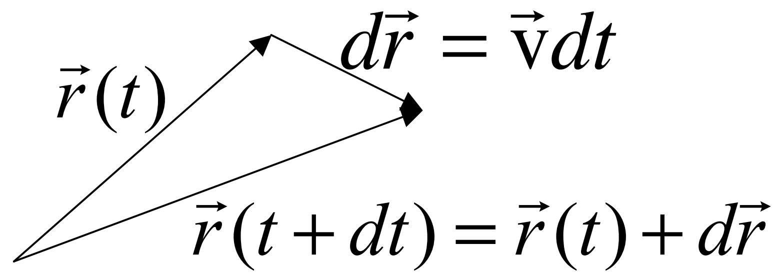
- A vector is a quantity that has a magnitude and a direction, denoted in drawings by an arrow.
- The symbol for a vector is a boldface character or the same letter with an arrow on top, for example, **v** ≡ $\vec{\mathrm{v}}$**.**
- Written without the boldface or the arrow, the same symbol means the magnitude of the quantity: v.
- Sometimes the vector will have a different name than the magnitude of the same quantity, for example, velocity **v** ≡ $\vec{\mathrm{v}}$ and speed v. This convention is not always followed, and the same name can mean either the vector or its magnitude, depending on context.
- A scalar quantity like mass m times a vector like velocity **v** ≡ $\vec{\mathrm{v}}$ is another vector **p ≡** $\vec{p} = m\vec{\mathrm{v}} = m\mathbf{v}$, in this case, the momentum, with the same direction as **v ≡** $\vec{\mathrm{v}}$, but with magnitude *m*v.

Some Vector Quantities (extra material)

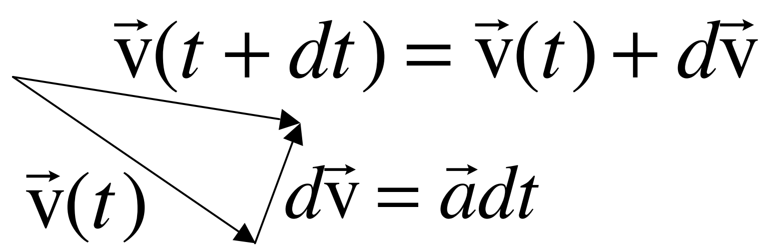
- Position **r** ≡ $\vec{r}$:



- Velocity **v** ≡ $\dfrac{d\vec{r}}{dt}$:



- Acceleration **a** ≡ $\dfrac{d\vec{\mathrm{v}}}{dt}$:



Inertial Frames

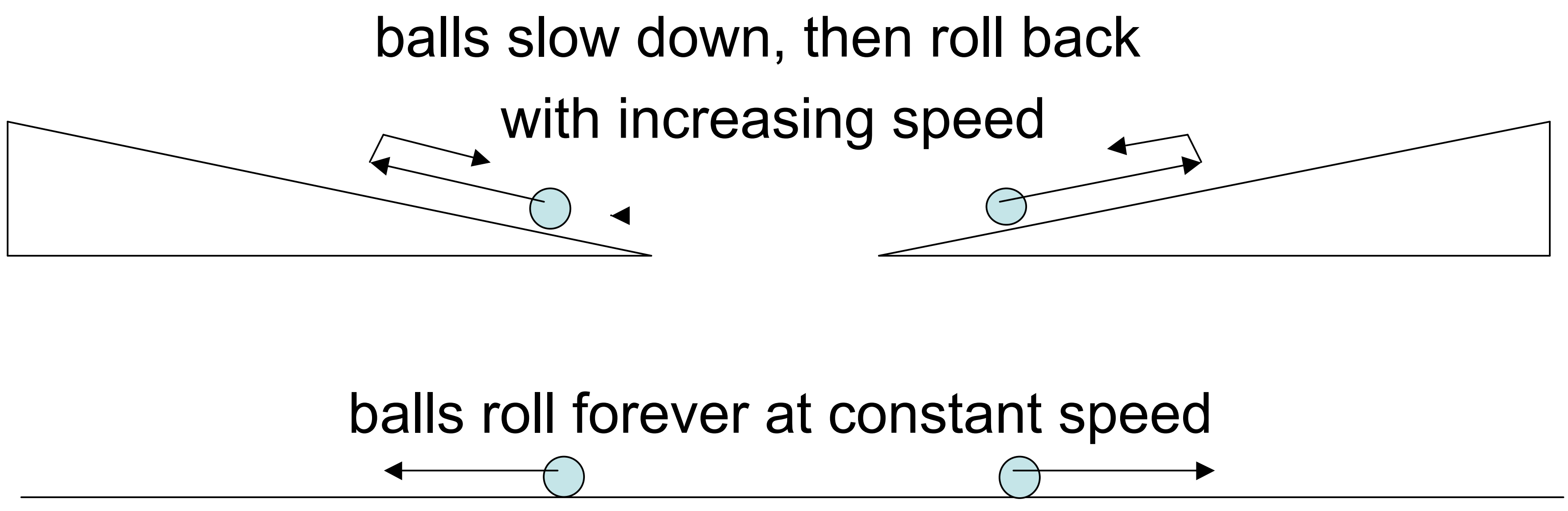
- Laws of mechanics hold only in inertial frames, when the observer is at rest with respect to the "fixed stars" or, at most, moves at constant velocity relative to them.
- Since stars are not truly "fixed," we might have to reassess what we mean ultimately by "inertial frames," but for the purposes of this course, we will ignore such complications.
- To the accuracy that one carries out fast, small-scale, experiments, a reference frame fixed to a laboratory on Earth usually constitutes a good enough approximation to an inertial frame ((but not always; cf. Foucault pendulum).

Michael Hart

James Kaler

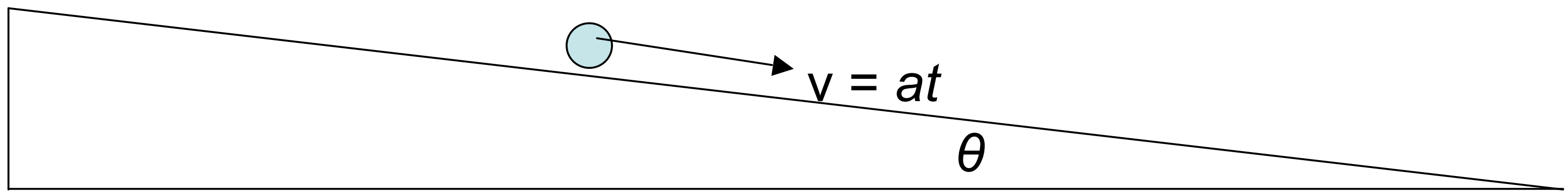
Galileo (1564-1642)

- After forced to abjure:
  - Placed under house arrest in hills of Florence.
  - Devoted last years to study of mechanics.
- First law of mechanics: When not subjected to any external forces, a body in motion remains in motion, at constant speed in a straight line. Thought experiment without force of friction:



Special Case of Second Law of Mechanics

- A body subjected to a constant force, such as that of gravity at the Earth's surface, will experience a uniform acceleration **a**, gaining equal increments of velocity in equal times.



- The greater is the inclination angle θ, the faster does the ball gain speed. Experimentally and theoretically (ignoring energy put into rotation and effect of friction),

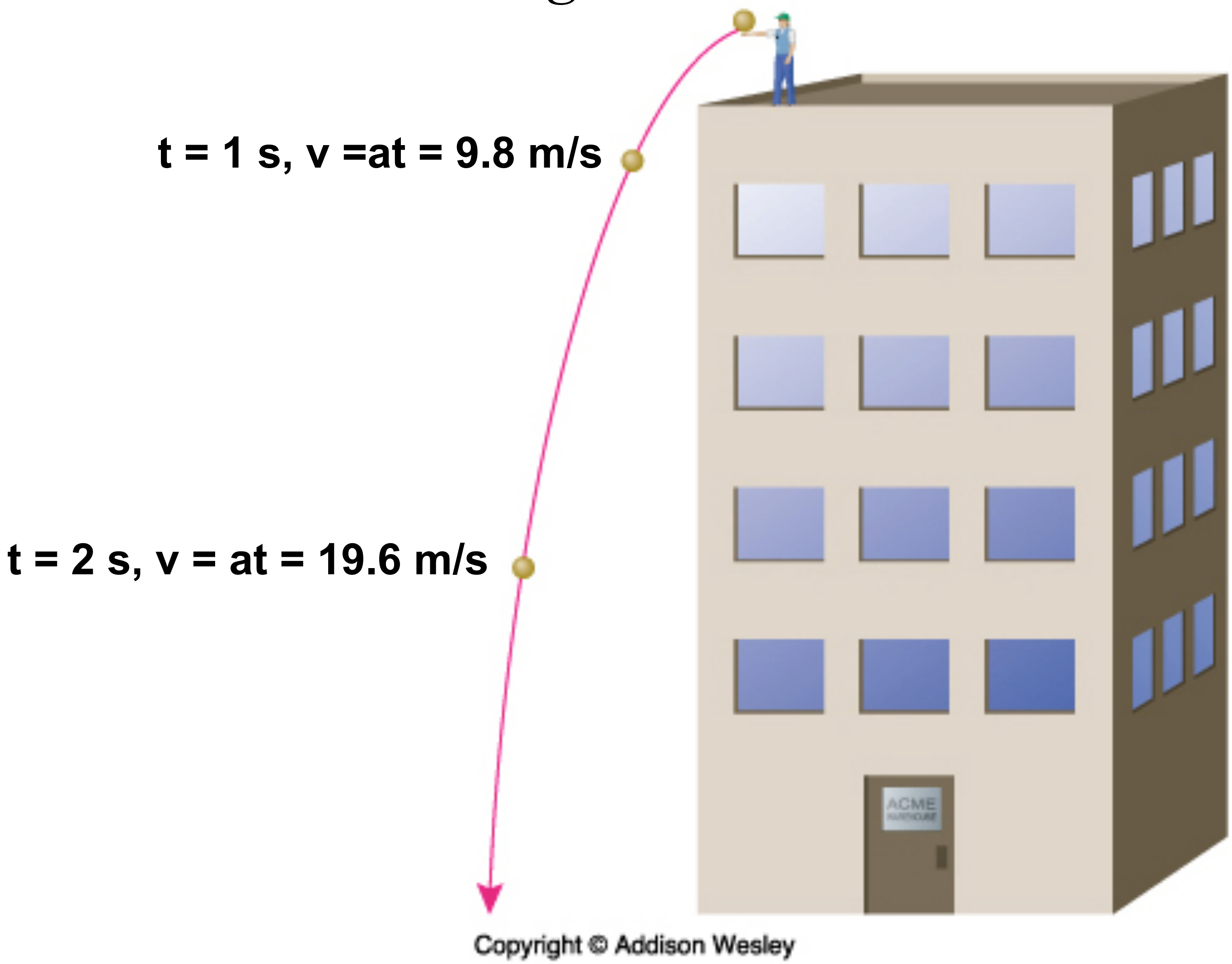
$$a = g\sin\theta,$$

where g = 9.8 m/s^2 is the gravitational field of the Earth.

- Notice for θ = 0º, i.e., a horizontal plane a = 0; whereas for θ = 90°, i.e., a vertical inclination, a = g. Thus, g is the acceleration produced by vertical free fall.

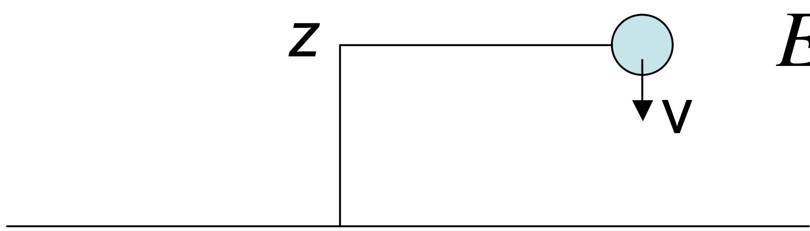
Uniform Acceleration and Velocity

$$a = g = 9.8\,\mathrm{m\,s^{-2}}$$



Conservation of Energy

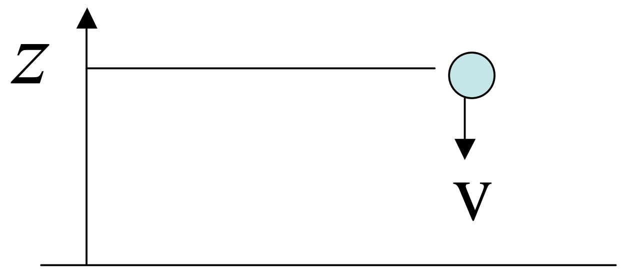
- In the free fall of a body of mass m, although the speed v and the vertical position *z* are both changing with time, a certain combination called the total energy *E* is a constant independent of time:



$$E = \frac{1}{2}m\mathrm{v}^2 + mgz = const.$$

- The quantity $m\mathrm{v}^2/2$ is called the kinetic energy. The quantity *mgz* is called the potential energy, and is obtained by multiplying the (constant) force *mg* by the distance *z* traversed in the direction of the force. It is the sum of kinetic energy and potential energy that is preserved in time, with the latter being converted to the former as the body falls.

Proof
(extra material)



$$E = \frac{1}{2} m\mathrm{v}^2 + mgz$$

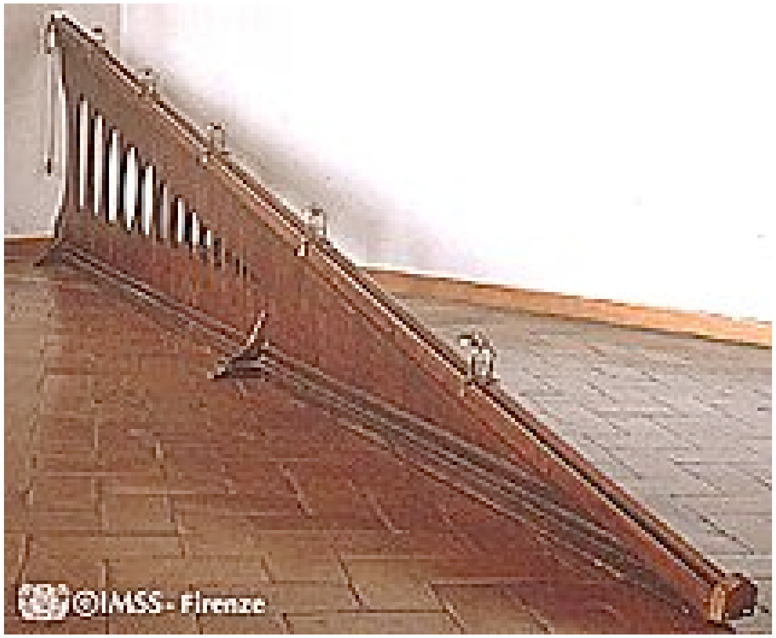
$$\frac{dE}{dt} = m\mathrm{v}\frac{d\mathrm{v}}{dt} + mg\frac{dz}{dt} = m\mathrm{v}(a - g) = 0$$

since $\mathrm{v} = \dfrac{dz}{dt}$ and $a = \dfrac{d\mathrm{v}}{dt} = g$.

$$\therefore E = \text{const.}$$

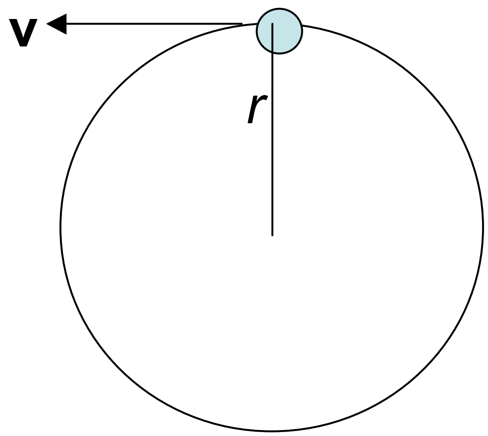
Fundamental Units

- Galileo first to realize primal importance of measurements of
  - length
  - mass
  - time
- Although he did not lack aesthetics (see his inclined plane), for Galileo, aspects such as color, smell, etc., are secondary.
- Today, we think everything physical can be assigned dimensions made of powers of m, kg, s (MKS or SI units). (Ultimately, sensations such as color, smell, etc. can be reduced to quantities measurable in m, kg, s.)
- Science majors might think quantities such as temperature and electric charge are exceptions, but we shall see later that these too always enter in fundamental combinations that involve only m, kg, s to various powers. This is a subtle point about the nature of physical reality that we will explore more deeply before the course ends.



Huygens (1629-1695)

- Motion in a circle at constant speed and radius *r* involves a change in direction of the velocity **v** (a vector), and therefore has nonzero acceleration **a** (also a vector with magnitude and direction).

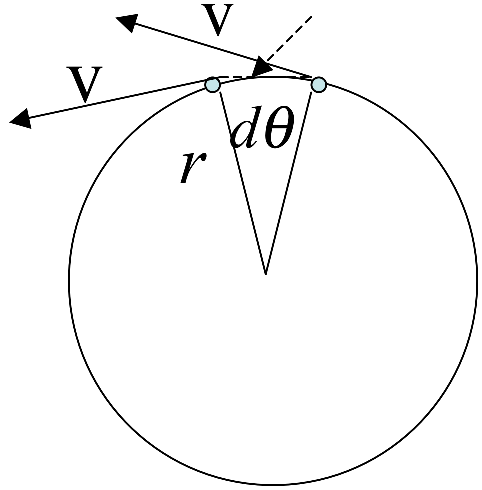


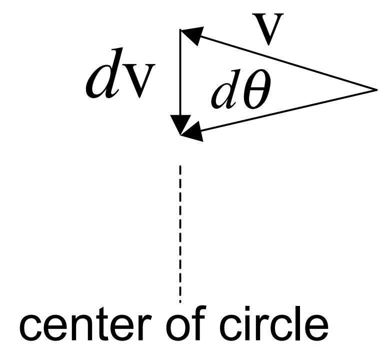
- The centripetal acceleration **a** is directed inward toward the center of the circle and has magnitude

$$a = \frac{\mathrm{v}^2}{r}.$$

Proof
(extra material)

$$r d\theta = \mathrm{v} dt$$



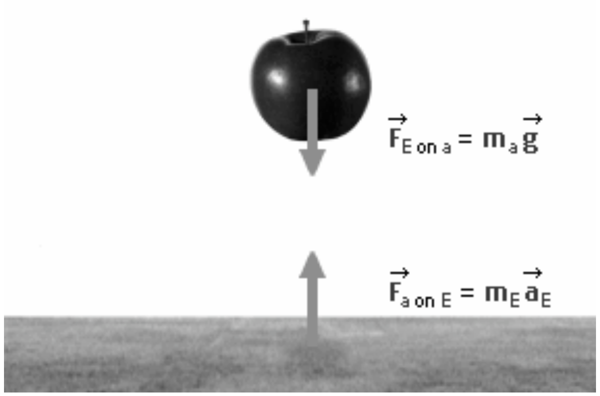


$$d\mathrm{v} = \mathrm{v} d\theta = \frac{\mathrm{v}^2}{r} dt$$

$$\therefore \quad a = \frac{d\mathrm{v}}{dt} = \frac{\mathrm{v}^2}{r} \quad \text{with} \quad \vec{a} \quad \text{directed toward center of circle}$$

Isaac Newton (1642-1727)

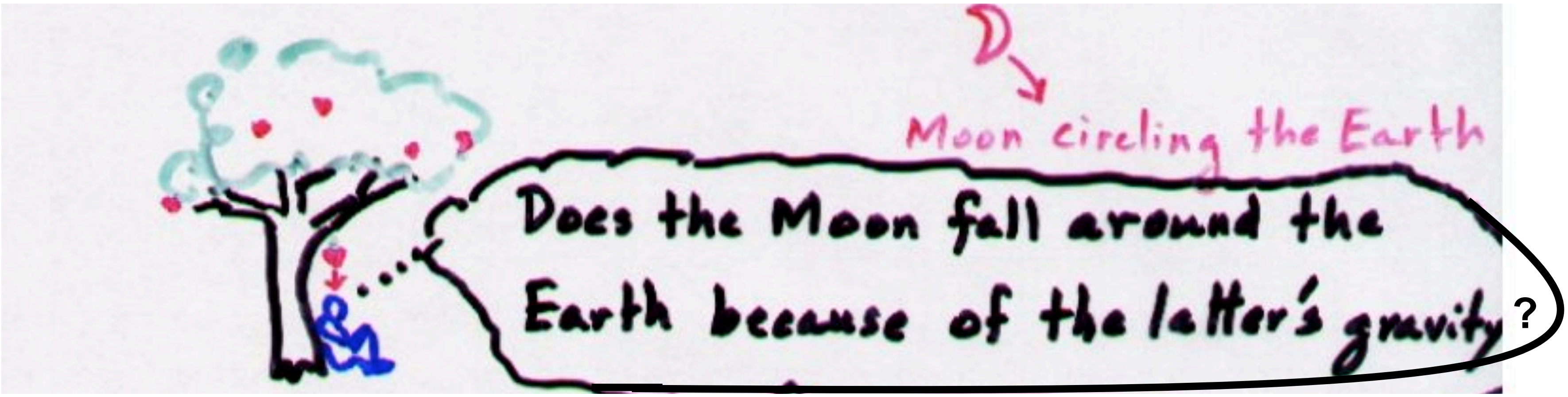
- More general statement of second law of mechanics: When force ***F*** is exerted on a body of mass m, it experiences an acceleration *a* given by ***a*** = ***F***/*m*.
- In order for 1st and 2nd laws to be consistent with each other, so that the total momentum of any complex closed system does not change with time, it is necessary for all internal forces to cancel. To guarantee this, Newton postulated the third law of mechanics: for every action, there is an equal but opposite reaction.
- In the case of the gravitational force that acts on a body of mass m in free fall, Galileo's experiments showed that ***a*** = ***F***/*m* = ***g***, independent of *m*, or indeed any other physical property of the body (e.g., its color, smell, etc.)!
- The gravitational field ***g*** of the Earth (at its surface) is a property only of the Earth, and not of the body being attracted by that field.



$$\vec{F}_{\text{a on E}} = -\vec{F}_{\text{E on a}}.$$

$m_E >> m_A$, $\vec{a}_E << \vec{a}_a$ and $\vec{a}_a \cong \vec{g}$, the same for all falling objects that are small compared to the Earth.

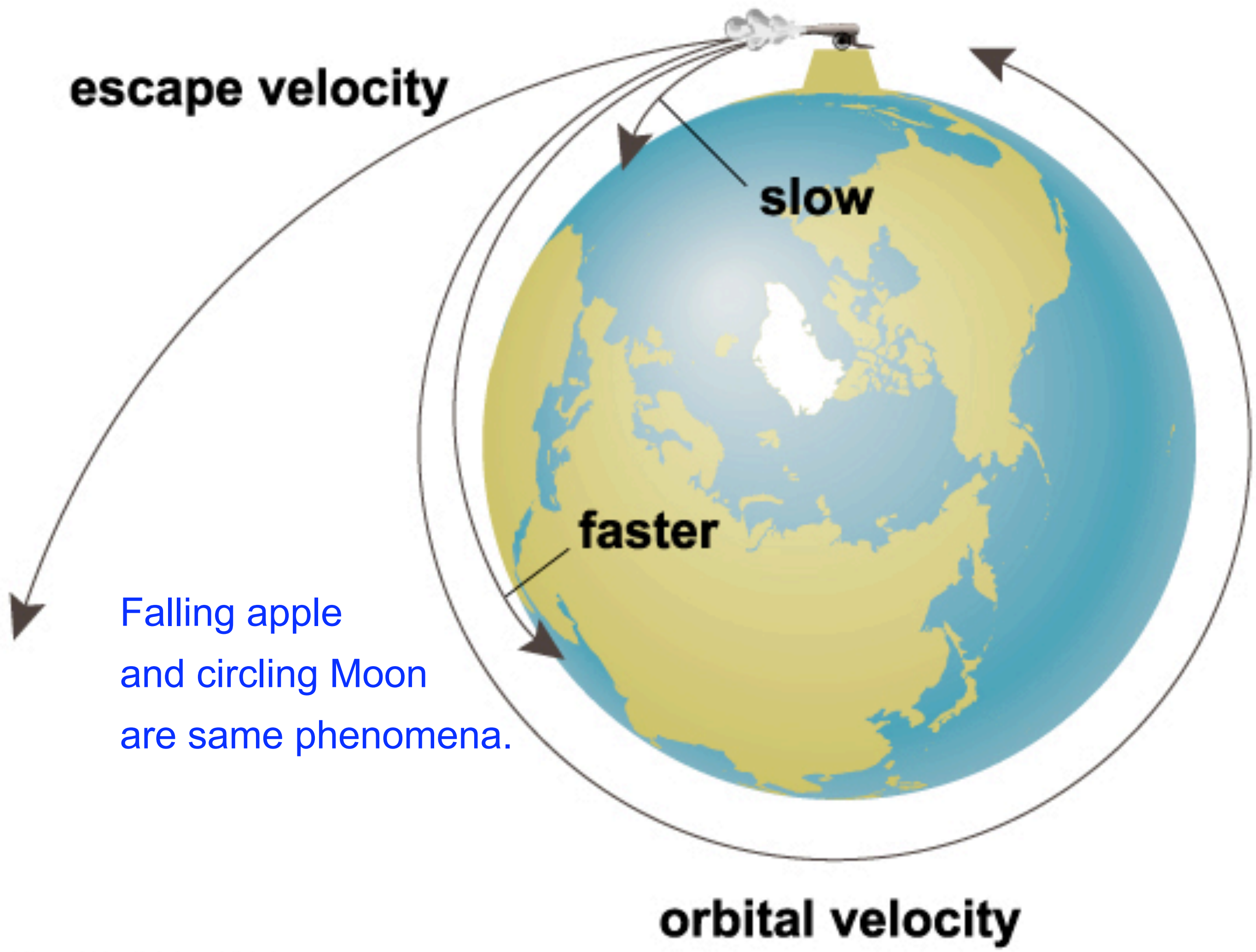
Newton and the Apple



$$a_A = 9.8 \text{ m s}^{-2}; \quad a_M = \frac{v_M{}^2}{r_M} = \frac{(2\pi r_M / P_M)^2}{r_M} = \frac{4\pi^2 r_M}{P_M{}^2}$$

Since $P_M = 27$ d (period relative to stars) and $r_M = 60 r_E$ with $r_E = 6400$ km according to Eratothenes, $a_M = 0.0029 \text{ m s}^{-2} = \dfrac{9.8 \text{ m s}^{-2}}{3600} = \dfrac{a_A}{60^2}$. Idea: Could it be that the Moon, being 60 times farther away from the center of the Earth, experiences an acceleration which is $\dfrac{1}{60^2}$ times as great because $g \propto \dfrac{1}{r^2}$?

Newton Imagines Putting a Cannonball into Orbit



Falling apple
and circling Moon
are same phenomena.

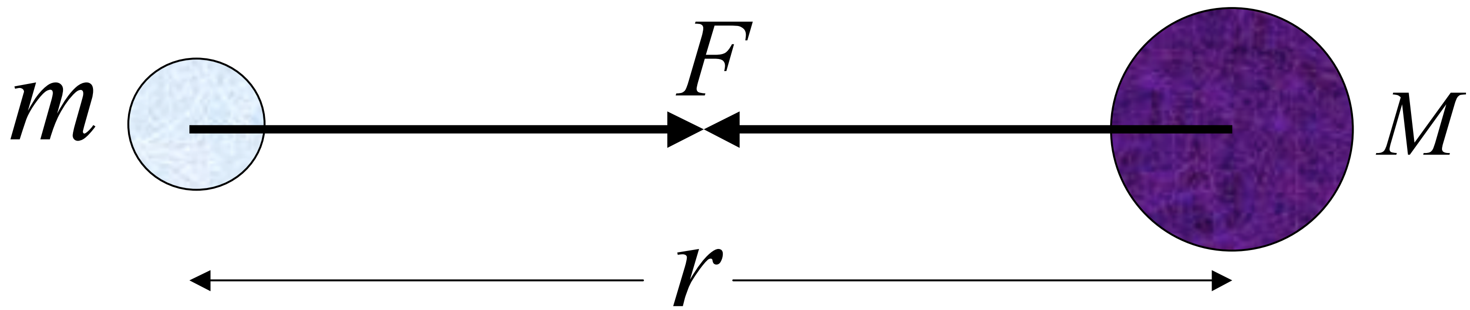
Universal Gravitation

- The force exerted by the Earth on a body of mass *m* is given in magnitude by *F* = *mg*, where *g* varies as the inverse square of the distance r of the body from the center of the Earth: $g \propto \frac{1}{r^2}$.

- By symmetry (or Newton's third law of mechanics, whereby every action has an equal but opposite reaction), the mass m must also exert a force on the Earth of mass *M* given by *F* = *Mf*, where $f \propto \frac{1}{r^2}$, with *r* being the separation distance between *m* and *M*.
- Putting everything together, $F \propto \frac{Mm}{r^2}$.
- Call the proportionality constant *G*, then

$$F = \frac{GMm}{r^2}.$$

Newton’s Law of Gravitation



$$F = \frac{GMm}{r^2}$$

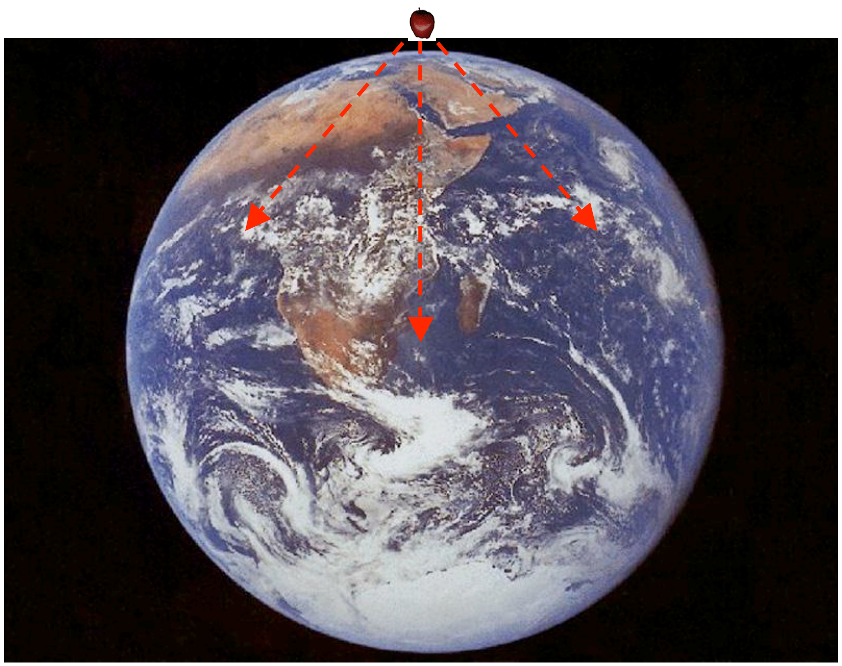
Two Flies in the Ointment

- Newton suspected that the formula

$$F = \frac{GMm}{r^2}$$

applies only to point masses and not fundamentally to bodies of finite size such as the Earth and the Moon. For his reasoning to work, spherical bodies like the Earth and the Moon must exert gravitational force as if their masses were all concentrated as a single point at their centers. (Issue is more severe for Earth attracting apple than Earth attracting Moon.) This he could not prove initially. He needed first to develop differential and integral calculus, which he proceeded to do in the ensuing months! (Reflect on enormity of this achievement.) In the interim, he put aside his results on universal gravitation and did not publish!

- What is the numerical value of G?



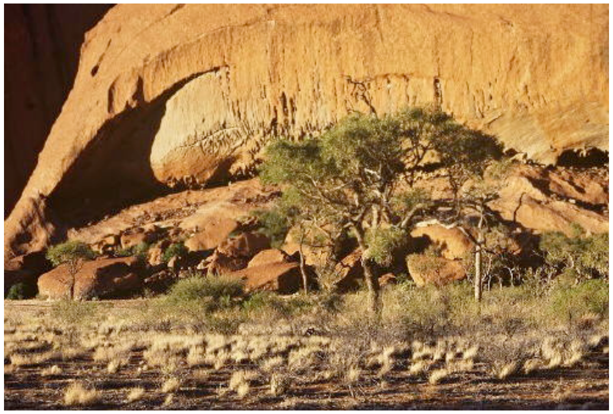
NASA

What is *G*?

- Force F exerted by Earth of mass M on mass m is $F = GMm/r^2 = mg$, which implies $g = GM/r^2$.
- But we know $g = 9.8$ m s^{-2} when $r = 6400$ km. Therefore, we can calculate G if we know the mass M of the Earth.
- Earth is made largely of rock. Density of surface rocks on Earth is about 2.7 times the density of liquid water, 1000 kg m^{-3}. From measurements of the increase in rock density in mine shafts, Newton guessed that the average density ρ of rocks in the Earth is twice the surface value: $\rho = 5.4 \times 10^3$ kg m^{-3} (very lucky guess, because his extrapolation method is completely invalid!)
- M = mean density (mass per unit volume) times volume $= \rho(4\pi R^3/3)$, where $R = 6400$ km $= 6.4 \times 10^6$ m. Hence, $M = 6.0 \times 10^{24}$ kg.
- Now, $g = GM/R^2 = 9.8$ m s^{-2} at surface of Earth implies $G = 6.7 \times 10^{-11}$ kg^{-1} m^3 s^{-2}. Modern value

$$G = 6.67 \times 10^{-11} \mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}.$$

- Moral of story: It pays to be lucky as well as good. But geniuses like Newton make their own luck.



D. B. Ayers

Application: What is Mass of Sun?

- Now let M = mass of Sun; m = mass of Earth; r = 1 AU = radius of Earth's orbit (assumed circular for simplicity), which has a period of 1 yr = 3.16×10^7 s and a speed $v = 2\pi r/P$.
- From variety of different techniques, 1 AU = 1.5×10^8 km (accurate value derived only in 19th century; Lecture 9).
- Centripetal acceleration $a = v^2/r = 4\pi^2 r/P^2$ must equal the gravitational field of the Sun at radius r, $g = GM/r^2$. Thus,

$$M = \frac{4\pi^2 r^3}{GP^2}.$$

- With G given in previous slide, and r and P given above, we obtain

$$M = 2.0 \times 10^{30}\,\mathrm{kg}.$$

- Compare with mass derived previously for the Earth, 6.0×10^{24} kg. The Sun is 330,000 times more massive than the Earth! No wonder the Earth circles the Sun, not the other way around. (Actually, both circle the center of mass, which is 450 km from center of the Sun.)

Relationship to Kepler’s Third Law (extra material)

- Formula $M = 4\pi^2 r^3/GP^2$ could have been applied to any planet of mass $m \ll M$ in a circular orbit at any radius r from Sun.
- Rewrite as

$$P^2 = \left(\frac{4\pi^2}{GM}\right) r^3,$$

  a special case of Kepler’s third law, when the mass of the Sun M does not change as we consider different planets of the solar system.
- A more refined derivation by Newton gives Kepler’s third law as

$$P^2 = \left[\frac{4\pi^2}{G(M+m)}\right] a^3,$$

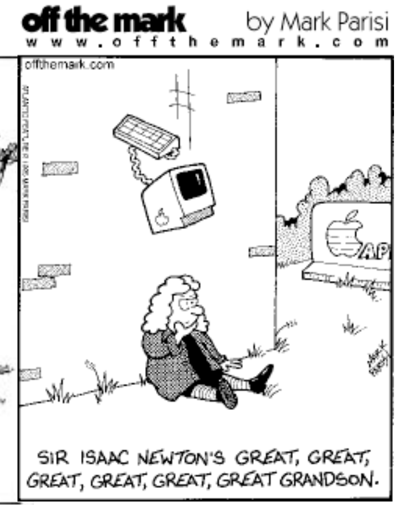
  where a is the semi-major axis of an *elliptical* orbit. Notice also that $M+m$ in Newton’s derivation replaces M. Newton’s derivation takes into account the gravitation of both Sun and planet and does not assume, as we have implicitly done above, that the Sun is perfectly stationary (or, more accurately, that its center is the origin of an inertial frame of reference). In fact, the wobbling of the central star in response to the orbit of the planet is the principal method by which astronomers discover planets around other stars.
- The appearance of the term, $M+m$, implies that the accurate proportionality “constant” does depend on a property of the planet, its mass m. However, the dependence is weak when $m \ll M$, as is the case for all the planets of the solar system.

Summary

- Falling apple and circling Moon are same phenomenon. This unifies the mechanics of Heaven and Earth. Same physical laws apply everywhere in the universe (e.g., to interacting galaxies).
- Kepler's Third Law ↔ Newton's $1/r^2$ law of universal gravitation.
- Kepler's other two laws are also derivable from Newton's laws of mechanics and gravitation.
- Logical relationship:

  Newton's laws (dynamical equations) → Kepler's laws (solutions of equations).
- Sets model for all subsequent scientific research.
- Newton = greatest scientific genius the world has ever seen.

Gary Larson



US One Dollar Bill

English One Pound Note