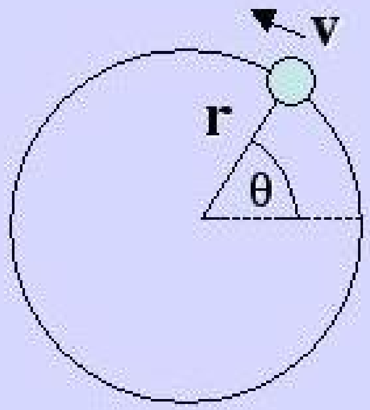
Circular Motion 2

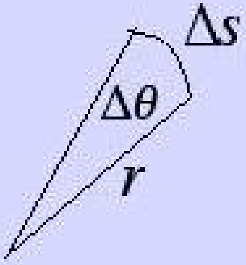
Lecture 5

Uniform Circular Motion



A body moving in a circle of radius r
with uniform speed v

In a time interval Δt the arc length traversed is



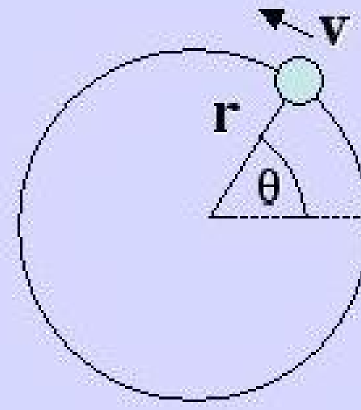
$\Delta s = r\Delta\theta$ θ in radians $0 \rightarrow 2\pi$

The limit as that time interval goes to zero is v

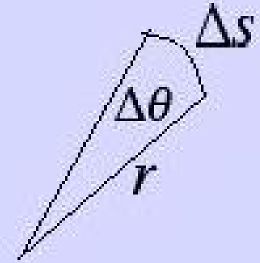
$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = r\,\frac{d\theta}{dt}$$

$\dfrac{d\theta}{dt}$ is called the angular velocity ω where $v = r\omega$

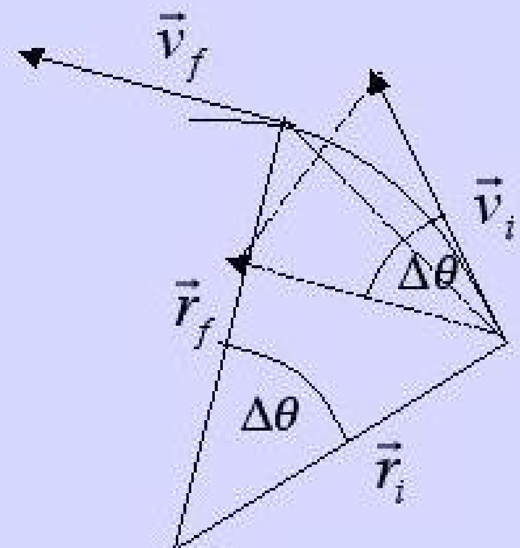
Instantaneous Acceleration



The direction of v is always tangent to the circle. So, in time Δt the vector v changes direction by the angle between v_i and v_f.



The angle made by r_i and r_f is equal to the angle made by v_i and v_f because the two triangles defined by the vectors are formally similar.



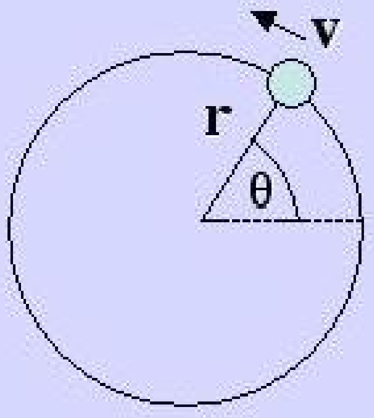
Thus the ratio of corresponding sides are equal:

$$\frac{\Delta\vec{v}}{|v_i|} = \frac{\Delta s}{r}$$

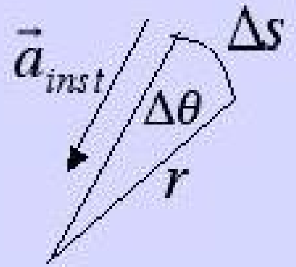
Giving an Average Acceleration:

$$a_{ave} = \frac{\Delta v}{\Delta t} = \frac{|v_i|}{r}\frac{\Delta s}{\Delta t}$$

Instantaneous Acceleration



In the same way as the limit as the time interval vanished defined the linear velocity and acceleration, and the angular velocity, the instantaneous acceleration is defined by this limit.

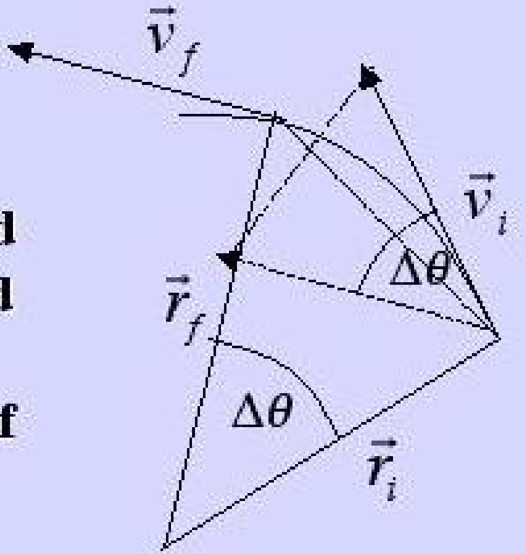


$$a_{ave} = \frac{|v_i|}{r}\frac{\Delta s}{\Delta t}$$

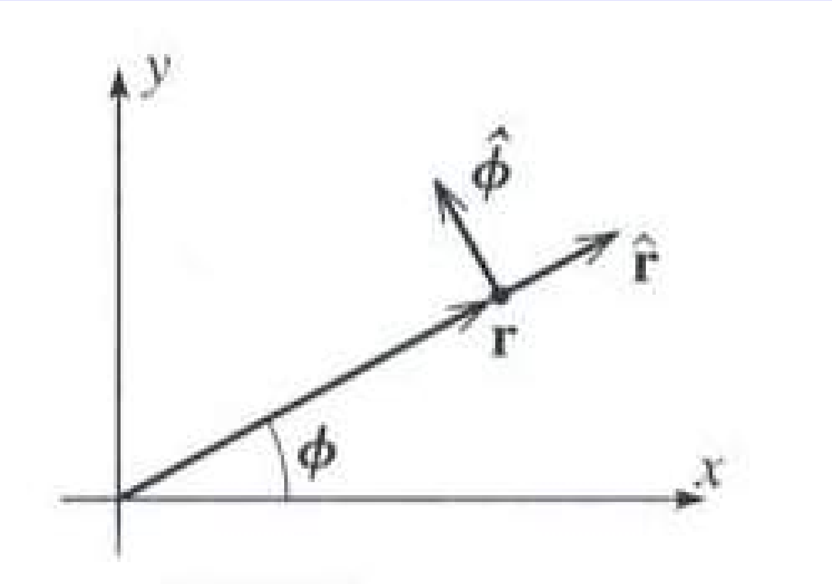
$$a_{inst} = \lim_{\Delta t \to 0} a_{ave}$$

$$a_{inst} = \frac{|v_i|}{r}\lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t} = \boxed{\frac{v^2}{r}}$$

a_{inst} is always pointed along the radius, and points toward the center of the circle of motion.



Circular Motion and Acceleration



Consider the two unit vectors that are convenient for angular motion:

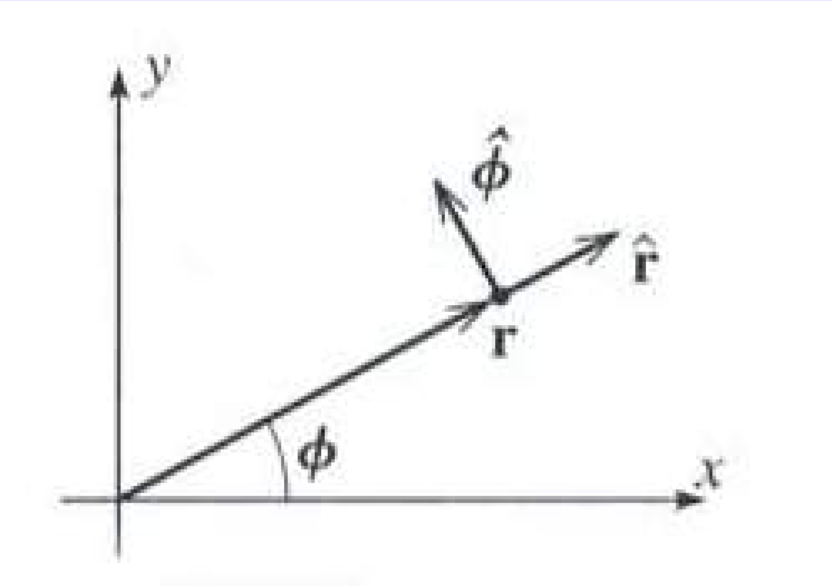
$$\hat{r} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

The position vector can now be written

$$\vec{r} = r\hat{r} = r\left(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}\right) = x\hat{i} + y\hat{j}$$

The velocity vector is simply the time derivative of the position vector. For circular motion the radius of curvature, *r*, is constant. However, $\phi = \phi(t)$ is time dependent. This means to find the velocity vector we must be careful when taking the derivative of the position vector – chain rule!

Circular Motion and Acceleration



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$$\hat{r} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

The velocity vector can now be expressed:

$$\vec{v} = \frac{d\vec{r}}{dt} = r\left(\frac{d\cos\phi}{dt}\hat{i} + \frac{d\sin\phi}{dt}\hat{j}\right)$$
$$\vec{v} = r\left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)\frac{d\phi}{dt} = \omega r\hat{\phi} = v\hat{\phi}$$

Taking the derivative of the velocity yields the acceleration. Assuming a **constant rate of rotation,** ω:

$$\vec{a} = \frac{d\vec{v}}{dt} = \omega r\left(-\frac{d\sin\phi}{dt}\hat{i} + \frac{d\cos\phi}{dt}\hat{j}\right)$$
$$\vec{a} = -\omega r\left(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}\right)\frac{d\phi}{dt} = -\omega^2 r\hat{r} = -\frac{v^2}{r}\hat{r}$$

The radial acceleration points radially **inward**! What happens when there is angular acceleration? Stay tuned!

Example: Space Shuttle Orbit

A space shuttle is in orbit at a height of 250km, where the Earth's gravitational acceleration is 93% of its surface value. What is the period of its orbit?

The radius for this circular orbit is $r = R_E + 250km$. The radial acceleration is due to the Earth gravitational pull. Hence:

$$.93g = \omega^2 r \rightarrow \omega = \sqrt{.93g/r}$$

$$\omega = 2\pi/T \rightarrow T = 2\pi\sqrt{r/.93g}$$

Since $r = 6.62x10^6\ m$ we have:

$$T = 5355\sec \;\; = 89.25\min$$

There is no choice here. The properties of low Earth orbits are determined by the size and mass of the Earth. As gravity weakens and the size of the orbits increases this period increases, e.g. the moon has a period of 27 days.

Example: Engineering a Road

A flat horizontal road is designed for an 80 km/h speed limit. If the maximum radial acceleration is $a_r = 1.5\ m/s^2$, what is the minimum radius for curves on his road?

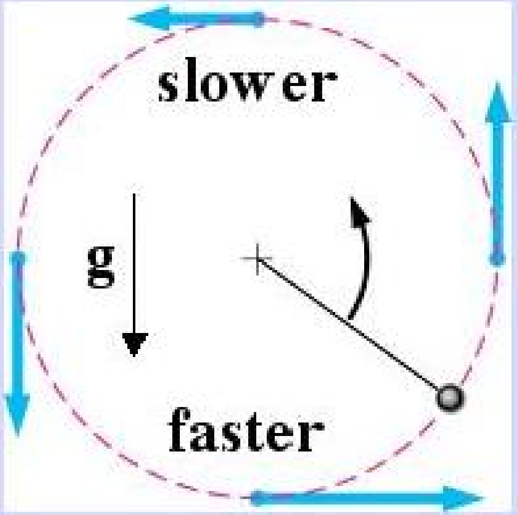
For this problem it is more convenient to use the expression that uses linear velocity versus angular velocity to find the radial acceleration.

$$a = \frac{v^2}{r} \rightarrow r = \frac{v^2}{a} = \frac{\left(80 \times 10^3/3600\right)^2}{1.5} = 329m$$

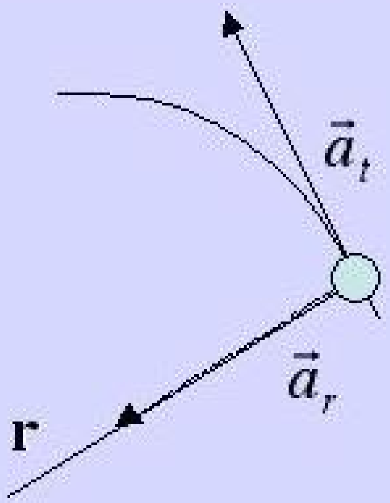
This radius is strongly dependent upon the amount of friction between the auto and the road (as we shall see in chapters 5 and 6). If friction cannot support this amount of acceleration then it will break free and slide off of the road.

Nonuniform Circular Motion

For a mass on a string moving vertically, gravity will add a component of acceleration in the vertical direction.



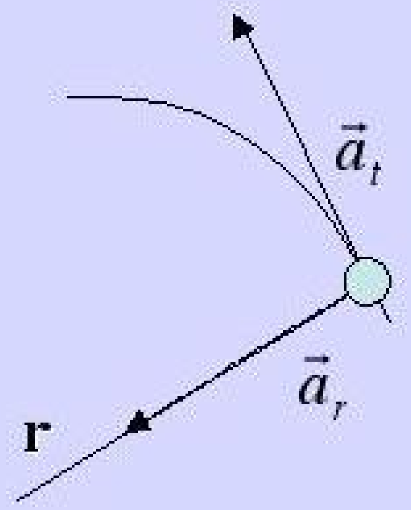
The ball will spin faster at the bottom than at the top.



Because the rotation is speeding up and slowing down along the circle, there must be a tangential acceleration in addition to the radial one.

Nonuniform Circular Motion

We can separate the acceleration vector into two parts. The radial (already discussed in Uniform Circular Motion), which changes the direction of the velocity, and the tangential, which changes the speed of rotation.



Using the limits we've defined previously, the tangential acceleration is simply:

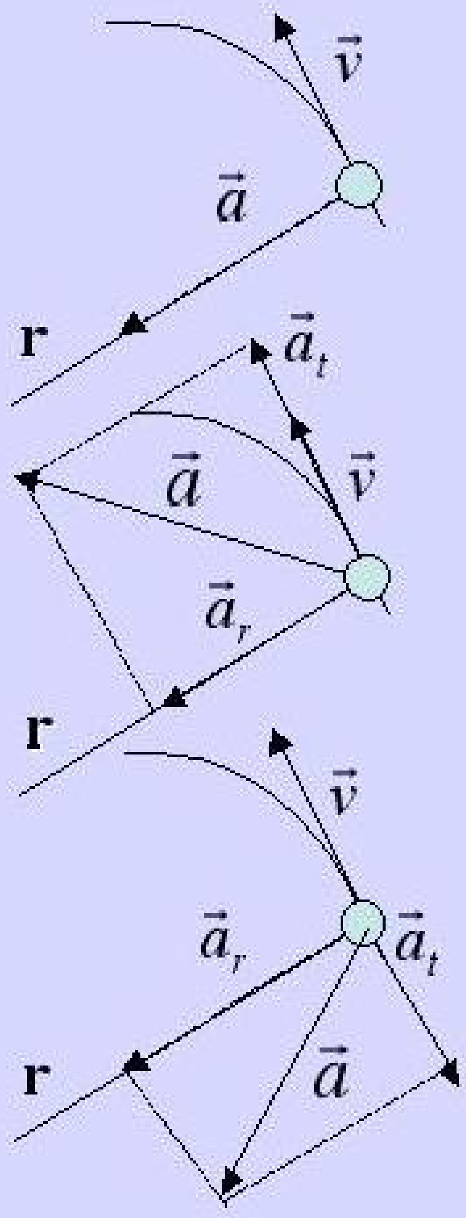
$$\boxed{a_t = \frac{dv}{dt}} \quad \text{while} \quad \boxed{a_r = \frac{v^2}{r}}$$

Nonuniform Circular Motion

When a body travels along a curved path with constant speed, $\vec{a}$ is $\perp$ to the path and $\perp$ to $\vec{v}$

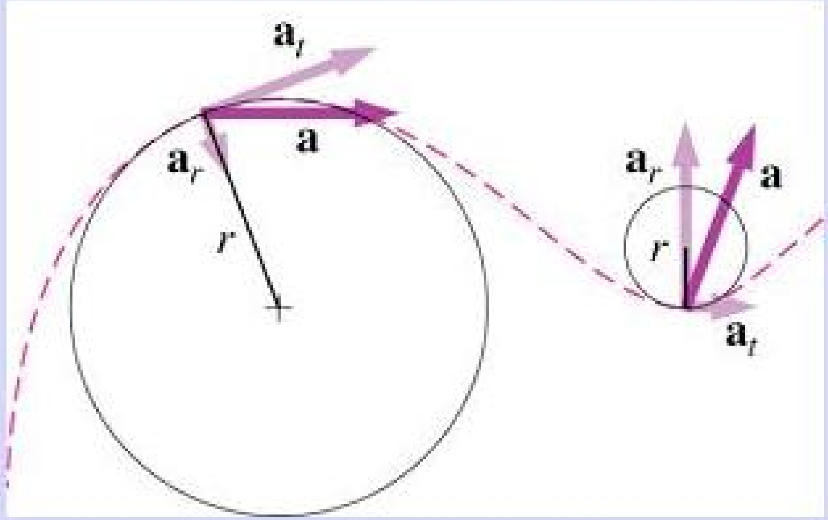
When the body travels along a curved path with **increasing** speed, $\vec{a}$ has both $\perp$ and $\parallel$ components, and points **ahead** of the radius to the path.

When the body travels along a curved path with **decreasing** speed, $\vec{a}$ has both $\perp$ and $\parallel$ components and points **behind** the radius to the path.



Nonuniform Circular Motion

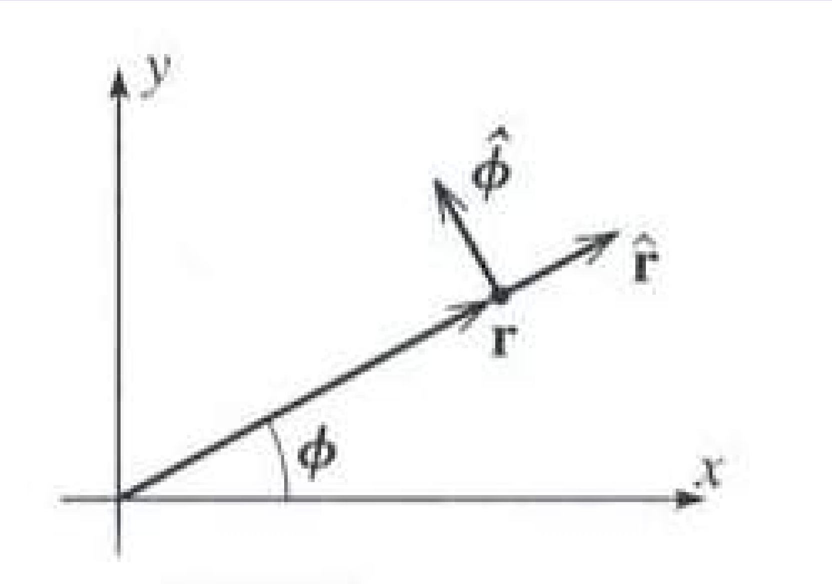
A body moving in any curved path has associated with it at any time t a **radius of curvature**, a tangential **angular velocity**, a **radial acceleration** and a **tangential acceleration**.



If the body has no tangential acceleration, and constant radial acceleration, it remains in Uniform Circular Motion.

For Nonuniform Circular Motion, the radius of curvature is constantly changing location, magnitude or both.

Circular Motion and Acceleration



Again consider the two unit vectors that are convenient for angular motion:

$$\hat{r} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

The position vector is written as:

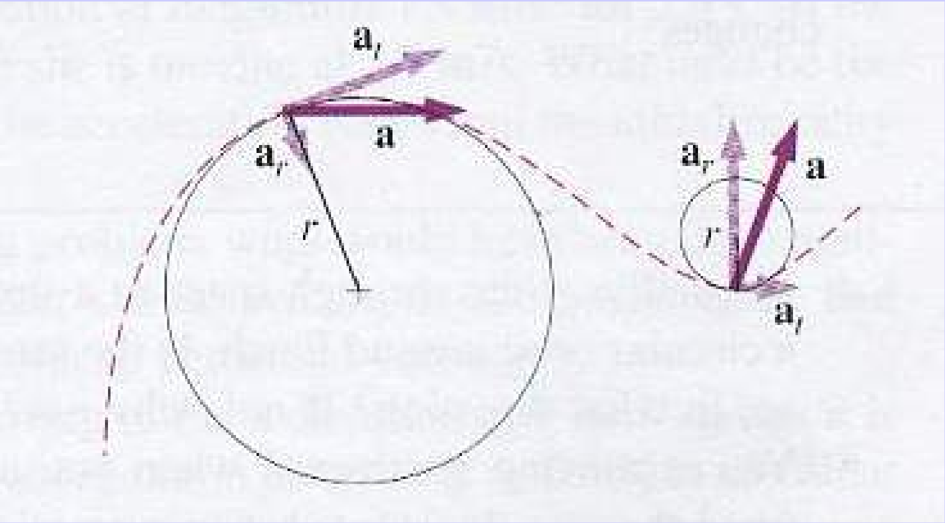
$$\vec{r} = r\hat{r} = r\left(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}\right) = x\hat{i} + y\hat{j}$$

Again we take the time derivative of the position vector assuming that the radius of curvature, *r*, is constant. Again, $\phi = \phi(t)$, is time dependent and velocity is given by:

$$\vec{v} = \omega r\left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right) = \omega r\hat{\phi} = v\hat{\phi}$$

Up to this point nothing has changed from our previous development, however now we will **not** assume that ω or equivalently v is a constant!

Circular Motion and Acceleration



Now we assume that $\omega = \omega(t)$ and $d\omega(t)/dt = \alpha$, the angular acceleration. The full acceleration now becomes:

$$\vec{a} = \frac{dv}{dt}\hat{\phi} - \omega^2 r\hat{r}$$

$$\vec{a} = a_t\hat{\phi} - \omega^2 r\hat{r} = a_t\hat{\phi} - \frac{v^2}{r}\hat{r}$$

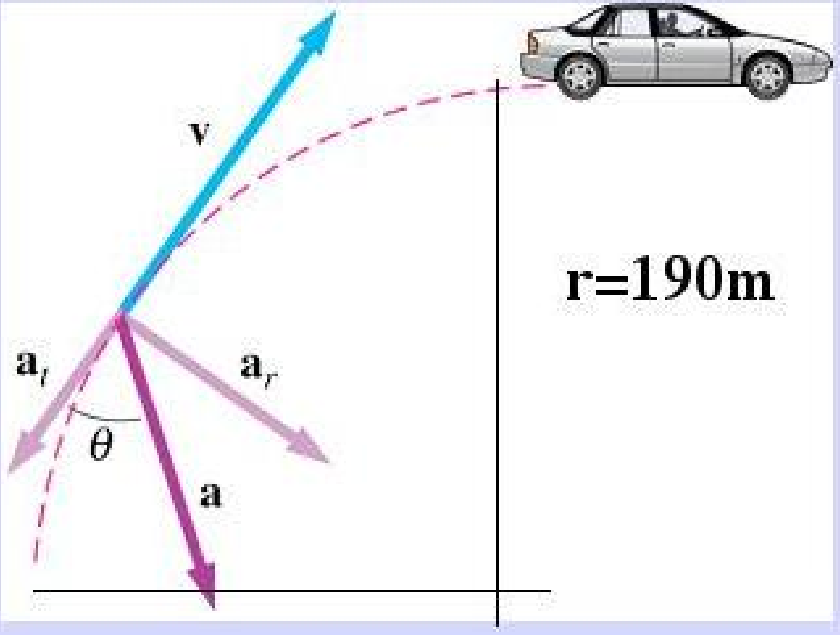
$$\vec{a} = a_t\hat{\phi} - a_r\hat{r}$$

This is the mathematical expression that includes both the tangential acceleration, $a_t\hat{\phi}$, and the radial acceleration, $-\omega^2 r\hat{r} = -\frac{v^2}{r}\hat{r}$.

When an object travels along a curved path its total acceleration is a vector sum of its radial acceleration and its tangential acceleration. As already pointed out if, a_t is positive the acceleration vector points ahead of the radius and if a_t is negative its acceleration vector points behind the radius.

Example: Nonuniform Circular Motion

4-10: A road makes a 90° bend with a radius of 190m. A car enters the bend moving at 20m/s. Finding this too fast, the driver decelerates at 0.92m/s². Determine the acceleration of the car when its speed rounding the bend has dropped to 15m/s.



The car has both radial and tangential accelerations. The tangential is given by the deceleration as a_t=0.92m/s². The radial acceleration when the speed is 15m/s:

$$a_r = \frac{v^2}{r} = \frac{(15m/s)^2}{190m} = 1.2m/s^2$$

Example: Nonuniform Circular Motion

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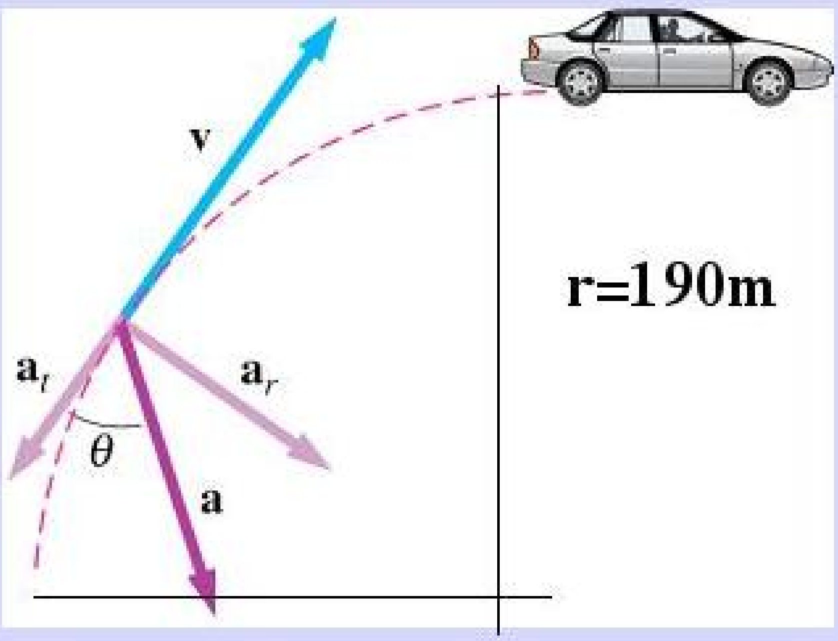


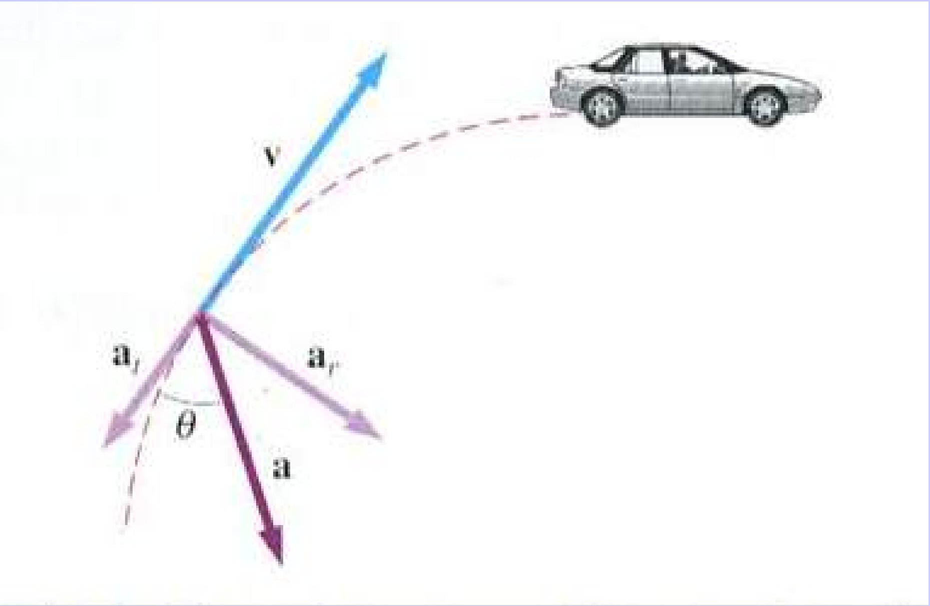
Summing the vector accelerations the magnitude is:

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{(1.2m/s^2) + (0.92m/s^2)} = \boxed{1.5m/s^2}$$

And it points an angle θ from tangential $\theta = \tan^{-1}\left(\frac{a_r}{a_t}\right) = \boxed{53^o}$

Example: Nonuniform Circular Motion

(a) When the car reaches 15m/s what arc length has it traversed? (b) Would the car stop on the turn if the driver continued to brake at this rate?



(a) The arc length that it traversed is:

$$s = \frac{v_f^2 - v_i^2}{2a} = \frac{15^2 - 20^2}{-2(.92)} = 95m$$

(b) For the final velocity to go to zero the arc length must be:

$$s = \frac{-v_i^2}{2a} = \frac{-20^2}{-2(.92)} = 217.4m$$

The angle the car traverses to come to a stop:

$$\phi = \frac{217.4}{190} = 1.14rad < \frac{\pi}{2}$$

So the car does come to a stop before turning 90°.

Example: Nonuniform Circular Motion

In the hammer throw a ball on the end of a *1.2 m* long wire is released from a height of 1.3 m while traveling at an angle of 24° above horizontal. If it travels *84 m* before landing, what was its radial acceleration upon release?

From the trajectory equation,

$$y = x\tan\theta - \frac{1}{2}\frac{g}{V_0^2\cos^2\theta}x^2$$

and the data given:

$$x = 84m,\ \theta = 24^\circ,$$
$$y = -1.3m$$

We can solve for $V_0{}^2$:

$$V_0^2 = \frac{1}{2\cos^2\theta}\frac{g}{x\tan\theta - y}x^2$$
$$V_0^2 = \frac{1}{2\cos^2 24^\circ}\frac{9.8}{84\tan 24^\circ + 1.3}(84)^2$$
$$V_0^2 = 1070.5(m/s)^2$$

Hence the radial acceleration is:

$$a_r = V_0^2/r = 1070.5/1.2 = 892m/s^2$$

Tomorrow

Lecture 6: Circular Motion Examples
Newton's Laws of Motion