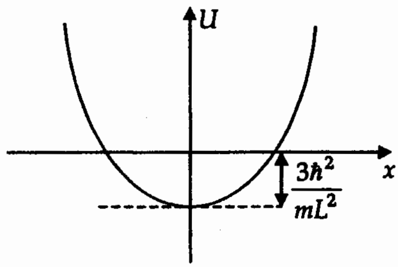
6-5 (a) Solving the Schrödinger equation for U with $E=0$ gives

$$U=\left(\frac{\hbar^2}{2m}\right)\frac{\left(\frac{d^2\psi}{dx^2}\right)}{\psi}.$$

If $\psi = Ae^{-x^2/L^2}$ then $\frac{d^2\psi}{dx^2}=\left(4Ax^3-6AxL^2\right)\left(\frac{1}{L^4}\right)e^{-x^2/L^2}$, $U=\left(\frac{\hbar^2}{2mL^2}\right)\left(\frac{4x^2}{L^2}-6\right)$.

(b) $U(x)$ is a parabola centered at $x=0$ with $U(0)=\frac{-3\hbar^2}{mL^2}<0$:



6-6

$$\psi(x)=A\cos kx+B\sin kx$$

$$\frac{\partial\psi}{\partial x}=-kA\sin kx+kB\cos kx$$

$$\frac{\partial^2\psi}{\partial x^2}=-k^2A\cos kx-k^2B\sin kx$$

$$\left(\frac{-2m}{\hbar^2}\right)(E-U)\psi=\left(\frac{-2mE}{\hbar^2}\right)(A\cos kx+B\sin kx)$$

The Schrödinger equation is satisfied if $\frac{\partial^2\psi}{\partial x^2}=\left(\frac{-2m}{\hbar^2}\right)(E-U)\psi$ or

$$-k^2(A\cos kx+B\sin kx)=\left(\frac{-2mE}{\hbar^2}\right)(A\cos kx+B\sin kx).$$

Therefore $E=\frac{\hbar^2k^2}{2m}$.

6-7 Since the particle is confined to the box, Δx can be no larger than L, the box length. With $n=0$, the particle energy $E_n=\frac{n^2h^2}{8mL^2}$ is also zero. Since the energy is all kinetic, this implies $\langle p_x^2\rangle=0$. But $\langle p_x\rangle=0$ is expected for a particle that spends equal time moving left as right, giving $\Delta p_x=\sqrt{\langle p_x^2\rangle-\langle p_x\rangle^2}=0$. Thus, for this case $\Delta p_x\Delta x=0$, in violation of the uncertainty principle.

6-8 $E=\frac{n^2h^2}{8mL^2}\rightarrow n^2=\frac{8mL^2E}{h^2}$. But $E=\frac{mv^2}{2}$ so $n^2=\frac{4m^2L^2v^2}{h^2}$ or $n=\frac{2mLv}{h}$. Now $v=0.10\ \text{nm/year}=3.17\times10^{-18}\ \text{m/s}$. So

$$n=2(0.005\ \text{kg})(0.2\ \text{m})\frac{3.17\times10^{-18}\ \text{m/s}}{6.63\times10^{-34}\ \text{Js}}=9.6\times10^{12}.$$

6-9 $E_n=\frac{n^2h^2}{8mL^2}$, so $\Delta E=E_2-E_1=\frac{3h^2}{8mL^2}$

$$\Delta E=(3)\frac{(1240\ \text{eV nm}/c)^2}{8(938.28\times10^6\ \text{eV}/c^2)(10^{-5}\ \text{nm})^2}=6.14\ \text{MeV}$$

$$\lambda=\frac{hc}{\Delta E}=\frac{1240\ \text{eV nm}}{6.14\times10^6\ \text{eV}}=2.02\times10^{-4}\ \text{nm}$$

This is the gamma ray region of the electromagnetic spectrum.

6-10 $E_n=\frac{n^2h^2}{8mL^2}$

$$\frac{h^2}{8mL^2}=\frac{(6.63\times10^{-34}\ \text{Js})^2}{8(9.11\times10^{-31}\ \text{kg})(10^{-10}\ \text{m})^2}=6.03\times10^{-18}\ \text{J}=37.7\ \text{eV}$$

(a) $E_1=37.7\ \text{eV}$

$E_2=37.7\times2^2=151\ \text{eV}$

$E_3=37.7\times3^2=339\ \text{eV}$

$E_4=37.7\times4^2=603\ \text{eV}$

(b) $hf=\frac{hc}{\lambda}=E_{n_i}-E_{n_f}$

$\lambda=\frac{hc}{E_{n_i}-E_{n_f}}=\frac{1\,240\ \text{eV}\cdot\text{nm}}{E_{n_i}-E_{n_f}}$

For $n_i=4$, $n_f=1$, $E_{n_i}-E_{n_f}=603\ \text{eV}-37.7\ \text{eV}=565\ \text{eV}$, $\lambda=2.19\ \text{nm}$

$n_i=4$, $n_f=2$, $\lambda=2.75\ \text{nm}$

$n_i=4$, $n_f=3$, $\lambda=4.70\ \text{nm}$

$n_i=3$, $n_f=1$, $\lambda=4.12\ \text{nm}$

$n_i=3$, $n_f=2$, $\lambda=6.59\ \text{nm}$

$n_i=2$, $n_f=1$, $\lambda=10.9\ \text{nm}$

6-11 In the present case, the box is displaced from $(0, L)$ by $\frac{L}{2}$. Accordingly, we may obtain the wavefunctions by replacing x with $x-\frac{L}{2}$ in the wavefunctions of Equation 6.18. Using

$$\sin\left[\left(\frac{n\pi}{L}\right)\left(x-\frac{L}{2}\right)\right]=\sin\left[\left(\frac{n\pi x}{L}\right)-\frac{n\pi}{2}\right]=\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi}{2}\right)-\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi}{2}\right)$$

we get for $-\frac{L}{2} \le x \le \frac{L}{2}$

$$\psi_1(x) = \left(\frac{2}{L}\right)^{1/2} \cos\left(\frac{\pi x}{L}\right);\ P_1(x) = \left(\frac{2}{L}\right)\cos^2\left(\frac{\pi x}{L}\right)$$
$$\psi_2(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{2\pi x}{L}\right);\ P_2(x) = \left(\frac{2}{L}\right)\sin^2\left(\frac{2\pi x}{L}\right)$$
$$\psi_3(x) = \left(\frac{2}{L}\right)^{1/2} \cos\left(\frac{3\pi x}{L}\right);\ P_3(x) = \left(\frac{2}{L}\right)\cos^2\left(\frac{3\pi x}{L}\right)$$

6-12 $\Delta E = \frac{hc}{\lambda} = \left(\frac{h^2}{8mL^2}\right)\left[2^2 - 1^2\right]$ and $L = \left[\frac{(3/8)h\lambda}{mc}\right]^{1/2} = 7.93\times10^{-10}\ \text{m} = 7.93\ \text{Å}$.

6-13 (a) Proton in a box of width $L = 0.200\ \text{nm} = 2\times10^{-10}\ \text{m}$

$$E_1 = \frac{h^2}{8m_pL^2} = \frac{\left(6.626\times10^{-34}\ \text{J}\cdot\text{s}\right)^2}{8\left(1.67\times10^{-27}\ \text{kg}\right)\left(2\times10^{-10}\ \text{m}\right)^2} = 8.22\times10^{-22}\ \text{J}$$
$$= \frac{8.22\times10^{-22}\ \text{J}}{1.60\times10^{-19}\ \text{J/eV}} = 5.13\times10^{-3}\ \text{eV}$$

(b) Electron in the same box:

$$E_1 = \frac{h^2}{8m_eL^2} = \frac{\left(6.626\times10^{-34}\ \text{J}\cdot\text{s}\right)^2}{8\left(9.11\times10^{-31}\ \text{kg}\right)\left(2\times10^{-10}\ \text{m}\right)^2} = 1.506\times10^{-18}\ \text{J} = 9.40\ \text{eV}.$$

(c) The electron has a much higher energy because it is much less massive.

6-14 (a) Still, $\frac{n\lambda}{2} = L$ so $p = \frac{h}{\lambda} = \frac{nh}{2L}$

$$K = \left[c^2p^2 + \left(mc^2\right)^2\right]^{1/2} - \left(mc^2\right) = E - mc^2$$
$$E_n = \left[\left(\frac{nhc}{2L}\right)^2 + \left(mc^2\right)^2\right]^{1/2},$$
$$K_n = \left[\left(\frac{nhc}{2L}\right)^2 + \left(mc^2\right)^2\right]^{1/2} - mc^2$$

(b) Taking $L = 10^{-12}$ m, $m = 9.11\times10^{-31}$ kg, and $n = 1$ we find $K_1 = 4.69\times10^{-14}$ J. The nonrelativistic result is

$$E_1 = \frac{h^2}{8mL^2} = \frac{\left(6.63\times10^{-34}\ \text{J}\cdot\text{s}\right)^2}{8\left(9.11\times10^{-31}\ \text{kg}\right)\left(10^{-24}\ \text{m}^2\right)} = 6.03\times10^{-14}\ \text{J}$$

Comparing this with K_1, we see that this value is too big by 29%.

6-15 (a) $$U=\left(\frac{e^2}{4\pi\varepsilon_0 d}\right)\left[-1+\frac{1}{2}-\frac{1}{3}+\left(-1+\frac{1}{2}\right)+(-1)\right]=\frac{(-7/3)e^2}{4\pi\varepsilon_0 d}=\frac{(-7/3)ke^2}{d}$$

(b) $$K=2E_1=\frac{2h^2}{8m\times 9d^2}=\frac{h^2}{36md^2}$$

(c) $E=U+K$ and $\frac{dE}{dd}=0$ for a minimum $\left[\frac{(+7/3)e^2k}{d^2}\right]-\frac{h^2}{18md^3}=0$

$$d=\frac{3h^2}{(7)(18ke^2m)} \text{ or } d=\frac{h^2}{42mke^2}$$

$$d=\frac{(6.63\times10^{-34}\ \mathrm{J\cdot s})^2}{(42)(9.11\times10^{-31}\ \mathrm{kg})(9\times10^9\ \mathrm{N\cdot m^2\cdot C^{-2}})(1.6\times10^{-19}\ \mathrm{C})^2}=0.5\times10^{-10}\ \mathrm{m}=0.050\ \mathrm{nm}$$

(d) Since the lithium spacing is a, where $Na^3=V$ and the density is $\frac{Nm}{V}$ where m is the mass of one atom, we get

$$a=\left(\frac{Vm}{Nm}\right)^{1/3}=\left(\frac{m}{\text{density}}\right)^{1/3}=\left(1.66\times10^{-27}\ \mathrm{kg}\times\frac{7}{530\ \mathrm{kg/m^3}}\right)^{1/3}\mathrm{m}=2.8\times10^{-10}\ \mathrm{m}$$
$$=0.28\ \mathrm{nm}$$

(2.8 times larger than $2d$)

6-16 (a) $\psi(x)=A\sin\left(\frac{\pi x}{L}\right)$, $L=3$ Å. Normalization requires

$$1=\int_0^L|\psi|^2dx=\int_0^L A^2\sin^2\left(\frac{\pi x}{L}\right)dx=\frac{LA^2}{2}$$

so $A=\left(\frac{2}{L}\right)^{1/2}$

$$P=\int_0^{L/3}|\psi|^2dx=\left(\frac{2}{L}\right)\int_0^{L/3}\sin^2\left(\frac{\pi x}{L}\right)dx=\frac{2}{\pi}\int_0^{\pi/3}\sin^2\phi\,d\phi=\frac{2}{\pi}\left[\frac{\pi}{6}-\frac{(3)^{1/2}}{8}\right]=0.195\,5.$$

(b) $\psi=A\sin\left(\frac{100\pi x}{L}\right)$, $A=\left(\frac{2}{L}\right)^{1/2}$

$$P=\frac{2}{L}\int_0^{L/3}\sin^2\left(\frac{100\pi x}{L}\right)dx=\frac{2}{L}\left(\frac{L}{100\pi}\right)\int_0^{100\pi/3}\sin^2\phi\,d\phi=\frac{1}{50\pi}\left[\frac{100\pi}{6}-\frac{1}{4}\sin\left(\frac{200\pi}{3}\right)\right]$$
$$=\frac{1}{3}-\left[\frac{1}{200\pi}\right]\sin\left(\frac{2\pi}{3}\right)=\frac{1}{3}-\frac{\sqrt{3}}{400\pi}=0.331\,9$$

(c) Yes: For large quantum numbers the probability approaches $\frac{1}{3}$.

Likewise, $P(x)$ is a *minimum* when $\frac{n\pi x}{L} = 0, \pi, 2\pi, 3\pi, \ldots = m\pi$ or when

$$x = \frac{Lm}{n} \qquad m = 0, 1, 2, 3, \ldots, n$$

6-19 The allowed energies for this system are given by Equation 6.17, or $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2}$. Using $E_n = 10^{-3}$ J, $m = 10^{-3}$ kg, $L = 10^{-2}$ m and solving for n gives

$$n = \frac{\left\{8(10^{-3}\ \text{kg})(10^{-2}\ \text{m})^2(10^{-3}\ \text{J})\right\}^{1/2}}{6.63\times10^{-34}\ \text{J}\cdot\text{s}} = 4.27\times10^{28}.$$

The excitation energy is $\Delta E = E_{n+1} - E_n$, or

$$\Delta E = \frac{h^2}{8mL^2}\left\{(n+1)^2 - n^2\right\} = \left(\frac{h^2}{8mL^2}\right)\{2n+1\} = E_n\left(\frac{2n+1}{n^2}\right) \approx \frac{2}{n}E_n \text{ for } n >> 1.$$

Thus, $\Delta E \approx \frac{(2)(10^{-3}\ \text{J})}{4.27\times10^{28}} = 4.69\times10^{-32}$ J.

6-20 The Schrödinger equation, after rearrangement, is $\frac{d^2\psi}{dx^2} = \left(\frac{2m}{\hbar^2}\right)\{U(x) - E\}\psi(x)$. In the well interior, $U(x) = 0$ and solutions to this equation are $\sin kx$ and $\cos kx$, where $k^2 = \frac{2mE}{\hbar^2}$. The waves symmetric about the midpoint of the well ($x = 0$) are described by

$$\psi(x) = A\cos kx \qquad -L < x < +L$$

In the region outside the well, $U(x) = U$, and the independent solutions to the wave equation are $e^{\pm\alpha x}$ with $\alpha^2 = \left(\frac{2m}{\hbar^2}\right)(U - E)$.

(a) The growing exponentials must be discarded to keep the wave from diverging at infinity. Thus, the waves in the exterior region, which are symmetric about the midpoint of the well are given by

$$\psi(x) = Ce^{-\alpha|x|} \qquad x > L \text{ or } x < -L.$$

At $x = L$ continuity of ψ requires $A\cos kL = Ce^{-\alpha L}$. For the slope to be continuous here, we also must require $-Ak\sin kL = -Ce^{-\alpha L}$. Dividing the two equations gives the desired restriction on the allowed energies: $k\tan kL = \alpha$.

(b) The dependence on E (or k) is made more explicit by noting that $k^2 + \alpha^2 = \frac{2mU}{\hbar^2}$, which allows the energy condition to be written $k\tan kL = \left\{\frac{2mU}{\hbar^2} - k^2\right\}^{1/2}$. Multiplying by L, squaring the result, and using $\tan^2\theta + 1 = \sec^2\theta$ gives

$(kL)^2 \sec^2(kL) = \frac{2mUL^2}{\hbar^2}$ from which the desired form follows immediately,

$k \sec(kL) = \frac{\sqrt{2mU}}{\hbar}$. The ground state is the symmetric waveform having the lowest energy. For electrons in a well of height $U = 5$ eV and width $2L = 0.2$ nm, we calculate

$$\frac{2mUL^2}{\hbar^2} = \frac{(2)(511\times10^3 \text{ eV}/c^2)(5 \text{ eV})(0.1 \text{ nm})^2}{(197.3 \text{ eV}\cdot\text{nm}/c)^2} = 1.3127.$$

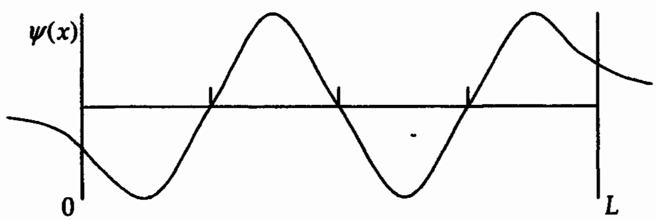
With this value, the equation for $\theta = kL$

$$\frac{\theta}{\cos\theta} = (1.3127)^{1/2} = 1.1457$$

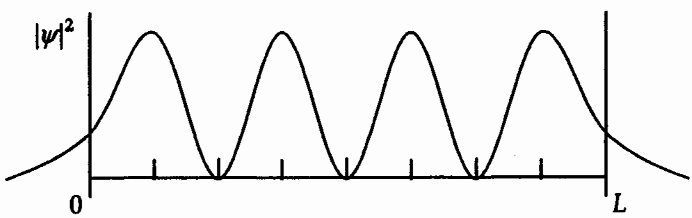
can be solved numerically employing methods of varying sophistication. The simplest of these is trial and error, which gives $\theta = 0.799$ From this, we find $k = 7.99 \text{ nm}^{-1}$, and an energy

$$E = \frac{\hbar^2 k^2}{2m} = \frac{(197.3 \text{ eV}\cdot\text{nm}/c)^2 (7.99 \text{ nm}^{-1})^2}{2(511\times10^3 \text{ eV}/c^2)} = 2.432 \text{ eV}.$$

6-21 $n = 4$



$n = 4$



Note that the $n = 4$ wavefunction has three nodes and is antisymmetric about the midpoint of the well.

6-22 See Multimedia Manager

6-23 Inside the well, the particle is free and the Schrödinger waveform is trigonometric with wavenumber $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$:

$$\psi(x) = A\sin kx + B\cos kx \quad 0 \le x \le L.$$

The infinite wall at $x=0$ requires $\psi(0)=B=0$. Beyond $x=L$, $U(x)=U$ and the Schrödinger equation $\frac{d^2\psi}{dx^2}=\left(\frac{2m}{\hbar^2}\right)\{U-E\}\psi(x)$, which has exponential solutions for $E<U$

$$\psi(x)=Ce^{-\alpha x}+De^{+\alpha x}, \qquad x>L$$

where $\alpha=\left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. To keep ψ bounded at $x=\infty$ we must take $D=0$. At $x=L$, continuity of ψ and $\frac{d\psi}{dx}$ demands

$$A\sin kL=Ce^{-\alpha L}$$
$$kA\cos kL=-\alpha Ce^{-\alpha L}$$

Dividing one by the other gives an equation for the allowed particle energies: $k\cot kL=-\alpha$. The dependence on E (or k) is made more explicit by noting that $k^2+\alpha^2=\frac{2mU}{\hbar^2}$, which allows the energy condition to be written $k\cot kL=-\left[\left(\frac{2mU}{\hbar^2}\right)-k^2\right]^{1/2}$. Multiplying by L, squaring the result, and using $\cot^2\theta+1=\csc^2\theta$ gives $(kL)^2\csc^2(kL)=\frac{2mUL^2}{\hbar^2}$ from which we obtain $\frac{kL}{\sin kL}=\left(\frac{2mUL^2}{\hbar^2}\right)^{1/2}$. Since $\frac{\theta}{\sin\theta}$ is never smaller than unity for *any* value of θ, there can be no bound state energies if $\frac{2mUL^2}{\hbar^2}<1$.

6-24 After rearrangement, the Schrödinger equation is $\frac{d^2\psi}{dx^2}=\left(\frac{2m}{\hbar^2}\right)\{U(x)-E\}\psi(x)$ with $U(x)=\frac{1}{2}m\omega^2x^2$ for the quantum oscillator. Differentiating $\psi(x)=Cxe^{-\alpha x^2}$ gives

$$\frac{d\psi}{dx}=-2\alpha x\psi(x)+C^{-\alpha x^2}$$

and

$$\frac{d^2\psi}{dx^2}=-\frac{2\alpha x d\psi}{dx}-2\alpha\psi(x)-(2\alpha x)Ce^{-\alpha x^2}=(2\alpha x)^2\psi(x)-6\alpha\psi(x).$$

Therefore, for $\psi(x)$ to be a solution requires $(2\alpha x)^2-6\alpha=\frac{2m}{\hbar^2}\{U(x)-E\}=\left(\frac{m\omega}{\hbar}\right)^2x^2-\frac{2mE}{\hbar^2}$. Equating coefficients of like terms gives $2\alpha=\frac{m\omega}{\hbar}$ and $6\alpha=\frac{2mE}{\hbar^2}$. Thus, $\alpha=\frac{m\omega}{2\hbar}$ and $E=\frac{3\alpha\hbar^2}{m}=\frac{3}{2}\hbar\omega$. The normalization integral is $1=\int_{-\infty}^{\infty}|\psi(x)|^2dx=2C^2\int x^2e^{-2\alpha x^2}dx$ where the second step follows from the symmetry of the integrand about $x=0$. Identifying a with 2α in the integral of Problem 6-32 gives $1=2C^2\left(\frac{1}{8\alpha}\right)\left(\frac{\pi}{2\alpha}\right)^{1/2}$ or $C=\left(\frac{32\alpha^3}{\pi}\right)^{1/4}$.

6-25 At its limits of vibration $x = \pm A$ the classical oscillator has all its energy in potential form: $E = \frac{1}{2}m\omega^2 A^2$ or $A = \left(\frac{2E}{m\omega^2}\right)^{1/2}$. If the energy is quantized as $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, then the corresponding amplitudes are $A_n = \left[\frac{(2n+1)\hbar}{m\omega}\right]^{1/2}$.

6-26 $P_c(x)dx$ is proportional to the time that the particle spends in the interval dx. This time dt is inversely related to its speed v as $dt = \frac{dx}{v}$, so that $P_c(x)dx = Cdt$ or $P_c(x) = \frac{C}{v}$. But the speed of the oscillator varies with its position in such a way as to keep the total energy constant:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 x^2 \qquad \text{or} \qquad v^2 = \frac{2E}{m} - \omega^2 x^2.$$

Writing E in terms of the classical amplitude as $E = \frac{1}{2}m\omega^2 A^2$ gives $v = \omega\left(A^2 - x^2\right)^{1/2}$ and $P_c(x) = \frac{C}{\omega}\left(A^2 - x^2\right)^{-1/2}$. The constant C is a normalizing factor chosen to ensure a total probability of one:

$$1 = \int_{-A}^{A} P_c(x)dx = \frac{C}{\omega}\int_{-A}^{A}\left(A^2 - x^2\right)^{-1/2}dx.$$

The integral is evaluated with the trigonometric substitution $x = A\sin\theta$ (so that $dx = A\cos\theta d\theta$) to get $1 = \frac{C}{\omega}\int_{-\pi/2}^{\pi/2} d\theta = \frac{\pi C}{\omega}$. Thus, $\frac{C}{\omega}$ is just $\frac{1}{\pi}$ and $P_c(x) = \frac{1/\pi}{\left(A^2 - x^2\right)^{1/2}}$ for a classical oscillator with amplitude of vibration equal to A.

6-27 See Multimedia Manager

6-28 A particle within the well is subject to no forces and, hence, moves with uniform speed, spending equal time in all parts of the well. Thus, for such a particle the probability density is *uniform*. That is, $P_c(x) = \text{constant}$. The constant is fixed by requiring the integrated probability to be unity, that is, $1 = \int_0^L P_c(x)dx = CL$ or $C = \frac{1}{L}$. To find $\langle x\rangle$ we weight the possible particle positions according to the probability density P_c to get $\langle x\rangle = \int_0^L xP_c(x)dx = \frac{1}{L}\left(\frac{x^2}{2}\right)\Big|_0^L = \frac{L}{2}$. Similarly, $\langle x^2\rangle$ is found by weighting the possible values of x^2 with P_c:

$$\langle x^2\rangle = \int_0^L x^2 P_c(x)dx = \frac{1}{L}\left(\frac{x^3}{3}\right)\Big|_0^L = \frac{L^2}{3}.$$

The classical and quantum results for $\langle x\rangle$ agree exactly; for $\langle x^2\rangle$ the quantum prediction is smaller by an amount $\frac{L^2}{2(n\pi)^2}$ which, however, goes to zero in the limit of large quantum numbers n, where classical and quantum results must coincide (correspondence principle).

$$\int_0^\pi \theta^2 \cos 2n\theta\, d\theta = -\frac{1}{n}\int_0^\pi \theta \sin 2n\theta\, d\theta = \left(\frac{1}{2n^2}\right)\theta \cos 2n\theta\Big|_0^\pi = \frac{\pi}{2n^2}.$$

Then $\left\langle x^2 \right\rangle = \frac{L^2}{\pi^3}\left\{\frac{\pi^3}{3} - \frac{\pi}{2n^2}\right\} = \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2}$.

6-31 The symmetry of $|\psi(x)|^2$ about $x = 0$ can be exploited effectively in the calculation of average values. To find $\langle x \rangle$

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx$$

We notice that the integrand is antisymmetric about $x = 0$ due to the extra factor of x (an odd function). Thus, the contribution from the two half-axes $x > 0$ and $x < 0$ cancel exactly, leaving $\langle x \rangle = 0$. For the calculation of $\left\langle x^2 \right\rangle$, however, the integrand is symmetric and the half-axes contribute equally to the value of the integral, giving

This equation is for <x^2>, not <x> as they have written.

$$\langle x \rangle = \int_0^\infty x^2 |\psi|^2 dx = 2C^2 \int_0^\infty x^2 e^{-2x/x_0} dx.$$

Two integrations by parts show the value of the integral to be $2\left(\frac{x_0}{2}\right)^3$. Upon substituting for C^2, we get $\left\langle x^2 \right\rangle = 2\left(\frac{1}{x_0}\right)(2)\left(\frac{x_0}{2}\right)^3 = \frac{x_0^2}{2}$ and $\Delta x = \left(\left\langle x^2 \right\rangle - \langle x \rangle^2\right)^{1/2} = \left(\frac{x_0^2}{2}\right)^{1/2} = \frac{x_0}{\sqrt{2}}$. In calculating the probability for the interval $-\Delta x$ to $+\Delta x$ we appeal to symmetry once again to write

$$P = \int_{-\Delta x}^{+\Delta x} |\psi|^2 dx = 2C^2 \int_0^{\Delta x} e^{-2x/x_0} dx = -2C^2\left(\frac{x_0}{2}\right)e^{-2x/x_0}\Big|_0^{\Delta x} = 1 - e^{-\sqrt{2}} = 0.757$$

or about 75.7% independent of x_0.

6-32 The probability density for this case is $|\psi_0(x)|^2 = C_0^2 e^{-ax^2}$ with $C_0 = \left(\frac{a}{\pi}\right)^{1/4}$ and $a = \frac{m\omega}{\hbar}$. For the calculation of the average position $\langle x \rangle = \int_{-\infty}^{\infty} x|\psi_0(x)|^2 dx$ we note that the integrand is an odd function, so that the integral over the negative half-axis $x < 0$ exactly cancels that over the positive half-axis ($x > 0$), leaving $\langle x \rangle = 0$. For the calculation of $\left\langle x^2 \right\rangle$, however, the integrand $x^2|\psi_0|^2$ is symmetric, and the two half-axes contribute equally, giving

$$\left\langle x^2 \right\rangle = 2C_0^2 \int_0^\infty x^2 e^{-ax^2} dx = 2C_0^2\left(\frac{1}{4a}\right)\left(\frac{\pi}{a}\right)^{1/2}.$$

Substituting for C_0 and a gives $\left\langle x^2 \right\rangle = \frac{1}{2a} = \frac{\hbar}{2m\omega}$ and $\Delta x = \left(\left\langle x^2 \right\rangle - \langle x \rangle^2\right)^{1/2} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$.

6-37 (a) Normalization requires

$$1=\int_{-\infty}^{\infty}|\Psi|^2dx=C^2\int_{-\infty}^{\infty}\{\psi_1^*+\psi_2^*\}\{\psi_1+\psi_2\}dx$$
$$=C^2\left\{\int|\psi_1|^2dx+\int|\psi_2|^2dx+\int\psi_2^*\psi_1dx+\int\psi_1^*\psi_2dx\right\}$$

The first two integrals on the right are unity, while the last two are, in fact, the same integral since ψ_1 and ψ_2 are both real. Using the waveforms for the infinite square well, we find

$$\int\psi_2\psi_1dx=\frac{2}{L}\int_0^L\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)dx=\frac{1}{L}\int_0^L\left\{\cos\left(\frac{\pi x}{L}\right)-\cos\left(\frac{3\pi x}{L}\right)\right\}dx$$

where, in writing the last line, we have used the trigonometric exponential identities of sine and cosine. Both of the integrals remaining are readily evaluated, and are zero. Thus, $1=C^2\{1+0+0+0\}=2C^2$, or $C=\frac{1}{\sqrt{2}}$. Since $\psi_{1,2}$ are stationary states, they develop in time according to their respective energies $E_{1,2}$ as $e^{-iEt/\hbar}$. Then $\Psi(x,\,t)=C\left\{\psi_1e^{-iE_1t/\hbar}+\psi_2e^{-iE_2t/\hbar}\right\}$.

(c) $\Psi(x,\,t)$ is a stationary state only if it is an eigenfunction of the energy operator $[E]=i\hbar\frac{\partial}{\partial t}$. Applying $[E]$ to Ψ gives

$$[E]\Psi=C\left\{i\hbar\left(\frac{-iE_1}{\hbar}\right)\psi_1e^{-iE_1t/\hbar}+i\hbar\left(\frac{-iE_2}{\hbar}\right)\psi_2e^{-iE_2t/\hbar}\right\}=C\left\{E_1\psi_1e^{-iE_1t/\hbar}+E_2\psi_2e^{-iE_2t/\hbar}\right\}.$$

Since $E_1\neq E_2$, the operations $[E]$ does *not* return a multiple of the wavefunction, and so Ψ is not a stationary state. Nonetheless, we may calculate the average energy for this state as

$$\langle E\rangle=\int\Psi^*[E]\Psi dx=C^2\int\left\{\psi_1^*e^{+iE_1t/\hbar}+\psi_2^*e^{+iE_2t/\hbar}\right\}\left\{E_1\psi_1e^{-iE_1t/\hbar}+E_2\psi_2e^{-iE_2t/\hbar}\right\}dx$$
$$=C^2\left\{E_1\int|\psi_1|^2dx+E_2\int|\psi_2|^2dx\right\}$$

with the cross terms vanishing as in part (a). Since $\psi_{1,2}$ are normalized and $C^2=\frac{1}{2}$ we get finally $\langle E\rangle=\frac{E_1+E_2}{2}$.

6-38 The average position at any instant is given by

$$\langle x\rangle=\int_{-\infty}^{\infty}x|\Psi|^2dx=C^2\int_{-\infty}^{\infty}x\left\{\psi_1^*e^{+iE_1t/\hbar}+\psi_2^*e^{+iE_2t/\hbar}\right\}\left\{\psi_1e^{-iE_1t/\hbar}+\psi_2e^{-iE_2t/\hbar}\right\}dx$$
$$=C^2\left\{\int_{-\infty}^{\infty}x|\psi_1|^2dx+\int_{-\infty}^{\infty}x|\psi_2|^2dx+e^{-i\Omega t}\int_{-\infty}^{\infty}x\psi_1^*\psi_2dx+e^{+i\Omega t}\int_{-\infty}^{\infty}x\psi_2^*\psi_1dx\right\}$$