8

Quantum Mechanics in Three Dimensions

8-1 $E=\frac{\hbar^2\pi^2}{2m}\left[\left(\frac{n_1}{L_x}\right)^2+\left(\frac{n_2}{L_y}\right)^2+\left(\frac{n_3}{L_z}\right)^2\right]$

$L_x=L$, $L_y=L_z=2L$. Let $\frac{\hbar^2\pi^2}{8mL^2}=E_0$. Then $E=E_0\left(4n_1^2+n_2^2+n_3^2\right)$. Choose the quantum numbers as follows:

n_1	n_2	n_3	$\frac{E}{E_0}$		
1	1	1	6		ground state
1	2	1	9	*	first two excited states
1	1	2	9	*	
2	1	1	18		
1	2	2	12	*	next excited state
2	1	2	21		
2	2	1	21		
2	2	2	24		
1	1	3	14	*	next two excited states
1	3	1	14	*	

Therefore the first 6 states are ψ_{111}, ψ_{121}, ψ_{112}, ψ_{122}, ψ_{113}, and ψ_{131} with relative energies $\frac{E}{E_0}=6, 9, 9, 12, 14, 14$. First and third excited states are doubly degenerate.

8-2 (a) $n_1=1, n_2=1, n_3=1$

$$E_0=\frac{3\hbar^2\pi^2}{2mL^2}=\frac{3h^2}{8mL^2}=\frac{3\left(6.626\times10^{-34}\text{ Js}\right)^2}{8\left(9.11\times10^{-31}\text{ kg}\right)\left(2\times10^{-10}\text{ m}\right)^2}=4.52\times10^{-18}\text{ J}=28.2\text{ eV}$$

(b) $n_1=2, n_2=1, n_3=1$ or
$n_1=1, n_2=2, n_3=1$ or
$n_1=1, n_2=1, n_3=2$

$$E_1=\frac{6h^2}{8mL^2}=2E_0=56.4\text{ eV}$$

8-3 $n^2 = 11$

(a) $E = \left(\frac{\hbar^2\pi^2}{2mL^2}\right)n^2 = \frac{11}{2}\left(\frac{\hbar^2\pi^2}{mL^2}\right)$

(b)

n_1	n_2	n_3	
1	1	3	
1	3	1	3-fold degenerate
3	1	1	

(c)

$$\psi_{113} = A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{3\pi z}{L}\right)$$
$$\psi_{131} = A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{3\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)$$
$$\psi_{311} = A\sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)$$

8-4 (a) $\psi(x, y) = \psi_1(x)\psi_2(y)$. In the two-dimensional case, $\psi = A(\sin k_1 x)(\sin k_2 y)$ where $k_1 = \frac{n_1\pi}{L}$ and $k_2 = \frac{n_2\pi}{L}$.

(b) $E = \frac{\hbar^2\pi^2(n_1^2 + n_2^2)}{2mL^2}$

If we let $E_0 = \frac{\hbar^2\pi^2}{mL^2}$, then the energy levels are:

n_1	n_2	$\frac{E}{E_0}$			
1	1	1	→	ψ_{11}	
1	2	$\frac{5}{2}$	→	ψ_{12}	doubly degenerate
2	1	$\frac{5}{2}$	→	ψ_{21}	
2	2	4	→	ψ_{22}	

8-5 (a) $n_1 = n_2 = n_3 = 1$ and $E_{111} = \frac{3h^2}{8mL^2} = \frac{3(6.63\times10^{-34})^2}{8(1.67\times10^{-27})(4\times10^{-28})} = 2.47\times10^{-13}\text{ J} \approx 1.54\text{ MeV}$

(b) States 211, 121, 112 have the same energy and $E = \frac{(2^2+1^2+1^2)h^2}{8mL^2} = 2E_{111} \approx 3.08\text{ MeV}$ and states 221, 122, 212 have the energy $E = \frac{(2^2+2^2+1^2)h^2}{8mL^2} = 3E_{111} \approx 4.63\text{ MeV}$.

(c) Both states are threefold degenerate.

8-9 $L=[l(l+1)]^{1/2}\hbar$

$$4.714\times10^{-34}\text{ Js}=[l(l+1)]^{1/2}\left(\frac{6.63\times10^{-34}\text{ Js}}{2\pi}\right)$$

$$l(l+1)=\frac{\left(4.714\times10^{-34}\right)^2(2\pi)^2}{\left(6.63\times10^{-34}\right)^2}=1.996\times10^1\approx20=4(4+1)$$

so $l=4$.

8-10 $n=4$, $l=3$, and $m_l=3$.

(a) $L=[l(l+1)]^{1/2}\hbar=[3(3+1)]^{1/2}\hbar=2\sqrt{3}\hbar=3.65\times10^{-34}$ Js

(b) $L_z=m_l\hbar=3\hbar=3.16\times10^{-34}$ Js

8-11 (a) $L=[l(l+1)]^{1/2}\hbar;\ 4.83\times10^{31}\text{ Js}=[l(l+1)]^{1/2}\hbar$, so

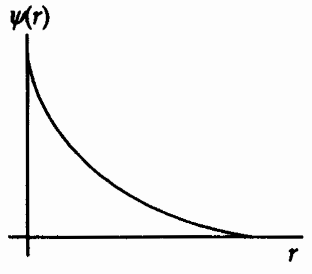
$$l^2+l=\frac{\left(4.83\times10^{31}\text{ Js}\right)^2}{\left(1.055\times10^{-34}\text{ Js}\right)^2}\approx\left(4.58\times10^{65}\right)^2\approx l^2$$

$$l\approx4.58\times10^{65}$$

(b) With $L\approx l\hbar$ we get $\Delta L\approx\hbar$ and $\frac{\Delta L}{L}\approx\frac{1}{l}=2.18\times10^{-66}$

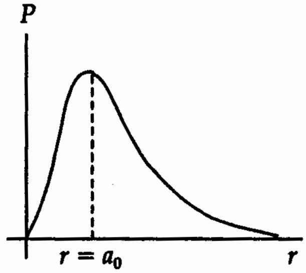
8-12 $\psi(r)=\left(\frac{1}{\pi}\right)^{1/2}\left(\frac{1}{a_0}\right)^{3/2}e^{-r/a_0}$

(a)



(b) The probability of finding the electron in a volume element dV is given by $|\psi|^2 dV$. Since the wave function has spherical symmetry, the volume element dV is identified here with the volume of a spherical shell of radius r, $dV=4\pi r^2 dr$. The probability of finding the electron between r and $r+dr$ (that is, within the spherical shell) is $P=|\psi|^2 dV=4\pi r^2|\psi|^2 dr$.

(c)



(d) $\int |\psi|^2 dV = 4\pi \int |\psi|^2 r^2 dr = 4\pi\left(\frac{1}{\pi}\right)\left(\frac{1}{a_0^3}\right)\int_0^\infty e^{-2r/a_0} r^2 dr = \left(\frac{4}{a_0^3}\right)\int_0^\infty e^{-2r/a_0} r^2 dr$

Integrating by parts, or using a table of integrals, gives

$$\int |\psi|^2 dV = \left(\frac{4}{a_0^3}\right)\left[2\left(\frac{a_0}{2}\right)^3\left(\frac{2}{a_0}\right)^3\right] = 1.$$

(e) $P = 4\pi \int_{r_1}^{r_2} |\psi|^2 r^2 dr$ where $r_1 = \frac{a_0}{2}$ and $r_2 = \frac{3a_0}{2}$

$$P = \left(\frac{4}{a_0^3}\right)\int_{r_1}^{r_2} r^2 e^{-2r/a_0} dr \qquad \text{let } z = \frac{2r}{a_0}$$
$$= \frac{1}{2}\int_1^3 z^2 e^{-z} dz$$
$$= -\frac{1}{2}\left(z^2 + 2z + 2\right)e^{-z}\Big|_1^3 \quad \text{(integrating by parts)}$$
$$= -\frac{17}{2}e^{-3} + \frac{5}{2}e^{-1} = 0.496$$

8-13 $Z = 2$ for He^+

(a) For $n = 3$, l can have the values of 0, 1, 2

$$l = 0 \rightarrow m_l = 0$$
$$l = 1 \rightarrow m_l = -1, 0, +1$$
$$l = 2 \rightarrow m_l = -2, -1, 0, +1, +2$$

(b) All states have energy $E_3 = \frac{-Z^2}{3^2}(13.6 \text{ eV})$

$$E_3 = -6.04 \text{ eV}.$$

8-14 $Z = 3$ for Li^{2+}

(a) $n = 1 \rightarrow l = 0 \rightarrow m_l = 0$
$n = 2 \rightarrow l = 0 \rightarrow m_l = 0$
and $l = 1 \rightarrow m_l = -1, 0, +1$

(b) For $n=1$, $E_1=-\left(\frac{3^2}{1^2}\right)(13.6)=-122.4\text{ eV}$

For $n=2$, $E_2=-\left(\frac{3^2}{2^2}\right)(13.6)=-30.6\text{ eV}$

8-15 (a) $E_n=-\left(\frac{ke^2}{2a_0}\right)\left(\frac{Z^2}{n^2}\right)$ from Equation 8.38. But $a_0=\frac{\hbar^2}{m_e ke^2}$ so with $m_e\to\mu$ we get

$E_n=-\left(\frac{\mu k^2e^4}{2\hbar^2}\right)\left(\frac{Z^2}{n^2}\right)$.

(b) For $n=3\to 2$, $E_3-E_2=\frac{hc}{\lambda}=\frac{\mu k^2e^4Z^2}{2\hbar^2}\left(\frac{1}{2^2}-\frac{1}{3^2}\right)$ with $\lambda=656.3$ nm for H ($Z=1$, $\mu\approx m_e$). For He^+, $Z=2$, and $\mu\approx m_e$, so, $\lambda=\frac{656.3}{2^2}=164.1$ nm (ultraviolet).

(c) For positronium, $Z=1$ and $\mu=\frac{m_e}{2}$, so, $\lambda=(656.3)(2)=1\,312.6$ nm (infrared).

8-16 For a d state, $l=2$. Thus, m_l can take on values −2, −1, 0, 1, 2. Since $L_z=m_l\hbar$, L_z can be $\pm 2\hbar$, $\pm\hbar$, and zero.

8-17 (a) For a d state, $l=2$

$$L=[l(l+1)]^{1/2}\hbar=(6)^{1/2}\left(1.055\times10^{-34}\text{ Js}\right)=2.58\times10^{-34}\text{ Js}$$

(b) For an f state, $l=3$

$$L=[l(l+1)]^{1/2}\hbar=(12)^{1/2}\left(1.055\times10^{-34}\text{ Js}\right)=3.65\times10^{-34}\text{ Js}$$

8-18 The state is $6g$

(a) $n=6$

(b) $E_n=-\frac{13.6\text{ eV}}{n^2}$ $\quad$ $E_6=\frac{-13.6}{6^2}\text{ eV}=-0.378\text{ eV}$

(c) For a g-state, $l=4$

$$L=[l(l+1)]^{1/2}\hbar=(4\times5)^{1/2}\hbar=\sqrt{20}\hbar=4.47\hbar$$

(d) m_l can be −4, −3, −2, −1, 0, 1, 2, 3, or 4

$L_z=m_l\hbar$; $\cos\theta=\frac{L_z}{L}=\frac{m_l}{[l(l+1)]^{1/2}}\hbar=\frac{m_l}{\sqrt{20}}$

m_l	−4	−3	−2	−1	0	1	2	3	4
L_z	$-4\hbar$	$-3\hbar$	$-2\hbar$	$-\hbar$	0	$\hbar$	$2\hbar$	$3\hbar$	$4\hbar$
θ	153.4°	132.1°	116.6°	102.9°	90°	77.1°	63.4°	47.9°	26.6°

8-19 When the principal quantum number is n, the following values of l are possible: $l=0, 1, 2, \ldots, n-2, n-1$. For a given value of l, there are $2l+1$ possible values of m_l. The maximum number of electrons that can be accommodated in the n^{th} level is therefore:

$$(2(0)+1)+(2(1)+1)+\ldots+(2l+1)+\ldots+(2(n-1)+1)=2\sum_{l=0}^{n-1}l+\sum_{l=0}^{n-1}l=2\sum_{l=0}^{n-1}l+n.$$

But $\sum_{l=0}^{k}l=\frac{k(k+1)}{2}$ so the maximum number of electrons to be accommodated is $\frac{2(n-1)n}{2}+n=n^2$.

8-20 The total electron energy in a circular orbit is (as given) $E=\frac{|\mathbf{L}|^2}{2m_e r^2}-\frac{Zke^2}{r}$. Substituting the quantization rules for angular momentum $|\mathbf{L}|^2=l(l+1)\hbar^2$ and $E=-\left\{\frac{Z^2}{n^2}\right\}\left(\frac{ke^2}{2a_0}\right)$ and using the Bohr result $r=\frac{n^2a_0}{Z}$ gives $-\left(\frac{Z^2}{n^2}\right)\left(\frac{ke^2}{2a_0}\right)=\frac{Z^2l(l+1)\hbar^2}{2m_e\left(n^2a_0\right)^2}-\frac{Z^2ke^2}{n^2a_0}$. Remembering that $a_0=\frac{\hbar^2}{m_eke^2}$ and canceling common factors in each term leaves $-\frac{1}{2}=\frac{l(l+1)}{2n^2}-1$. Thus, any orbital quantum number l greater than $l_{\text{max}}=n-1$ will produce a total energy larger than that prescribed by n, i.e., $E_n=-\frac{\left(Z^2/n^2\right)ke^2}{2a_0}$.

8-21 (a) $\psi_{2s}(r)=\frac{1}{4(2\pi)^{1/2}}\left(\frac{1}{a_0}\right)^{3/2}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}$. At $r=a_0=0.529\times10^{-10}$ m we find

$$\psi_{2s}(a_0)=\frac{1}{4(2\pi)^{1/2}}\left(\frac{1}{a_0}\right)^{3/2}(2-1)e^{-1/2}=(0.380)\left(\frac{1}{a_0}\right)^{3/2}$$

$$=(0.380)\left[\frac{1}{0.529\times10^{-10}\text{ m}}\right]^{3/2}=9.88\times10^{14}\text{ m}^{-3/2}$$

(b) $|\psi_{2s}(a_0)|^2=\left(9.88\times10^{14}\text{ m}^{-3/2}\right)^2=9.75\times10^{29}\text{ m}^{-3}$

(c) Using the result to part (b), we get $P_{2s}(a_0)=4\pi a_0^2|\psi_{2s}(a_0)|^2=3.43\times10^{10}\text{ m}^{-1}$.

8-22 $R_{2p}(r)=Are^{-r/2a_0}$ where $A=\frac{1}{2(6)^{1/2}a_0^{5/2}}$

$$P(r)=r^2R_{2p}^2(r)=A^2r^4e^{-r/a_0}$$

$$\langle r\rangle=\int_0^\infty rP(r)dr=A^2\int_0^\infty r^5e^{-r/a_0}dr=A^2a_0^6 5!=5a_0=2.645\text{ Å}$$

8-23 (a) $\dfrac{1}{\alpha}=\dfrac{\hbar c}{ke^2}=\dfrac{(6.63\times10^{-34}\text{ Js})(3\times10^8\text{ m/s})}{2\pi(9\times10^9\text{ N m}^2/\text{C}^2)(1.6\times10^{-19}\text{ C})^2}=137.036$

(b) $\dfrac{\lambda_c}{r_0}=\dfrac{h/m_ec}{ke^2/m_ec^2}=\dfrac{hc}{ke^2}=\dfrac{2\pi}{\alpha}=2\pi\times137$

(c) $\dfrac{a_0}{\lambda_c}=\dfrac{\hbar^2/m_eke^2}{h/m_ec}=\dfrac{1}{2\pi}\dfrac{\hbar c}{ke^2}=\dfrac{1}{2\pi\alpha}=\dfrac{137}{2\pi}$

(d) $\dfrac{1}{Ra_0}=\left(\dfrac{m_eke^2}{\hbar^2}\right)\left(\dfrac{4\pi c\hbar^3}{m_ek^2e^4}\right)=\dfrac{4\pi\hbar c}{ke^2}=\dfrac{4\pi}{\alpha}=4\pi(137)$

8-24 $P_{1s}(r)=\dfrac{4}{a_0^3}r^2e^{-2r/a_0}$ for hydrogen ground state, $U(r)=-\dfrac{ke^2}{r}$ is potential energy $(Z=1)$

$$\langle U\rangle=\int_0^\infty U(r)P_{1s}(r)dr=-\frac{4ke^2}{a_0^3}\int_0^\infty re^{-2r/a_0}dr$$
$$=-\frac{4ke^2}{a_0^3}\left(\frac{a_0}{2}\right)^2\int_0^\infty ze^{-z}dz\quad\text{where } z=\frac{2r}{a_0}$$
$$=\frac{-ke^2}{a_0}=-2(13.6\text{ eV})=-27.2\text{ eV}.$$

To find $\langle K\rangle$, we note that $\langle K\rangle+\langle U\rangle=\langle E\rangle=-\dfrac{ke^2}{2a_0}=-13.6\text{ eV}$ so, $\langle K\rangle=\dfrac{ke^2}{a_0}=+13.6\text{ eV}$.

8-25 The most probable distance is the value of r which maximizes the radial probability density $P(r)=|rR(r)|^2$. Since $P(r)$ is largest where $rR(r)$ reaches its maximum, we look for the most probable distance by setting $\dfrac{d\{rR(r)\}}{dr}$ equal to zero, using the functions $R(r)$ from Table 8.4. For clarity, we measure distances in bohrs, so that $\dfrac{r}{a_0}$ becomes simply r, etc. Then for the $2s$ state of hydrogen, the condition for a maximum is

$$0=\frac{d}{dr}\{(2r-r^2)e^{-r/2}\}=\left\{2-2r-\frac{1}{2}(2r-r^2)\right\}e^{-r/2}$$

or $0=4-6r+r^2$. There are two solutions, which may be found by completing the square to get $0=(r-3)^2-5$ or $r=3\pm\sqrt{5}$ bohrs. Of these $r=3+\sqrt{5}=5.236a_0$ gives the largest value of $P(r)$, and so is the most probable distance. For the $2p$ state of hydrogen, a similar analysis gives $0=\dfrac{d}{dr}\{r^2e^{-r/2}\}=\left\{2r-\dfrac{1}{2}r^2\right\}e^{-r/2}$ with the obvious roots $r=0$ (a minimum) and $r=4$ (a maximum). Thus, the most probable distance for the $2p$ state is $r=4a_0$, in agreement with the simple Bohr model.

8-26 The probabilities are found by integrating the radial probability density for each state, $P(r)$, from $r=0$ to $r=4a_0$. For the 2s state we find from Table 8.4 (with $Z=1$ for hydrogen)

$P_{2s}(r)=|rR_{2s}(r)|^2=(8a_0)^{-1}\left(\frac{r}{a_0}\right)^2\left(2-\frac{r}{a_0}\right)^2 e^{-r/a_0}$ and $P=(8a_0)^{-1}\int_0^{4a_0}\left(\frac{r}{a_0}\right)^2\left(2-\frac{r}{a_0}\right)^2 e^{-r/a_0}dr$.

Changing variables from r to $z=\frac{r}{a_0}$ gives $P=8^{-1}\int_0^4\left(4z^2-4z^3+z^4\right)e^{-z}dz$. Repeated integration by parts gives

$$P=8^{-1}\left\{-\left(4z^2-4z^3+z^4\right)-\left(8z-12z^2+4z^3\right)-\left(8-24z+12z^2\right)-(-24+24z)-(24)\right\}e^{-z}\Big|_0^4$$
$$=8^{-1}\left\{-(64+96+104+72+24)e^{-4}+8\right\}=0.176$$

For the 2p state of hydrogen $P_{2p}(r)=\left|rR_{2p}(r)\right|^2=(24a_0)^{-1}\left(\frac{r}{a_0}\right)^4 e^{-r/a_0}$ and

$P=(24a_0)^{-1}\int_0^{4a_0}\left(\frac{r}{a_0}\right)^4 e^{-r/a_0}dr=24^{-1}\int_0^4 z^4e^{-z}dz$. Again integrating by parts, we get

$P=24^{-1}\left\{-z^4-4z^3-12z^2-24z-24\right\}e^{-z}\Big|_0^4=24^{-1}\left\{-824e^{-4}+24\right\}=0.371$. The probability for the 2s electron is much smaller, suggesting that this electron spends more of its time in the outer regions of the atom. This is in accord with classical physics, where the electron in a lower angular momentum state is described by orbits more elliptic in shape.

8-27 See Multimedia Manager

8-28 See Multimedia Manager

8-29 To find Δr we first compute $\left\langle r^2\right\rangle$ using the radial probability density for the 1s state of hydrogen: $P_{1s}(r)=\frac{4}{a_0^3}r^2e^{-2r/a_0}$. Then $\left\langle r^2\right\rangle=\int_0^\infty r^2P_{1s}(r)dr=\frac{4}{a_0^3}\int_0^\infty r^4e^{-2r/a_0}dr$. With $z=\frac{2r}{a_0}$, this is $\left\langle r^2\right\rangle=\frac{4}{a_0^3}\left(\frac{a_0}{2}\right)^5\int_0^\infty z^4e^{-z}dz$. The integral on the right is (see Example 8.9) $\int_0^\infty z^4e^{-z}dz=4!$ so that $\left\langle r^2\right\rangle=\frac{4}{a_0^3}\left(\frac{a_0}{2}\right)^5(4!)=3a_0^2$ and $\Delta r=\left(\left\langle r^2\right\rangle-\langle r\rangle^2\right)^{1/2}=\left[3a_0^2-(1.5a_0)^2\right]^{1/2}=0.866a_0$. Since Δr is an appreciable fraction of the average distance, the whereabouts of the electron are largely unknown in this case.

8-30 The averages $\langle r\rangle$ and $\left\langle r^2\right\rangle$ are found by weighting the probability density for this state $P_{1s}(r)=4\left(\frac{Z}{a_0^3}\right)r^2e^{-2Zr/a_0}$ with r and r^2, respectively, in the integral from $r=0$ to $r=\infty$:

$$\langle r\rangle=\int_0^\infty rP_{1s}(r)dr=4\left(\frac{Z}{a_0^3}\right)\int_0^\infty r^3e^{-2Zr/a_0}dr$$
$$\left\langle r^2\right\rangle=\int_0^\infty r^2P_{1s}(r)dr=4\left(\frac{Z}{a_0^3}\right)\int_0^\infty r^4e^{-2rZ/a_0}dr$$

Substituting $z = \frac{2Zr}{a_0}$ gives

$$\langle r \rangle = 4\left(\frac{Z}{a_0}\right)^3\left(\frac{a_0}{2Z}\right)^4\int_0^\infty z^3 e^{-z}dz = \frac{3!}{4}\left(\frac{a_0}{Z}\right) = \frac{3}{2}\left(\frac{a_0}{Z}\right)$$

$$\left\langle r^2 \right\rangle = 4\left(\frac{Z}{a_0}\right)^3\left(\frac{a_0}{2Z}\right)^5\int_0^\infty z^4 e^{-z}dz = \frac{4!}{8}\left(\frac{a_0}{Z}\right)^2 = 3\left(\frac{a_0}{Z}\right)^2$$

and $\Delta r = \left(\left\langle r^2\right\rangle - \langle r\rangle^2\right)^{1/2} = \frac{a_0}{Z}\left[3 - \frac{9}{4}\right]^{1/2} = 0.866\left(\frac{a_0}{Z}\right)$. The momentum uncertainty is deduced from the average potential energy

$$\langle U\rangle = -kZe^2\int_0^\infty \frac{1}{r}P_{1s}(r)dr = -4kZe^2\left(\frac{Z}{a_0}\right)^3\int_0^\infty re^{-2Zr/a_0} = -4kZe^2\left(\frac{Z}{a_0}\right)^3\left(\frac{a_0}{2Z}\right)^2 = -\frac{k(Ze)^2}{a_0}.$$

Then, since $E = -\frac{k(Ze)^2}{2a_0}$ for the 1s level, and $a_0 = \frac{\hbar^2}{m_e ke^2}$, we obtain

$$\left\langle p^2\right\rangle = 2m_e\langle K\rangle = 2m_e(E - \langle U\rangle) = \frac{2m_e k(Ze)^2}{2a_0} = \left(\frac{Z\hbar}{a_0}\right)^2.$$

With $\langle \mathrm{p}\rangle = 0$ from symmetry, we get $\Delta p = \left(\left\langle p^2\right\rangle\right)^{1/2} = \frac{Z\hbar}{a_0}$ and $\Delta r\Delta p = 0.866\hbar$ for any Z, consistent with the uncertainty principle.

8-31 Outside the surface, $U(x) = -\frac{A}{x}$ (to give $F = -\frac{dU}{dx} = -\frac{A}{x^2}$), and Schrödinger's equation is $-\left(\frac{\hbar^2}{2m_e}\right)\frac{d^2\psi}{dx^2} + \left(-\frac{A}{x}\right)\psi(x) = E\psi(x)$. From Equation 8.36 $g(r) = rR(r)$ satisfies a one-dimensional Schrödinger equation with effective potential $U_{eff}(r) = U(r) + \frac{l(l+1)\hbar^2}{2m_e r^2}$. With $l = 0$ (s states) and $U(r) = -\frac{kZe^2}{r}$ the equation for $g(r)$ has the same form as that for $\psi(x)$. Furthermore, $\psi(0) = 0$ if no electrons can cross the surface, while $g(0) = 0$ since $R(0)$ must be finite. It follows that the functions $g(r)$ and $\psi(x)$ are the same, and that the energies in the present case are the hydrogenic levels $E_n = -\left(\frac{Z^2ke^2}{2a_0}\right)\left(\frac{1}{n^2}\right)$ with the replacement $kZe^2 \to A$.

Remembering that $a_0 = \frac{\hbar^2}{m_e ke^2}$, we get $E_n = -\left(\frac{mA^2}{2\hbar^2}\right)\left(\frac{1}{n^2}\right)$ $n = 1, 2, \ldots$

8-32 See Multimedia Manager

8-33 See Multimedia Manager

8-34 See Multimedia Manager