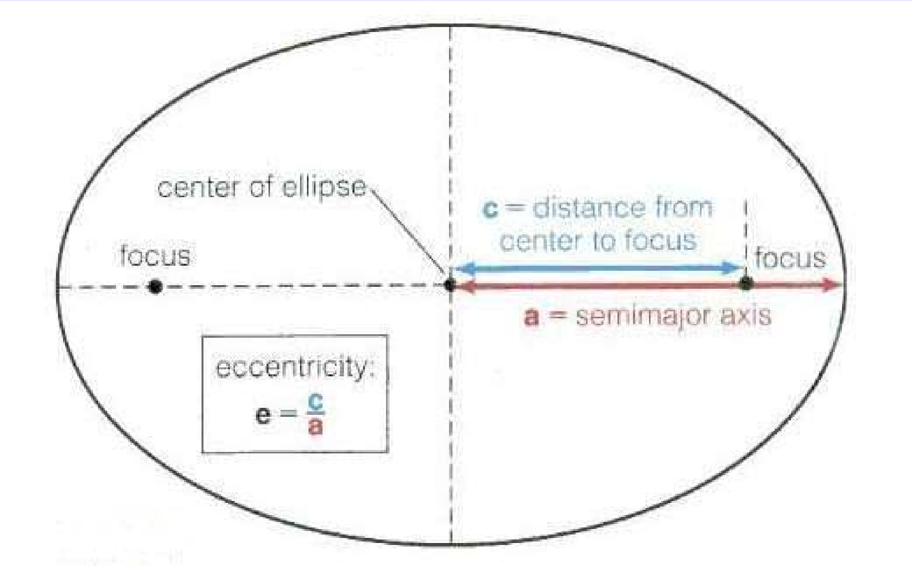
Chapter 9

Gravitation II

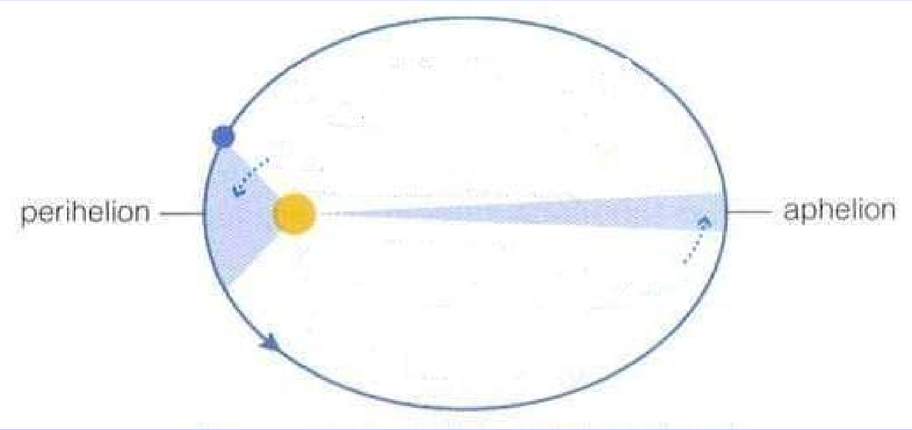
Kepler's Laws of Planetary Motion

Kepler's three laws make no effort to explain the planetary motion. Instead, they are mathematical descriptions of the planet's motion.



1. The planets orbit the Sun in ellipses with the Sun at one focus.

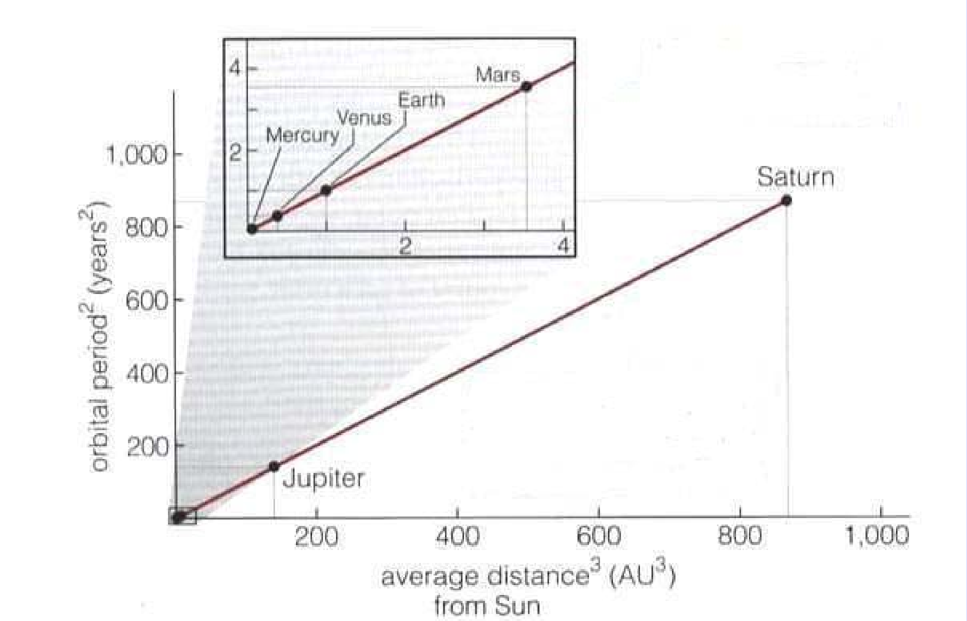
Discuss Keper's breakthrough with Brahe's data.



2. A line joining the Sun and a planet sweeps out equal areas in equal times.

This is a straightforward result of the conservation of angular momentum.

Kepler's Laws of Planetary Motion



3. The square of the planet's orbital period is proportional to the cube of the semimajor axis of its orbit.

In units of Earth years and Astronomical Units, the average distance from the Earth to the Sun, this law is expressed as $\mathbf{T^2 = \mathit{a}^3}$.

This final observation occurred several years after the first two.

It was Newton's prediction of these observations using his law of gravity that resulted in a basic understanding of orbital motion and (weak) gravity in general. In fact Kepler's third law (in SI units) is a straightforward extension of our knowledge of the angular velocity of an orbiting object.

$$\varpi^2 = \frac{4\pi^2}{T^2} = \frac{GM}{r_o^3} \rightarrow T^2 = \frac{4\pi^2}{GM} r_o^3$$

Newton's Law of Gravity

Newton realized that the motion of the falling apple and the motion of the moon around the Earth were due to the same force. They were both falling toward the Earth due to the force of gravity.

$$F_g = -\frac{GMm}{r^2}$$ Universal Gravitation

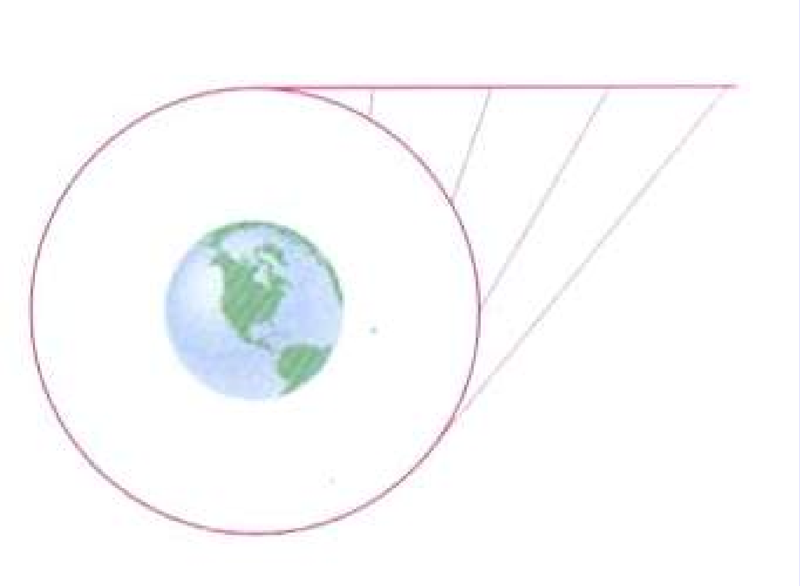
This force obeys the inverse square law. Also the minus sign indicates that this force is attractive.

G is the universal constant of gravitational attraction and is given by

$$\boldsymbol{G = 6.673 \times 10^{-11} Nm^2/kg^2}$$

Strictly speaking it only applies to point objects. However, for spherically symmetrical objects ***r*** is the distance between their centers. As long as the size of the object is small compared to ***r***, then it is simply the distance between them.

Orbital Motion



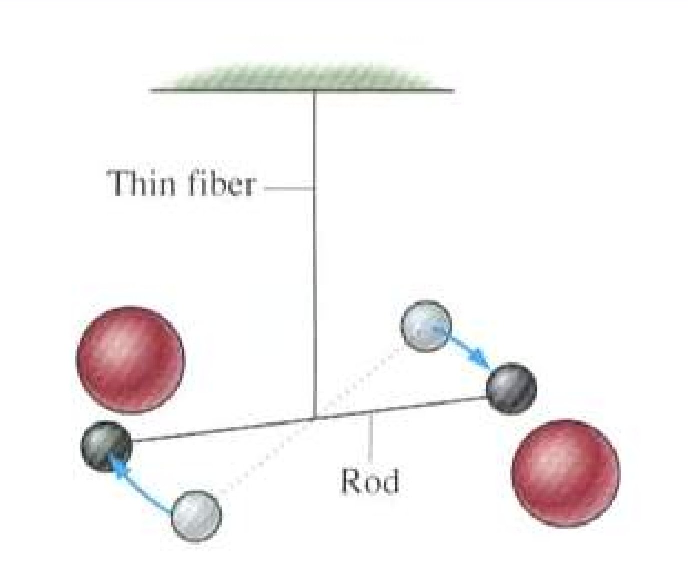
An object orbiting the Earth (or any other object orbiting a large massive object) is accelerating toward the center of the Earth. The blue lines indicate the path of an object in the absence of gravity. From our study of circular motion we know that gravity must provide the force for radial acceleration. This leads to the period for a circular orbit:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} = m\omega^2 r \rightarrow \omega^2 = \frac{GM}{r^3}$$

$$\frac{4\pi^2}{T^2} = \frac{GM}{r^3} \rightarrow T^2 = \frac{4\pi^2}{GM} r^3$$

We have proved Kepler's 3rd law for circular orbits. Note that this expression is independent of the object's **mass**. This law is the primary way astronomers measure the product ***GM*** of objects throughout our galaxy.

Cavendish Experiment



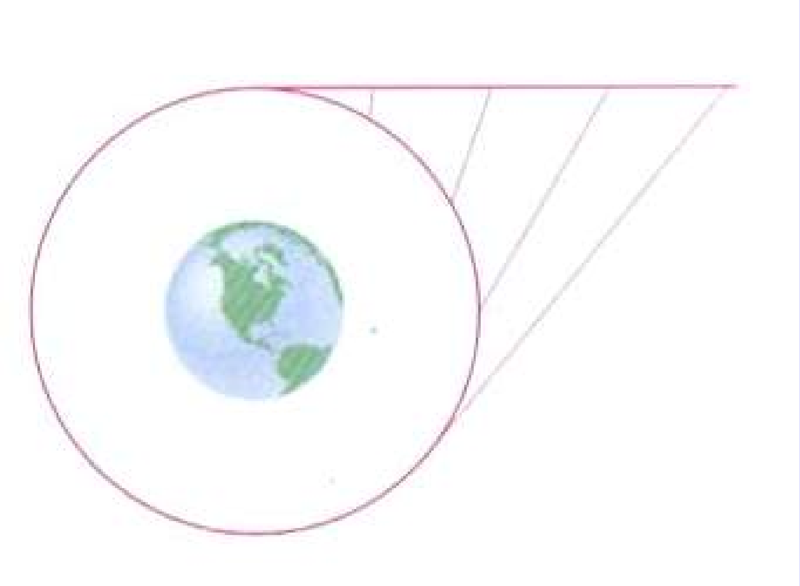
Astronomers measure the product ***GM*** for orbiting objects, but what about ***G*** itself? It is extremely weak but Cavendish found a way. He suspended two ***5cm*** diameter lead spheres connected by a thin rod on the end of a thin fiber. He then brought two ***30cm*** lead spheres close to the suspended lead spheres and measured the small rotation of the rod. Knowing the torsional property, κ, of the fiber, he was able to calculate ***G***.

Actually Cavendish's purpose was to measure the mass of the Earth. Knowing the gravitational acceleration at the surface of the Earth as well as the rotational period of the moon, he knew the product ***GM***. His final result for ***G*** in SI units was

$$G = 6.74 \times 10^{-11} Nm^2/kg^2.$$

It is the product ***GM*** that determines the dynamical properties of objects in a gravitational field. Astronomers routinely measure this product to ***5*** parts in 10^8 while ***G*** itself is only known to ***5*** parts in 10^4.

Example: Geosynchronous Orbit



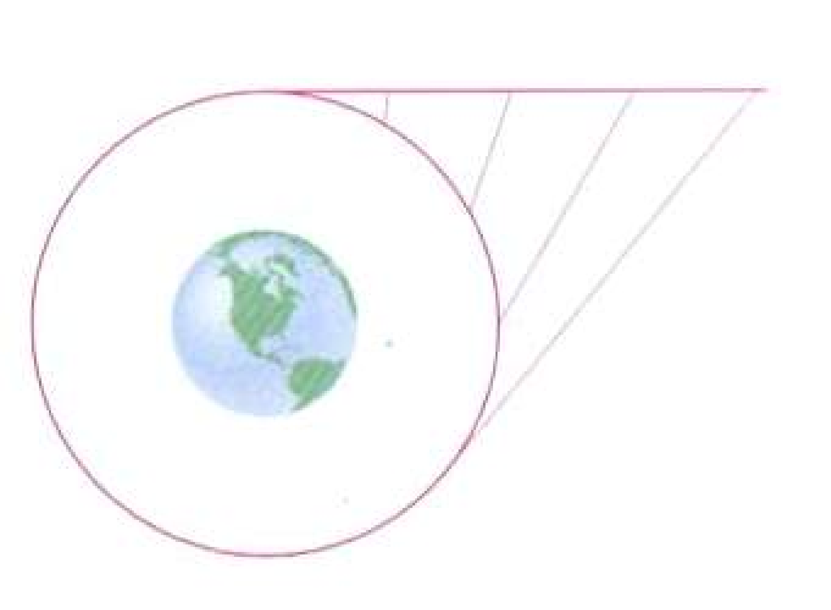
Geosynchronous orbits are important for communications satellites. For example, Direct Dish TV. We are now in a position to determine the altitude for a geosynchronous orbit. The period must be one day! From Kepler's 3rd law:

$$r^3 = \frac{GM_E T^2}{4\pi^2} = 7.538 \times 10^{22} m^3$$

$$r = 4.224 \times 10^7 m$$

However this quantity is the distance from the center of the Earth, not its altitude. The altitude is $\boldsymbol{r - R_E = 3.59 \times 10^7 m}$.

Orbital Motion

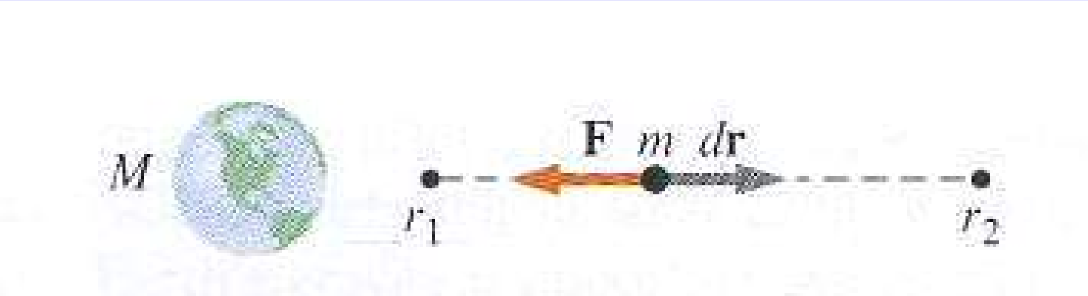


From our study of circular motion we know that gravity must provide the force for radial acceleration. This leads to the orbital velocity for a circular orbit:

$$\frac{GMm}{r^2} = \frac{mv_{orb}^2}{r} \rightarrow v_{orb}^2 = \frac{GM}{r}$$

Since the orbital velocity is independent of the object's mass, all objects in orbit are traveling at the same speed. For example, an astronaut on a space walk is not left behind by the space shuttle.

Gravitational Potential Energy



The change in gravitational potential that occurs when moving an object of mass ***m*** from $\boldsymbol{r_1}$ to $\boldsymbol{r_2}$ is:

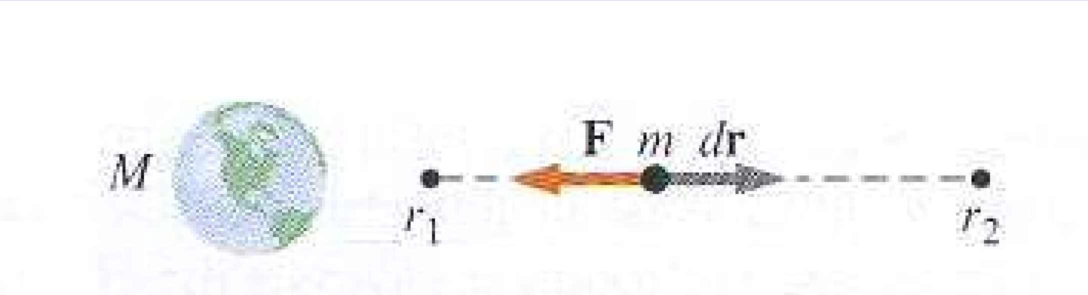
$$\Delta U = -\int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = \int_{r_1}^{r_2} \frac{GMm}{r^2} dr = -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$\Delta U = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

So the potential energy increases as an object's distance from the Earth (or any massive object) is increased. By convention the potential energy is defined to be zero when the object is infinity far from the gravitational source. Hence setting $\boldsymbol{r_1}$ to infinity we find

$$U(r) = -\frac{GMm}{r}$$

Gravitational Potential Energy



$$U(r) = -\frac{GMm}{r}$$

Is this definition for gravitational potential energy consistent with ***mgh***?

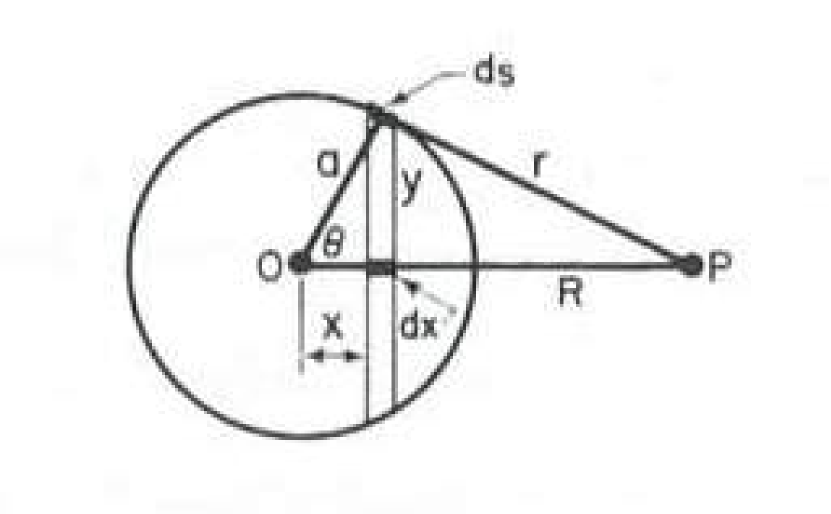
$$\Delta U = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = GMm\left(\frac{1}{R_E} - \frac{1}{R_E + h}\right)$$

$$\Delta U = GMm\frac{R_E + h - R_E}{R_E(R_E + h)} \simeq GMm\frac{h}{R_E^2} = m\frac{GM}{R_E^2}h$$

$$\Delta U = mgh!$$

As long as the change in elevation, ***h***, is small compared to the radius of the Earth, then the expression we have used (assumed that ***g*** is constant) is fine!

Gravitational Potential Energy



$$U(r) = -\frac{GMm}{r}$$

We mentioned that strictly speaking Newton's law of gravity only applies to point particles. With this in mind consider a thin shell of uniform mass density.

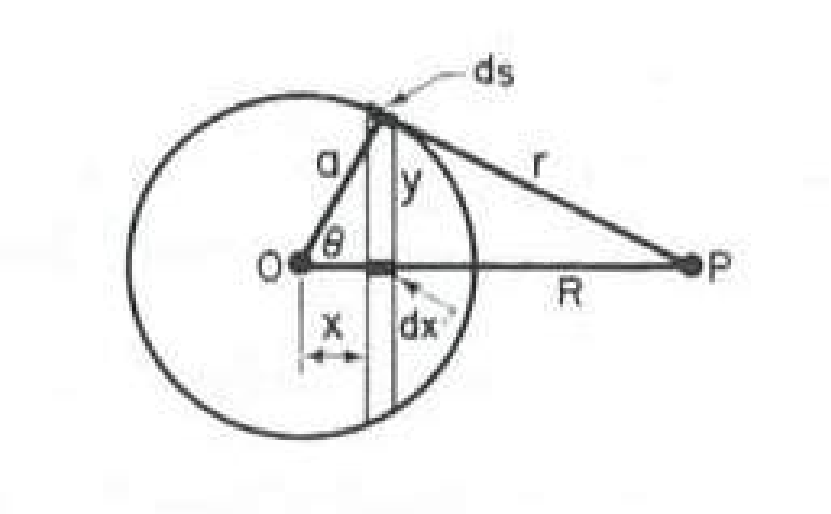
Consider the gravitational potential energy for a particle of mass ***m*** located at the point **P** due to a ring on the shell of radius ***a***. The potential energy at P:

$$dU = -\frac{Gm}{r}dM = -\frac{Gm}{r}\frac{M}{A}dA = -\frac{GmM}{4\pi a^2}\frac{1}{r}2\pi a\sin\theta a d\theta$$

$$dU = -\frac{GmM}{2}\frac{\sin\theta}{r(\theta)}d\theta$$

All that remains is to determine ***r(θ)*** and perform the integration.

Gravitational Potential Energy



From the Pythagorean theorem we know:

$$r^2 = (R - x)^2 + y^2$$

$$r^2 = (R - a\cos\theta)^2 + a^2\sin^2\theta$$

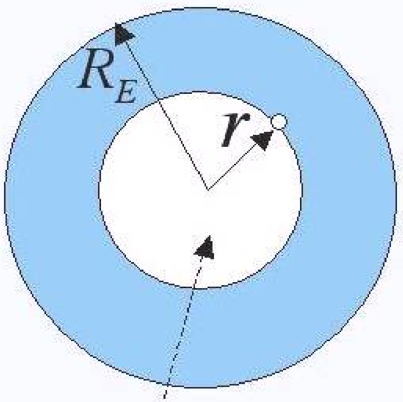
$$r^2 = R^2 + a^2 - 2aR\cos\theta$$

Performing the integral:

$$U = -\frac{GmM}{2}\int_0^{\pi}\frac{\sin\theta}{\sqrt{R^2 + a^2 - 2aR\cos\theta}}d\theta = -GMm\begin{bmatrix} 1/R & R > a \\ 1/a & R < a \end{bmatrix}$$

Outside the shell the gravitational potential is the same as though all the mass were concentrated at the center of the shell. Inside the shell the potential is constant and independent of the location inside the shell. Taking the derivative we would find that there is zero force on an object inside the shell.

Gravitational Potential Inside a Uniform Sphere



The total mass inside a uniform density sphere of radius ***r*** is:

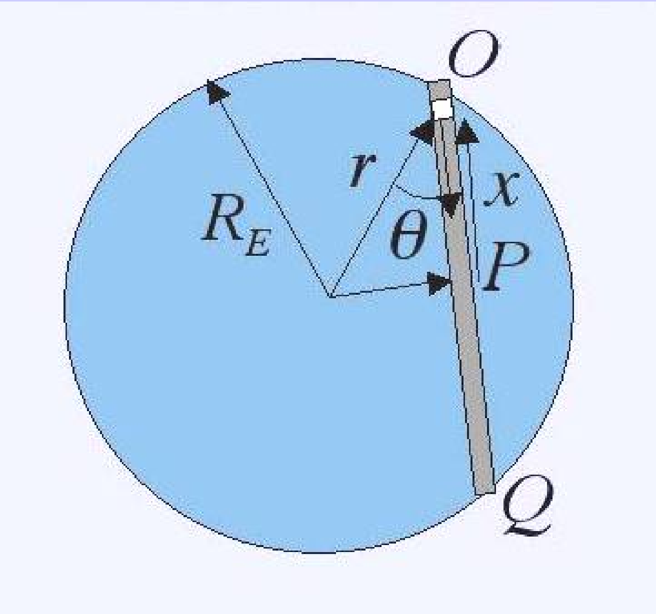
$$m(r) = \rho \frac{4}{3}\pi r^3 = M\frac{r^3}{R_E^3}$$

From Newton's law of gravity the force on a particle of mass ***m*** is:

$$F_g = -\frac{GmMr^3}{r^2 R_E^2} = -\frac{GmM}{R_E^3} r$$

So the force due to gravity increases linearly from zero at the center (although the pressure is maximum there) to the usual $1/R_E^2$ at the surface of the Earth. For $r > R_E$ the gravitational field behaves via the inverse square law.

Simple Harmonic Oscillations Due to Gravity



Consider a tunnel through the Earth. Then measure ***θ*** from the point ***P***, the point of closest approach. When an object is located a distance ***x*** from ***P*** the force parallel to the tunnel is:

$$F = -\frac{GMm}{R_E^3} r\cos\theta = -\frac{GMm}{R_E^3} x$$

This is a linear restoring force! The angular frequency and period are:

$$\omega^2 = \frac{GM_E}{R_E^3} \quad \text{and} \quad T = 2\pi R_E\sqrt{R_E/GM_E} = 5147.0\,\text{sec} = 85.8\,\text{min}$$

This is the roundtrip time. One way from any location on Earth to any other is ***T* = *43*min**. Starting from rest, what is the maximum velocity at the center?

$$v_{\text{max}} = \omega R_E = \sqrt{GM_E/R_E} = 7.79\,km/\sec$$

Gravitational Energy

The total energy for an object in a gravitational field is:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

For a circular orbit there is only a tangential velocity and it is gravity that provides the radial acceleration:

$$m\frac{v^2}{r} = \frac{GMm}{r^2} \rightarrow \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r}$$

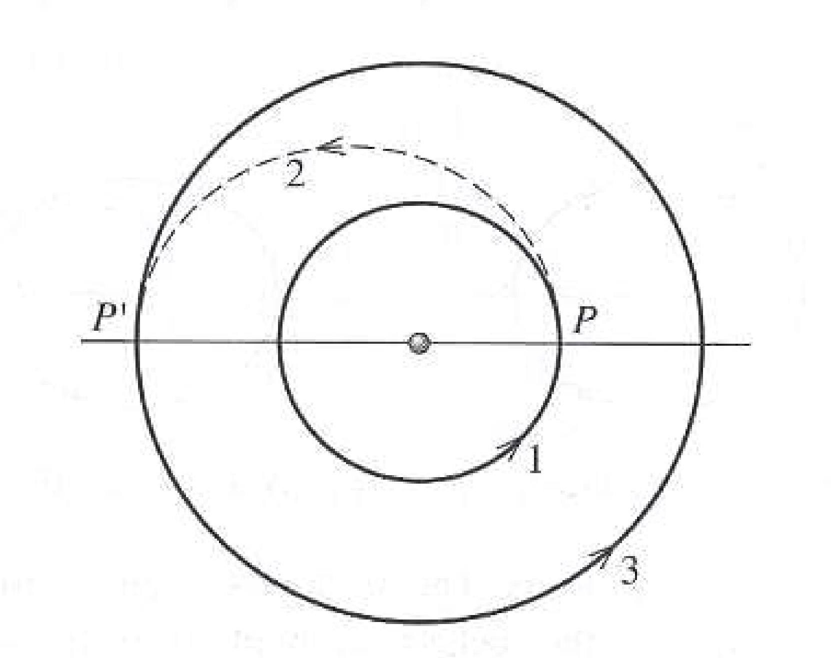
The total energy is then:

$$E = \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r}$$

$$E = -\frac{1}{2}\frac{GMm}{r} = \frac{1}{2}U = -K$$

Higher kinetic energy corresponds to a lower total energy! To get into a faster circular orbit a spacecraft must *lose* energy! To get into a slower circular orbit a spacecraft must *gain* energy!

Change Circular Orbits



To change circular orbit from 1 to 3 a spacecraft fires its rockets at ***P***. This changes the circular orbit to an elliptical orbit, orbit 2. When the spacecraft reaches its apogee in the elliptical orbit, ***P'*,** it fires its rockets again to change into a larger circular orbit.

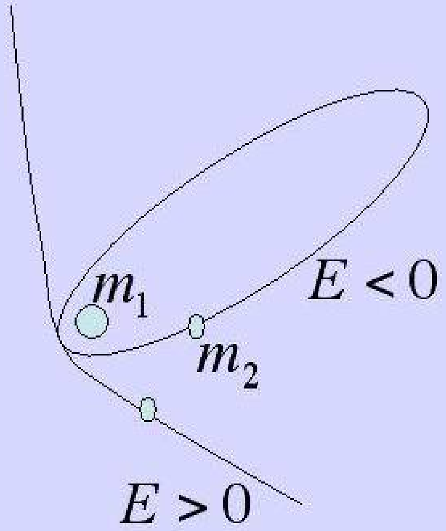
The tangential velocity in a circular orbit is found from

$$K = \frac{1}{2}mv_t^2 = \frac{1}{2}|U| = \frac{1}{2}\frac{GMm}{r} \rightarrow v_t = \sqrt{GM/r}$$

The craft speeds up when it fires its rockets at ***P***. Then it slows down during its trip to ***P'*.** When it is tangent to the outer circular orbit, the craft is traveling to slow to remain in a circular orbit at that ***r***. It must fire its rockets again.

Closed and Open Orbits

Based on total energy, we can determine if an object will or will not be trapped in a closed orbit.

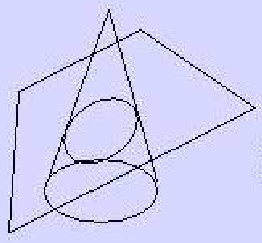


$$E = K + U$$ **is conserved**

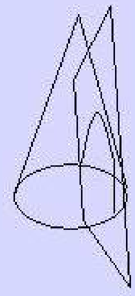
$$E = \frac{1}{2} m_2 v^2 - \frac{G m_1 m_2}{r}$$

E<0: elliptical (or circular)
E=0: parabolic
E>0: hyperbolic

Orbits can be thought of as conical sections:



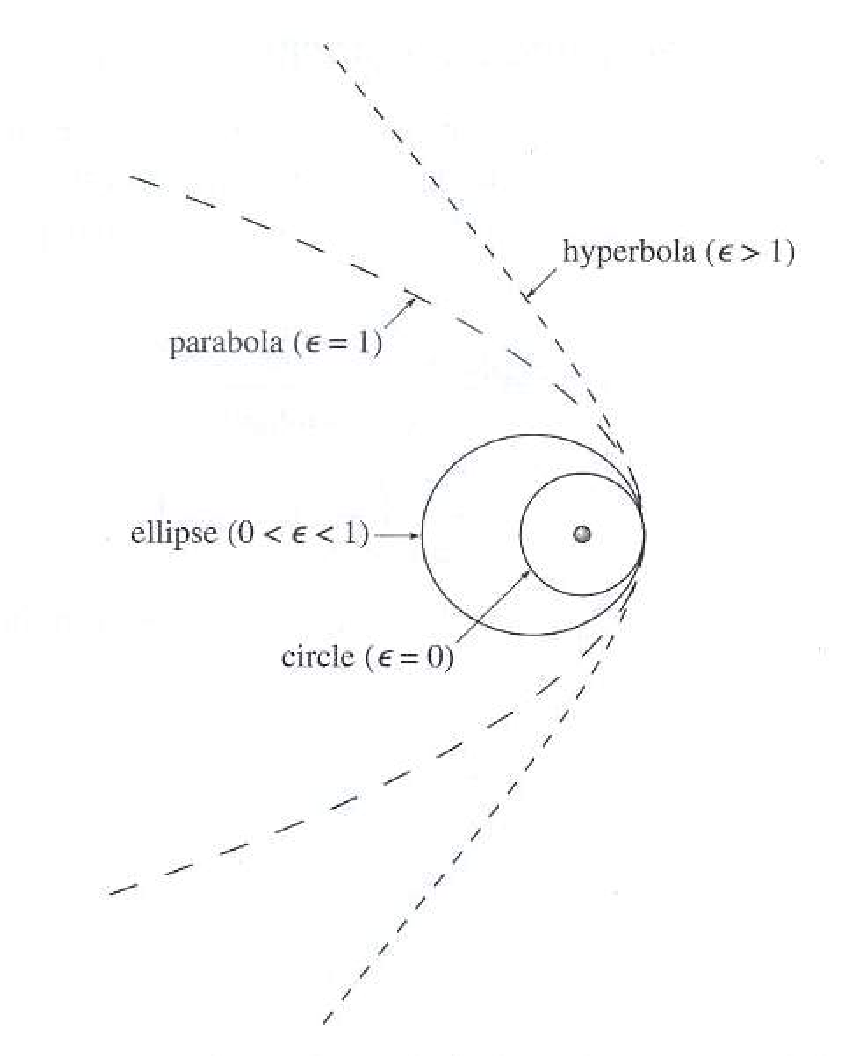
Elliptical



Hyperbolic

Starting from any point if the kinetic energy is larger than the gravitational potential, then the object can escape the gravity (in other words: can go to infinity with kinetic energy left over). Otherwise, the orbit must be closed, in which case it forms an ellipse.

Escape Velocity



To escape the gravitational field the spacecraft must have ***E* > 0**. It must be in a hyperbolic orbit, or at least a parabolic orbit for which ***E* = 0**. For that case:

$$E = \frac{1}{2} m v_{esc}^2 - \frac{GMm}{r} = 0$$

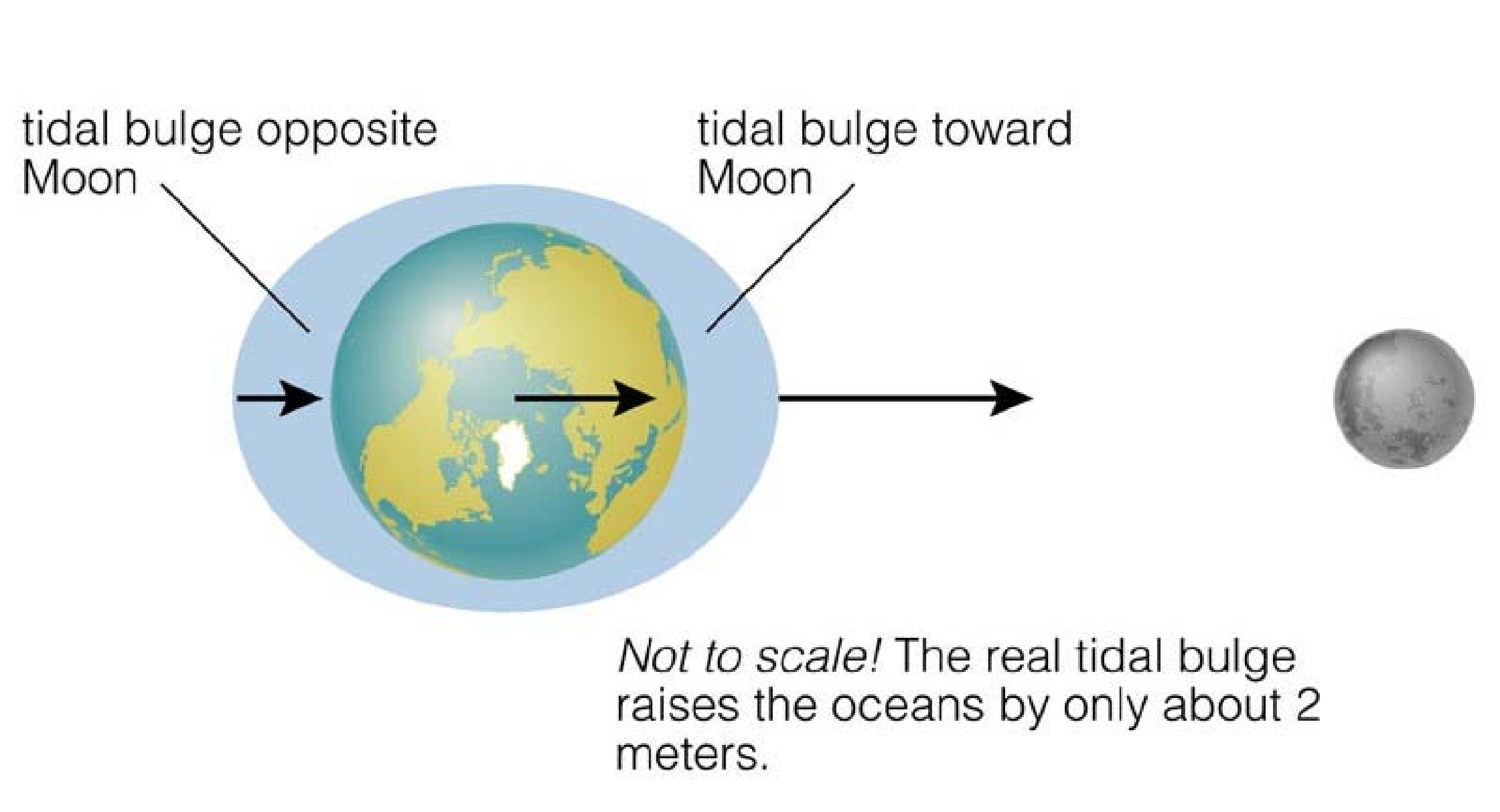
$$v_{esc} = \sqrt{2GM/r}$$

Does the direction matter? No!

At the surface of the Earth:

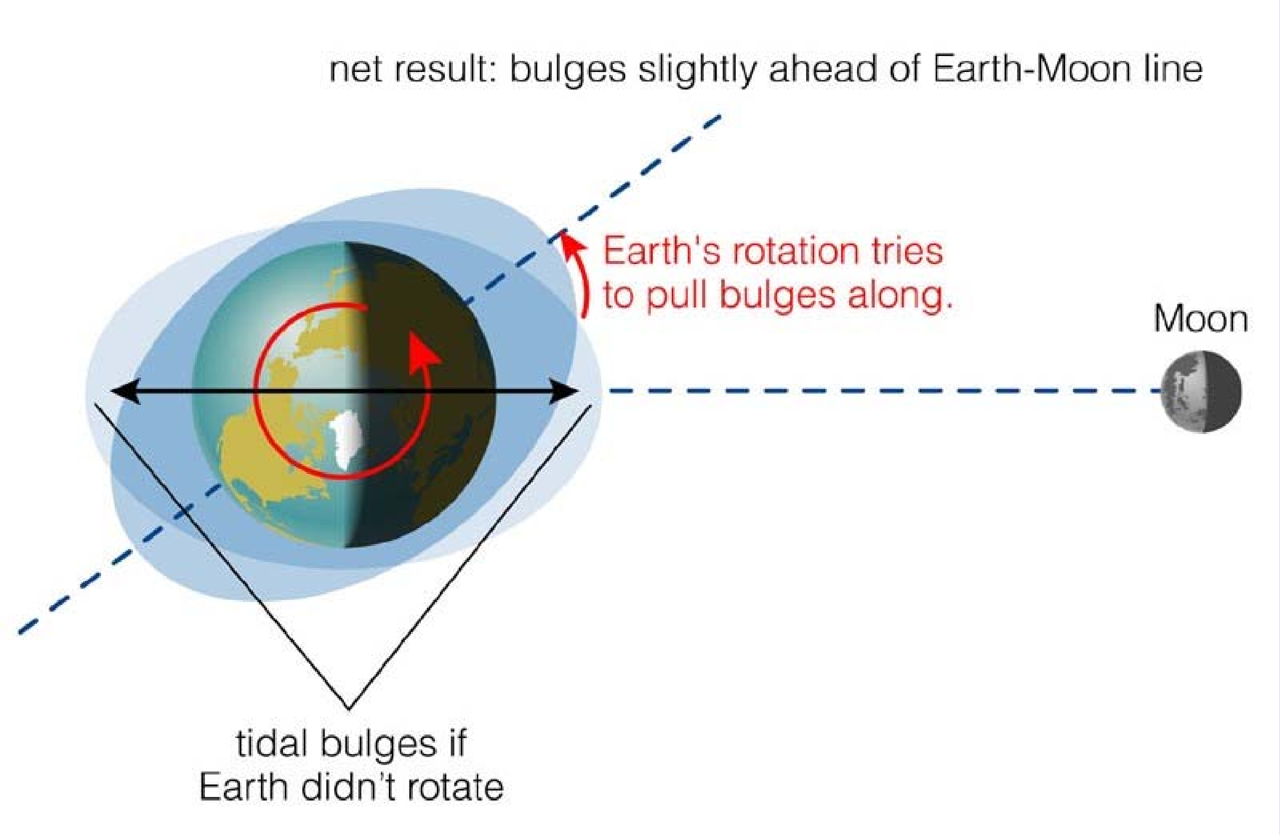
$$v_{esc} = \sqrt{2GM_E/R_E} = 40 km/hr$$

Tidal Forces



The change in the gravitational acceleration between the side of the Earth closest to the Moon and the side farthest from the Moon results in the water bulging toward (and away from) the moon. This explains why there are approximately two tides per day.

Tidal Friction

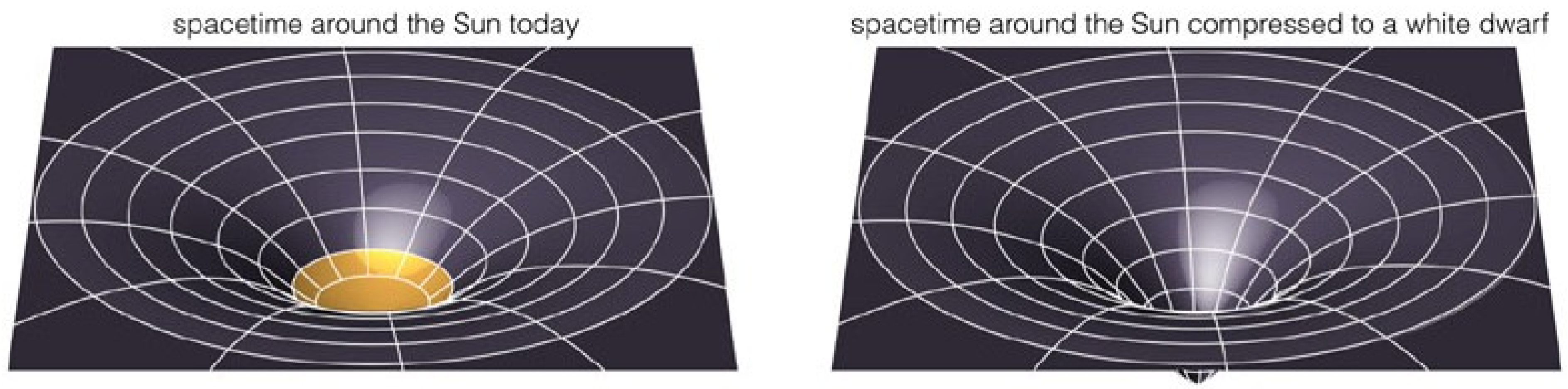


The resulting friction from the flow of the water trying to stay on the Earth-Moon line slows the Earth's rotation.

Tidal friction in the past (when the Moon was molten) slowed the Moon to the point where its orbital period and it period of rotation on its axis became synchronized. We now only see one face (slightly bulged toward us).

Spacetime Curvature

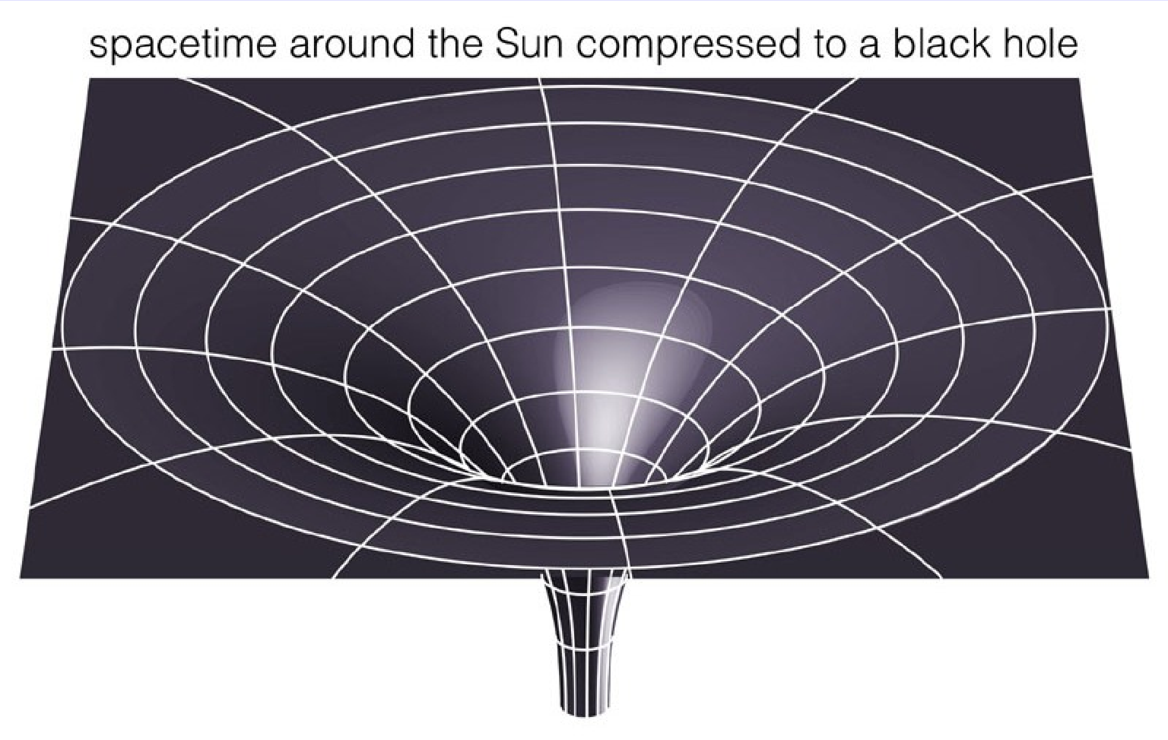
In General Relativity the effects of massive objects are to curve space-time. This curvature causes light to bend, bound orbits (ellipses) to precess, and time to slow down.



It was GR's prediction of the precession of Mercury's orbit and the bending of light (observed during a solar eclipse) that made Einstein famous to the general public. The weaker gravitational field experienced by orbiting GPS satellites means their rate of time is faster than for clocks on Earth. Without taking this into account those satellites would not know their correct position!

Black Holes

General Relativity also allows for the possibility of Black Holes. In this case the mass of a large star has collapsed to such a small volume that nothing can stop it from proceeding to infinite density. A horizon forms about this "singularity" that does not allow anything to escape even light.



The radius of an "event horizon" from which inside nothing can escape is:

$$R_{hor} = 2GM/c^2 \rightarrow R_{Sun} \simeq 3km$$

That’s All Folks!

This completes Physics 2A. I hope you enjoyed at least some of it!

There will be a review session tomorrow morning that should be helpful in preparing for the final! Good Luck to Everyone!