Lecture 13

Special Relativity

(General Relativity

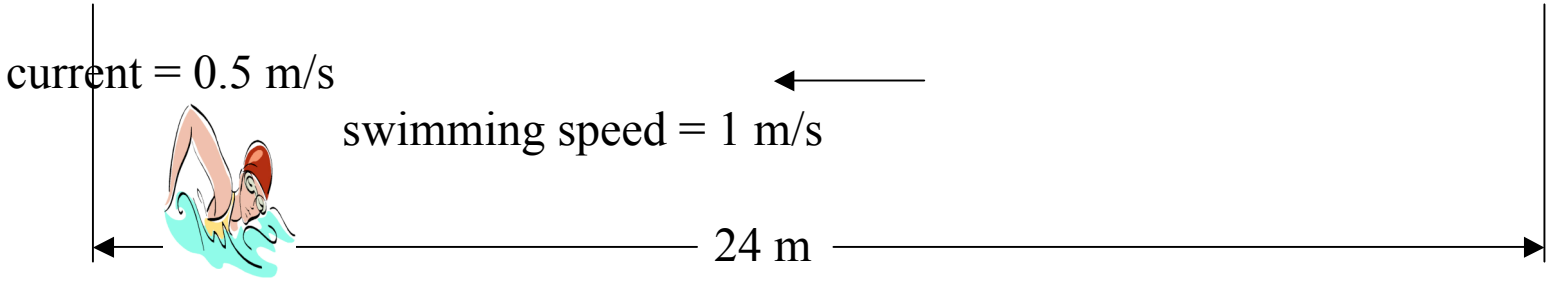
Postponed to Lecture 15)

Outline of Lecture 13

- Einstein's two postulates for special relativity:
  - The laws of physics should have the same form in all inertial frames (which are stationary or move, at most, at a constant vector velocity with respect to the "fixed" stars.)
  - The speed of light in a vacuum, c = 300,000 km/s, is the same no matter how fast an observer moves with respect to the source of the light.
- As consequences:
  - Time appears dilated and lengths seem to suffer (Lorentz) contraction for moving objects.
  - There is an equivalence between mass and energy, $E = mc^2$, the second most famous equation in all of science.
- When Dirac married special relativity to the ideas of quantum mechanics, he obtained the concepts of electron spin and anti-particles.

Prelude to Relativity Theory

- Resurrection of concept of "aether" (quintessence) – something mechanical in which **E** and **B** oscillate for a light wave. From perspective of how light can propagate in a vacuum from stars to us, light as "corpuscles" suffers fewer conceptual difficulties than light as electromagnetic waves.
- Michelson-Morley experiment in 1881: Basic idea – time to swim given distance against and with current greater than if current is not there. Example:



Round-trip lap time, without current, = 24+24 = 48 s.

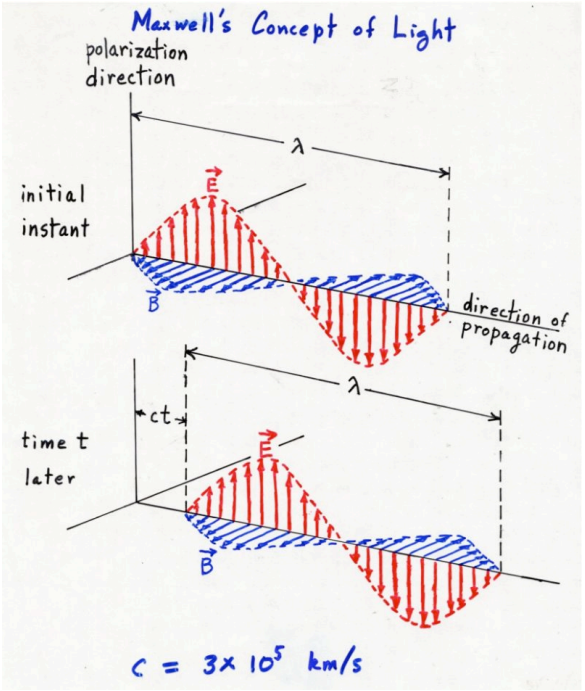
Lap time, with current of 0.5 m/s, = 48 +16 = 64 s.

- Similarly, Earth's motion around Sun of 30 km/s ought to be detectable (with an interferometer) as 1 part in 10,000 variation in the effective speed of light, c = 300,000 km/s. Not seen! Many explanations, none satisfactory until Einstein.

Image from www.morrismeadows.us

History of Relativistic Ideas

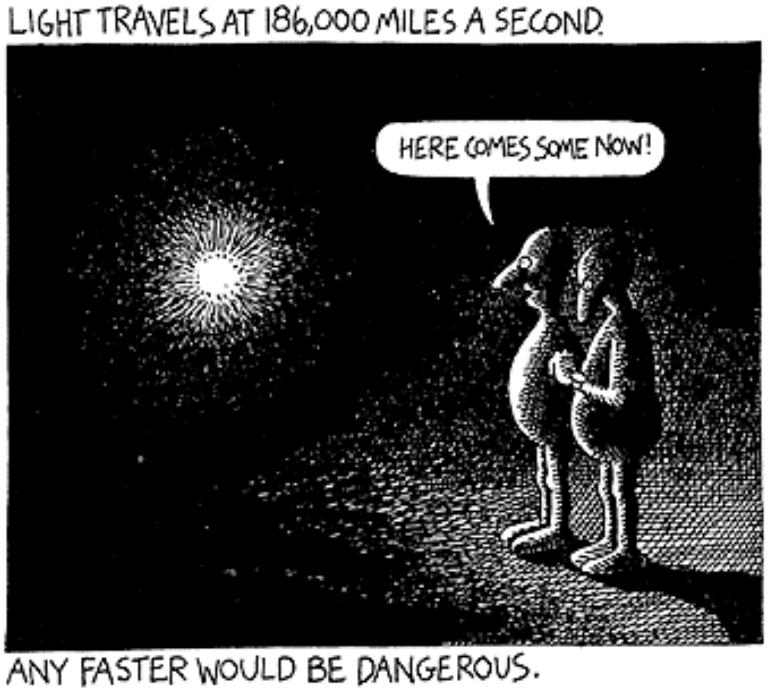
- Hendrik Lorentz (1853-1928)
- Henri Poincare (1854-1912)
- Albert Einstein (1879-1955)
  - Conflict between Newtonian mechanics and Maxwell's theory of electromagnetism.
  - Epitomized by Einstein's thought experiment as a boy of 16: What would one see if one were to race along a beam of light at the speed of light?



According to Newtonian (common-sense) ideas, if we race alongside light at same speed in a vacuum, we would see a wave of ***E*** and ***B*** that oscillates in space, but not in time. But Maxwell's equations have no solutions of this kind. In the absence of charges or currents (a vacuum), electric and magnetic fields must oscillate in time in order to sustain one another. Either Maxwell is wrong, or Newton is wrong.

Theory of Special Relativity: Two Basic Postulates

- Valid laws of physics are the same for all inertial observers (people who move at constant velocity, maybe zero, relative to the "fixed stars").
- The speed of light in a vacuum, c = 300,000 km/s, is the same for all observers, independent of their motion relative to the source of light.



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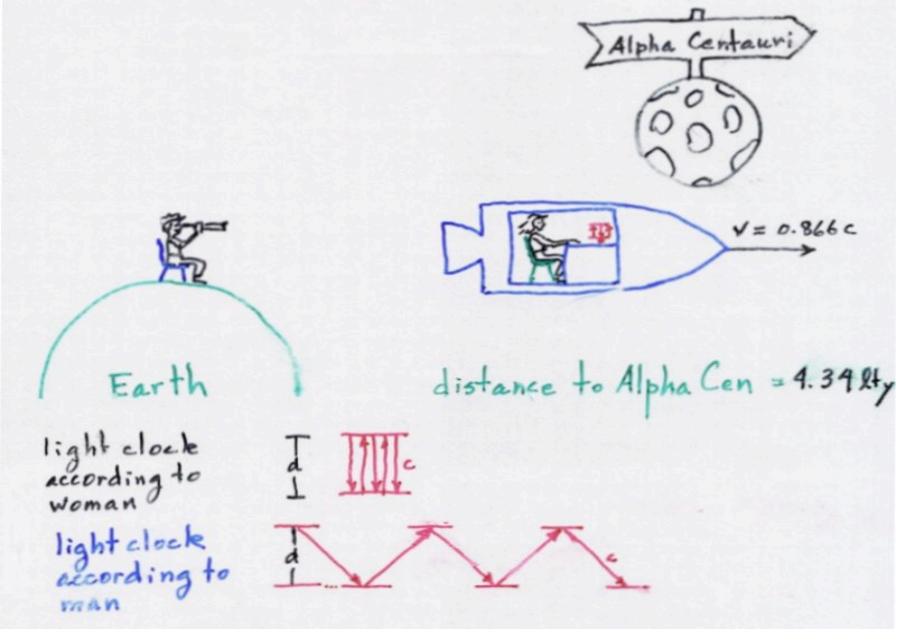
Time Dilation

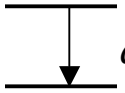


Time for woman to get to Alpha Centauri according to man = 4.34 lt-yr/0.866 c = 5 yr

Time to get to Alpha Centauri according to woman = 2.5 yr, shorter by factor $\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2}$.

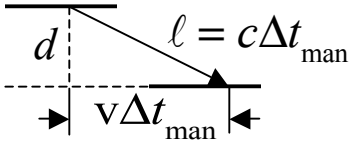
Proof (extra material):

- One tick of light clock according to woman:



$$d = c\Delta t_{woman}$$

- One tick of light clock according to man:



- Relationship between time intervals is given by Pythagoras's theorem:

$$\left(c\Delta t_{man}\right)^2 = \ell^2 = d^2 + v(\Delta t_{man})^2 = (c\Delta t_{woman})^2 + (v\Delta t_{man})^2.$$

Solving for Δt_{woman}, we get

$$\Delta t_{woman} = \Delta t_{man}\sqrt{1-\frac{v^2}{c^2}},$$

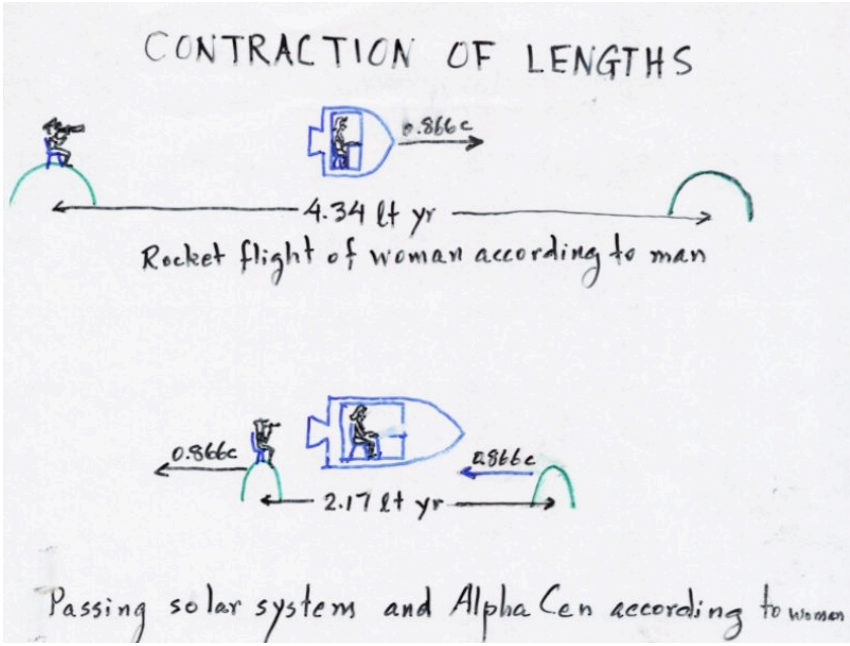
i.e., time passed for woman is shorter by half than time passed for man.

Lorentz Contraction

- Let L_0 be the length of a rod seen by an observer at rest with respect to the rod.
- Let this rod be moving at speed v along its length with respect to a second observer, and let L be the length of the rod measured by the second observer.
- Relationship:

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}.$$

- Moving rod appears to have contracted in length.

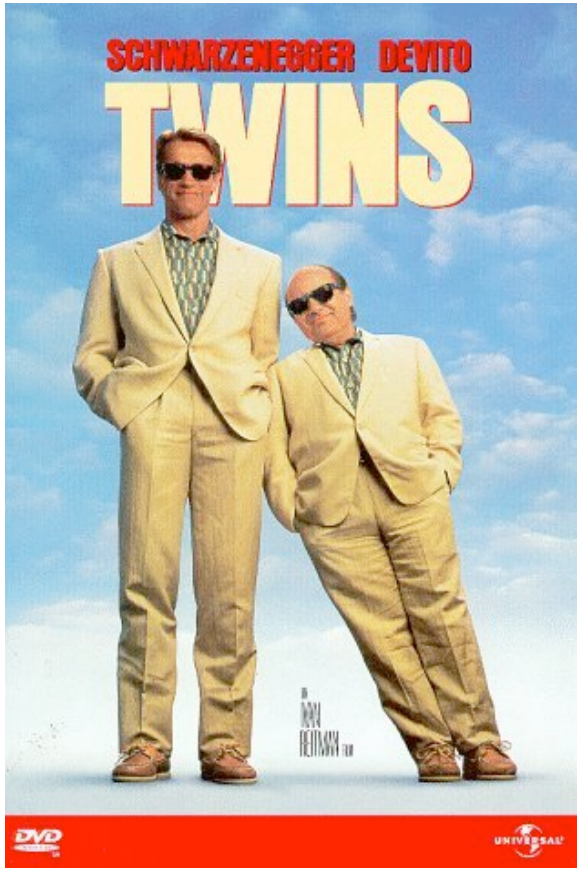


Time for woman to fly to Alpha Centauri = 4.34 lt-yr/0.866 c = 5 yr according to man.

Time for Earth and Alpha Centauri to pass = 2.17 lt-yr/0.866 c = 2.5 yr according to woman.

Twin Paradox and Its Resolution

- Can't the woman regard herself to be at rest and her brother to be moving? Thus, shouldn't the woman age more slowly according to the man, but the man age more slowly according to the woman?
- And so they do think that. However, as long as she's moving away from him, there's no way for them to compare wrinkles. For them to compare wrinkles, she has to decelerate at Alpha Centauri, stop, reaccelerate back to the Earth, decelerate again, and come to a stop at Earth. When they compare wrinkles, she will indeed have fewer; i.e., she will have aged less.
- In the latter case, the woman is not in an inertial frame; i.e., it would be unreasonable for her to assume that she is at rest, and it is her brother and the rest of the universe which are doing all that accelerating and decelerating.



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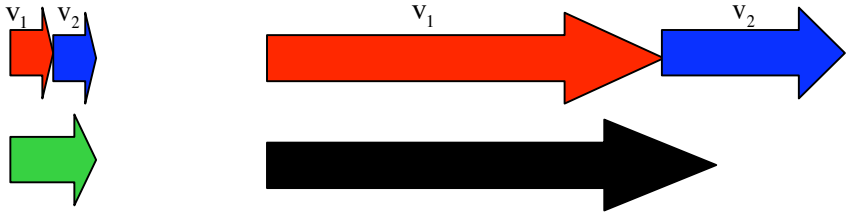
Addition of Velocities and Relativistic Space Travel

- In a straight line, add v_2 to v_1, what is resulting v?

$$v = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.$$

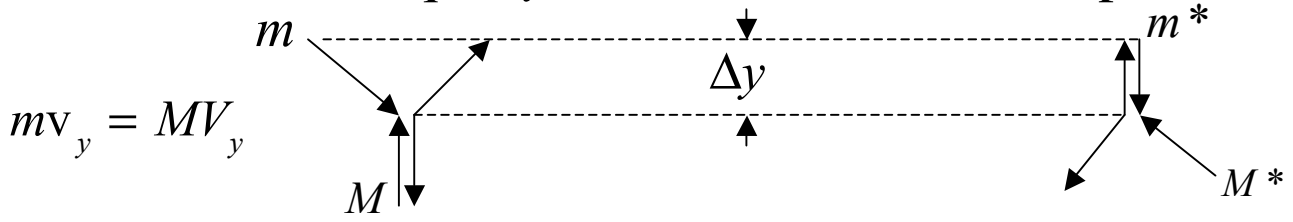
- Add velocity by continuous acceleration a = 1 gee = g = 9.80 m s^{-2}. Accelerate half-way, decelerate half-way; how long does it take to go a certain distance?



Target	Distance (lt-yr)	Earth time (yr)	Ship time (yr)
Alpha Centauri	4.34	6.02	3.64
Galactic Center	25,000	25,000	16
Farther than any telescope	100 billion	100 billion	51

Relativistic Increase in Mass (extra material)

Elastic collision with equal y-momentum, extra component of velocity v_x



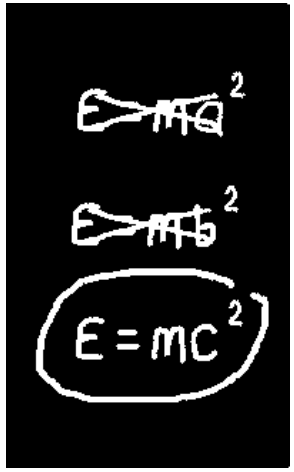
Stationary observer A Moving observer A* $\rightarrow$ $v = v_x$

- M and m have identical mass m_0 when they are at rest with respect to each other.
- Distance Δy seen by A and A* is unaffected by motion of A* in x-direction at velocity v, chosen to equal v_x, which is why $v_x^* = 0$.
- Time Δt^* for collision measured by A* is shorter than Δt measured by A by a factor of $\sqrt{1 - v^2/c^2}$. This implies $v_y^* = \dfrac{\Delta y}{\Delta t^*} > \dfrac{\Delta y}{\Delta t} = v_y$.
- But by momentum balance and symmetry: $mv_y = MV_y = m^* v_y^*$.
- Thus, $m > m^* = M$ by the inverse factor $\gamma \equiv \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

  The reason is that m as seen by A has an extra component of velocity v in the x-direction compared to M seen by the same observer A.
- In the limit when v_y and V_y go to zero, M and m^* go to the rest mass m_0, and we conclude that the mass m of a ball in motion at speed v is larger than its rest mass m_0 by the factor $\gamma > 1$ defined above: $m = \gamma m_0$.

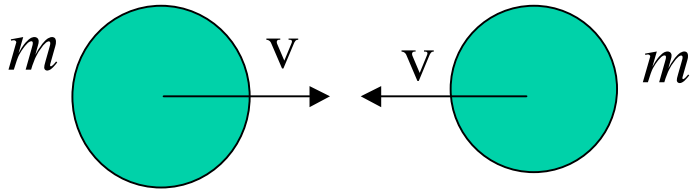
Equivalence of Mass and Energy

- According to Newton, what a moving ball has extra compared to a stationary ball of mass m_0 is
$$\text{kinetic energy} = \frac{1}{2} m_0 \mathrm{v}^2.$$
- According to Einstein, what a moving ball has extra compared to a nonmoving ball of mass m_0 is *more mass*:
$$m = \gamma m_0 \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \mathrm{v}^2 / c^2}} > 1.$$
- Perhaps energy and mass are the same thing!
$$E = mc^2.$$
- In the above equation, E = total energy. For $\mathrm{v} << c$,
$$\gamma \cong 1 + \frac{\mathrm{v}^2}{2c^2} \quad \text{and} \quad E \cong m_0 c^2 + \frac{1}{2} m_0 \mathrm{v}^2$$
where the term $m_0 c^2$ is called the *rest energy* and $m_0 \mathrm{v}^2 / 2$ is the kinetic energy as defined by Newton. More generally,
$$\text{KE} = E - m_0 c^2 = (\gamma - 1) m_0 c^2.$$
- Looking ahead, according to Newton, when a ball is placed in a gravitational field (or more generally, any attractive field of force), what it has is (negative) potential energy. For Einstein, what it has is *less mass*.

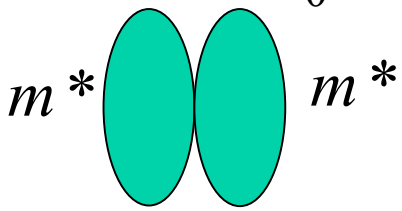


Possibility of Converting Energy into Mass and Vice-Versa

$m = \gamma m_0$



$m^* = m_0{}^*$



- Energy of putty balls before collision = $2mc^2 = 2\gamma m_0 c^2$.
- Energy of putty balls after collision = $2m^* c^2 = 2m_0{}^* c^2$.
- Energy is conserved in process $\Rightarrow 2\gamma m_0 c^2 = 2m_0{}^* c^2$.

$$\therefore\ m_0{}^* = \gamma m_0 > m_0 \quad \text{since} \quad \gamma > 1.$$

- Rest mass after collision is greater than rest mass before collision! There has been conversion of energy into mass.
- Conversion of mass into energy must also be possible.
- Basis of nuclear power. Although less well known, also basis of chemical power; indeed, basis of all power.

Creation of Matter from Energy

- In actual experiment carried out at non-relativistic energies, we get heat not more putty:
  - Kinetic energy of bulk motion becomes converted to kinetic energy of microscopic motion (of putty molecules) = heat.
  - If we cool putty down, we still have same number of putty molecules.
  - However, the agitated putty molecules have slightly more mass than putty molecules at rest, so in the sense of Lavoisier, we do have more putty.
- What happens if we carry out experiment by slamming putty balls together nearly at the speed of light?
  - Then we do have the possibility of creating more putty particles.
  - But only at the expense of producing an equal amount of anti-putty!
- How then did the universe manage to create a net amount of matter without a corresponding amount of antimatter?

Marriage of Special Relativity & Quantum Mechanics

- Relationship between energy, momentum, and rest mass (extra material):

$$E = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - \mathrm{v}^2 / c^2}} = \frac{m_0 c^2}{\sqrt{m_0{}^2 c^2 / (p^2 + m_0{}^2 c^2)}} \Rightarrow$$

$$E^2 = p^2 c^2 + m_0{}^2 c^4 .$$

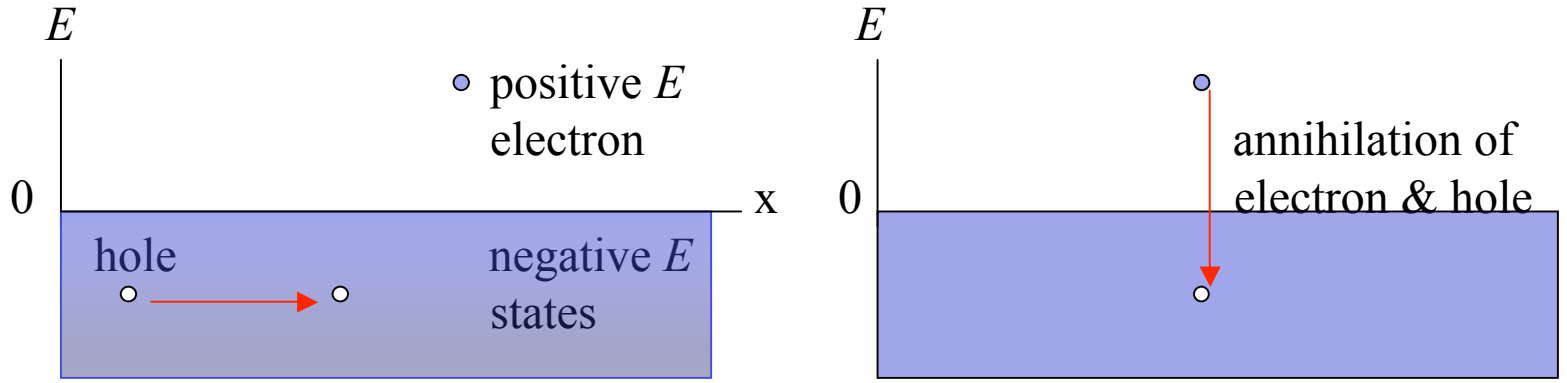
- Dirac (1902-1984) writes down an analogous equation to incorporate relativity into the quantum mechanics of electrons. This equation implies that electrons must have *spin angular momentum.* (The reasoning involves very elegant mathematics which is, unfortunately, too complicated to explain here.)
- He also finds that his equation has two solutions for the energy of a free electron:

$$E = \pm\sqrt{p^2 c^2 + m_0{}^2 c^4} .$$

- Moreover, the nature of Dirac's equation is such that he cannot simply discard one of the solutions without discarding the other. But what can be meant by a solution where the total energy, including rest energy, is *negative*? For example, for a particle at rest, $p = 0$, and $E = \pm m_0 c^2$. Such solutions could spell catastrophe for real electrons. (They could continually lose energy and fall into more and more negative states.)

Dirac's Concept of Anti-Electron as Hole in Infinite Sea of Electrons

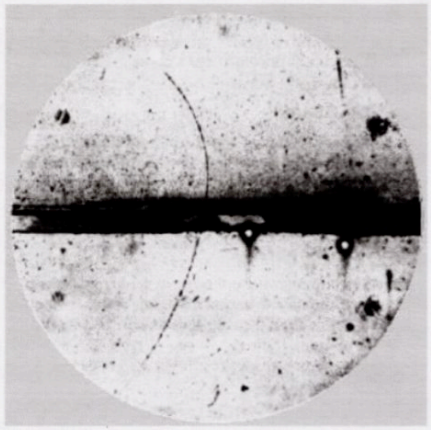
- Dirac speculates that usually all of the negative energy states are filled by an *infinite sea of unseen electrons*. Seen electrons cannot then fall into the unseen sea because of Pauli's exclusion principle.
- On the other hand, if a *hole* develops in this sea, neighboring electrons can fall into the hole, and the sea would move like a positive charge opposite to that of a real electron (anti-electron = positron).



- Moreover, a seen electron not in the negative sea can fall into the unoccupied hole and fill it. Such an event would be seen as the *annihilation of an electron and a positron*.

Positron in Cosmic Rays as Anti-electron

Discovery of positron



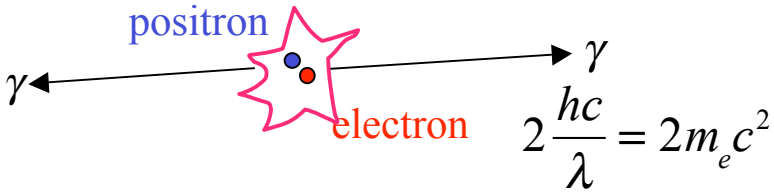
Dirac equation (1928) has negative energy solutions. Antiparticles.

1932 Anderson discovers e^+ in cosmic ray event in cloud chamber. Curvature opposite to e^-. Magnet polarity checked! Direction of track given by energy lost as lead plate traversed.

- Problems with Dirac's picture: Charge and gravity of infinite sea of electrons? His theory is developed only for a single particle, not an infinite sea of them. Nevertheless, this idea and its confirmation in cosmic rays deserved the Nobel Prize.
- Today, we think of positrons as anti-particles, fully equivalent to electrons (with positive total energies) but with opposite charges. More correct description of positrons is in terms of quantum field theory.
- With possibility of each particle of matter being partnered by a counterpart of anti-matter, it becomes possible to create and destroy particles just as its possible to create and destroy photons.
- Possibility of relationship between fermions and bosons? Ideas of supersymmetry.

Matter-Antimatter Annihilation in Medical Imaging

- Positron Emission Tomography (PET scan):
  - Cancerous cells have a high rate of metabolism, i.e., a high rate of using glucose (a simple sugar).
  - Inject glucose carrying radio-isotope that is a positron emitter.
  - Emitted positron slows down in tissue and annihilates with body's electron at low speed, producing two gamma rays with nearly zero total linear momentum:

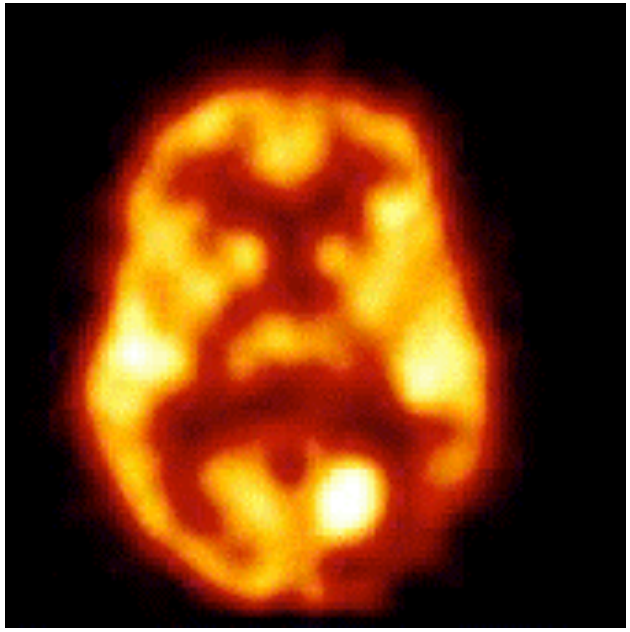


$$2\frac{hc}{\lambda} = 2m_e c^2$$

fly off in opposite directions

- Detect two gamma rays and draw straight line through source.
- Intersection of multiple lines gives 3-D location of tumor.

Bright spot indicates location of brain tumor



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