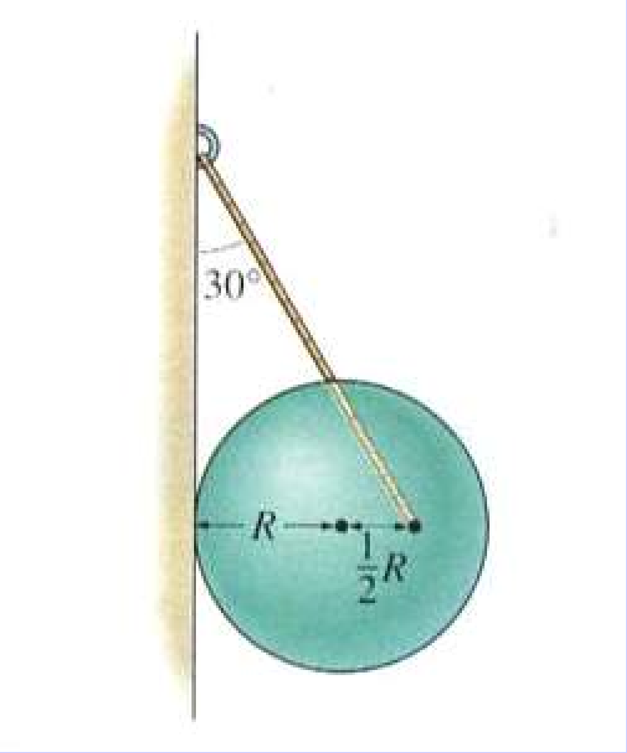
Chapter 14

Static Equilibrium III

Chapter 15

Oscillatory Motion II

Example: Slipping Sphere



A uniform sphere is supported by a rope. The point where the rope is attached to the sphere is located so a continuation of the rope would intersect a horizontal line through the sphere's center a distance ***R*/2** beyond the center, as shown. What is the smallest value for μ_s between the wall and the sphere?

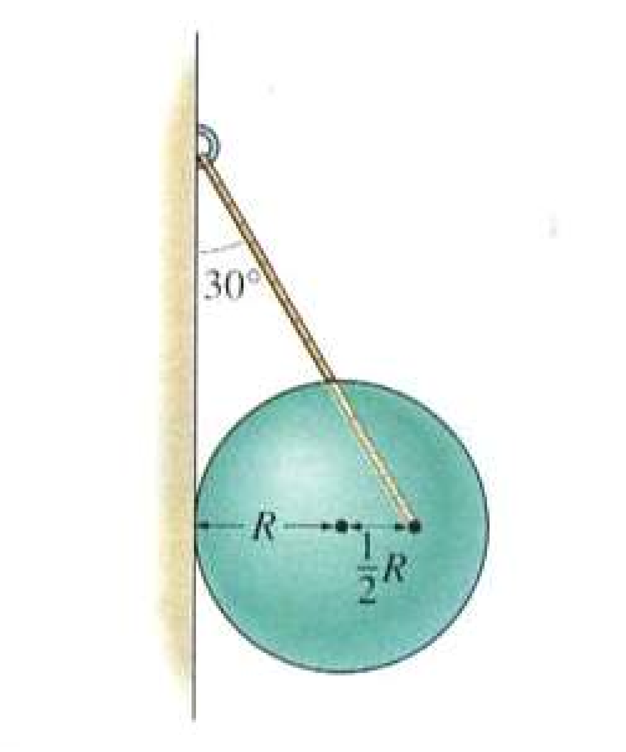
Taking torques about the contact between the sphere and the wall, assuming a mass ***m***, yields:

$$\frac{3}{2}RT\cos 30^\circ = mgR$$

Summing forces in both the ***x*** and ***y*** directions:

$$N = T\sin 30^\circ \quad \text{and} \quad \mu_s N + T\cos 30^\circ = mg$$

Example: Slipping Sphere



A uniform sphere is supported by a rope. The point where the rope is attached to the sphere is located so a continuation of the rope would intersect a horizontal line through the sphere's center a distance ***R*/2** beyond the center, as shown. What is the smallest value for μ_s between the wall and the sphere?

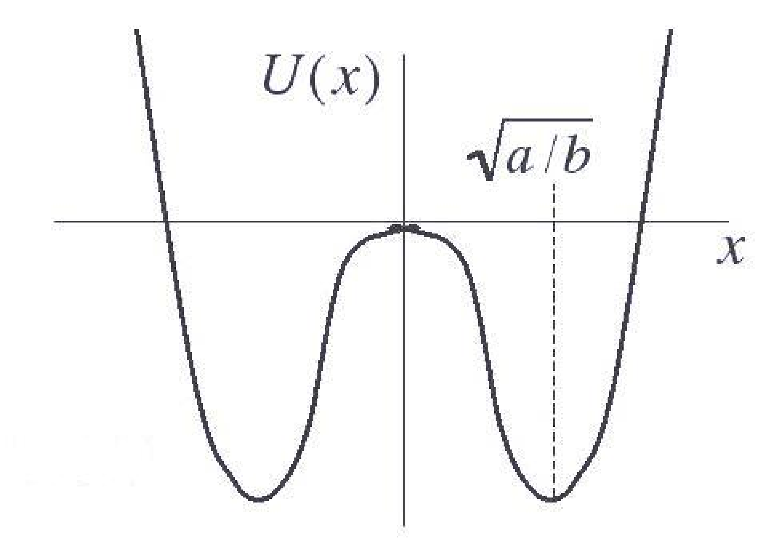
There are three unknowns, μ_s, T, and N. Substituting into the force equation in the y direction T from the torque equation and N from the x force equation:

$$mg = (\mu_s \sin 30^\circ + \cos 30^\circ)\frac{2mg}{3\cos 30^\circ}$$

$$\frac{3}{2} = \mu_s \tan 30^\circ + 1 \rightarrow \mu_s = \frac{\cos 30^\circ}{2\sin 30^\circ}$$

$$\mu_s = \cos 30^\circ = \sqrt{3}/2 = .866$$

Example: Double Welled Potential



Consider the potential given by

$$U(x) = -\frac{ax^2}{2} + \frac{bx^4}{4}$$

Find the equilibrium points and determine if they are stable.

The equilibrium points are found from:

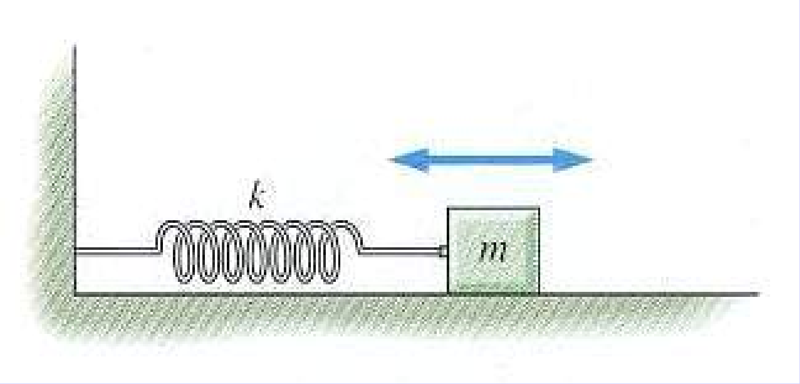
$$\frac{dU}{dx} = -ax + bx^3 = 0 \rightarrow x = 0, \pm\sqrt{a/b}$$

Stability is determined by the second derivative of the potential at equilibrium.

$$\left.\frac{d^2U}{dx^2}\right|_0 = -a + 3bx^2\big|_0 = -a < 0 \quad \text{unstable}$$

$$\left.\frac{d^2U}{dx^2}\right|_{\pm\sqrt{a/b}} = -a + 3bx^2\big|_{\pm\sqrt{a/b}} = -a + 3a = 2a > 0 \quad \text{stable}$$

Simple Harmonic Motion



Simple harmonic motion results when an object is subject to a linear restoring force and is called **simple harmonic motion**, SHO.

Mathematically such a force is described as:

$$F = -kx$$

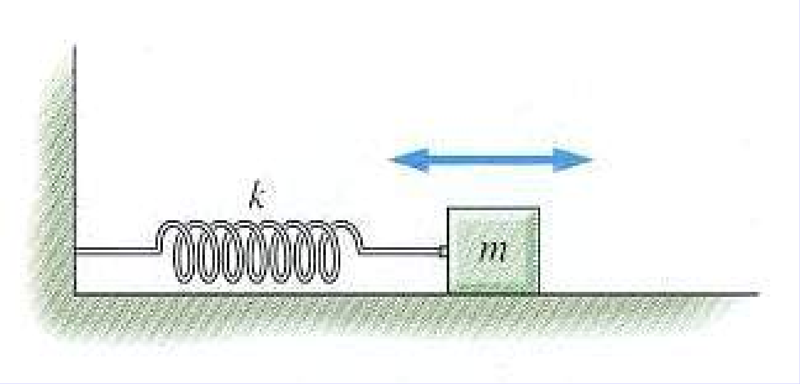
The is the force exerted by an ideal spring of spring constant ***k***.
From Newton's 2nd we can write:

$$F = m\frac{d^2x}{dt^2} = -kx$$

An object experiencing such a force means that when it is displaced from equilibrium there is a force proportional to the distance from equilibrium that accelerates the object back towards its equilibrium position.

How do we describe such motion?

Simple Harmonic Motion



Simple harmonic motion results when an object is subject to a linear restoring force and is called **simple harmonic motion**, SHO.

From Newton's 2nd:

$$F = m\frac{d^2x}{dt^2} = -kx$$

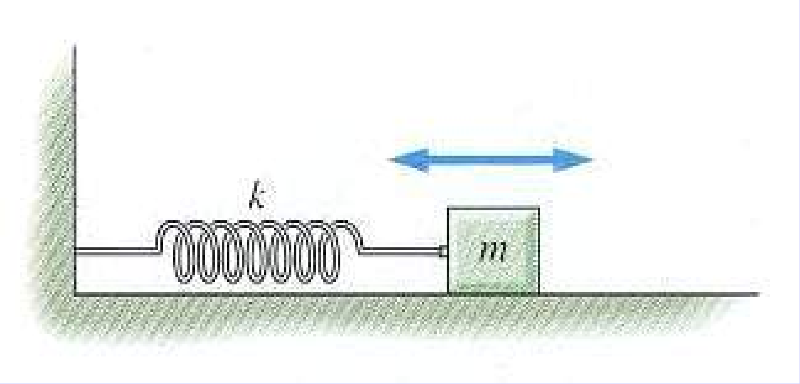
The solution to this equation is of the form ***x = A cos(ωt) + B sin(ωt)***. To see this we simply substitute this function into the "differential equation" represented by Newton's 2nd. Taking the derivatives:

$$\frac{dx}{dt} = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$\frac{d^2x}{dt^2} = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Simple Harmonic Motion



Simple harmonic motion results when an object is subject to a linear restoring force and is called **simple harmonic motion**, SHO.

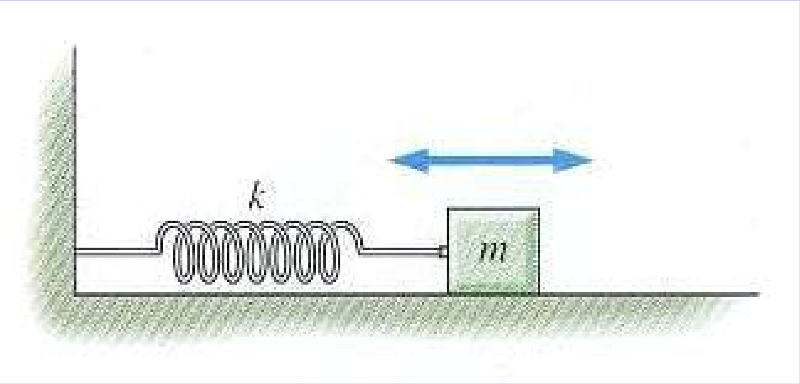
Substituting this result into Newton's 2nd:

$$F = m\frac{d^2x}{dt^2} = -m\omega^2 x = -kx$$

This equation is satisfied if $\boldsymbol{\omega^2 = k/m}$ **!** But what is this $\boldsymbol{\omega}$? We know from the form of $\boldsymbol{x}$ that when $\boldsymbol{\omega T = 2\pi}$ the sine or cosine function returns to its value it had when $\boldsymbol{t = 0}$. So $\boldsymbol{T}$ is the period of the sine or cosine function. Hence $\boldsymbol{\omega = 2\pi /T = 2\pi f}$, where $\boldsymbol{f}$ is the frequency of oscillation.

Just as with angular velocity, $\boldsymbol{\omega}$ is measured in radians per second. When $\boldsymbol{\omega T = 2\pi}$ radians the trignometric functions repeat. The motion is **oscillatory** with frequency $\boldsymbol{f = \omega/2\pi}$ where $\omega = \sqrt{k/m}$.

Simple Harmonic Motion



Simple harmonic motion results when an object is subject to a linear restoring force and is called **simple harmonic motion**, SHO. The general solution is then:

$$x = A\cos\omega t + B\sin\omega t \quad \text{with} \quad \omega = \sqrt{k/m}$$

What about the unknown constants ***A*** and ***B***? They are determined by the initial conditions. For example if at $\boldsymbol{t = 0}$ the system satisfies $\boldsymbol{x = x_o}$ and $\boldsymbol{v = 0}$, then:

$$x(t = 0) = A = x_o \quad \text{and} \quad v = dx/dt = \omega B = 0$$

The solution for this initial condition becomes:

$$x(t) = x_o \cos\omega t$$

Simple Harmonic Motion

Simple harmonic motion results when an object is subject to a linear restoring force and is called **simple harmonic motion**, SHO. An equivalent solution is given by:

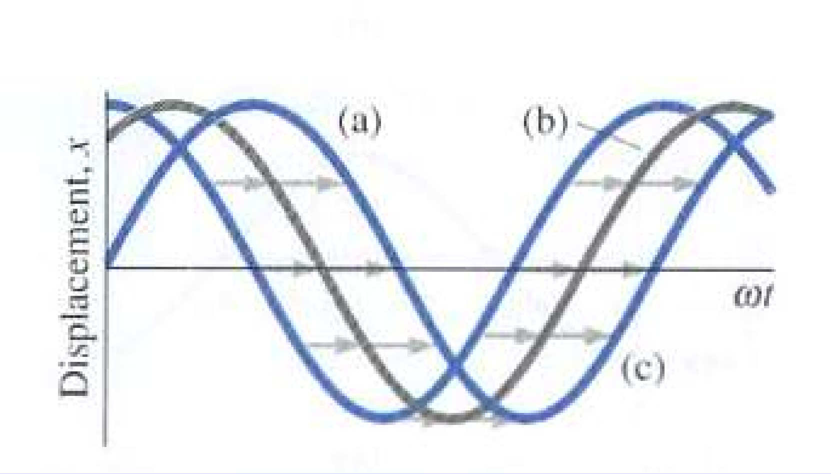
$$x = C\cos(\omega t + \phi)$$

Expanding the cosine function:

$$x = C\cos\phi\cos\omega t - C\sin\phi\sin\omega t$$

This is an equivalent solution with:

$$A = C\cos\phi \quad \text{and} \quad B = -C\sin\phi$$



Either form of the solution can be used. Convenience is what determines the choice. The figure shows three different phases (a) $\boldsymbol{\phi} = \mathbf{0}$, (b) $\boldsymbol{\phi} = \mathbf{-\pi/4}$,
(c) $\boldsymbol{\phi} = \mathbf{-\pi/2}$.

S H O - Velocity and Acceleration

It is useful to consider the velocity and acceleration as it relates to the displacement. For this we will use the solution that includes the phase.

$$x(t) = A\cos(\omega t + \phi)$$

The velocity is the first derivative:

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

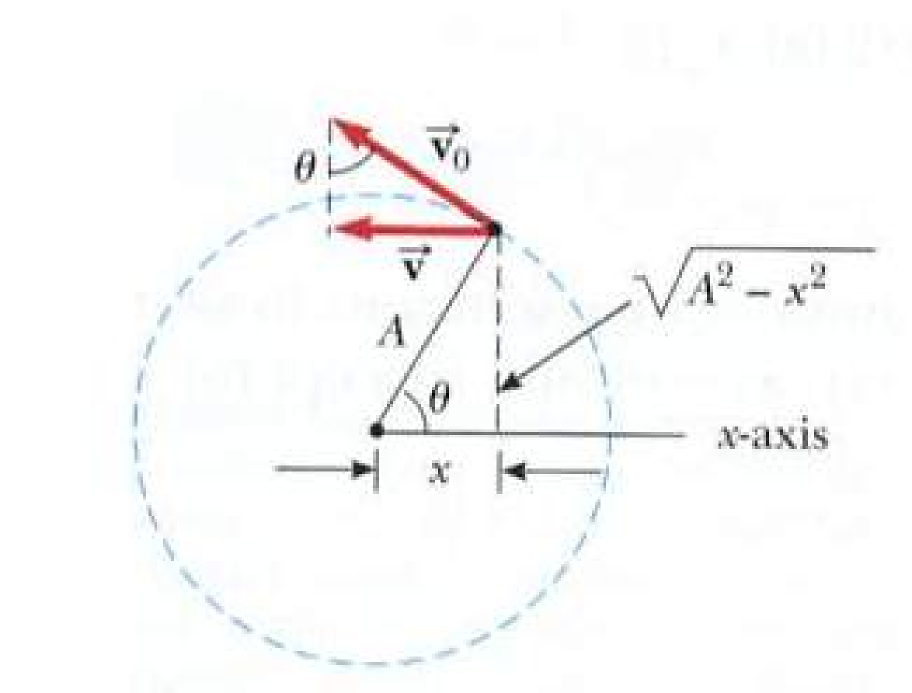
From this we see that the velocity is out of phase with the displacement. When the displacement is maximum, the velocity is zero. Similarly when the velocity is maximum the displacement is zero.

The acceleration is the second derivative:

$$a(t) = \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi)$$

The acceleration **always** has the opposite of the displacement, i.e. the object is under the influence of a restoring force!

Uniform Circular Motion and SHO

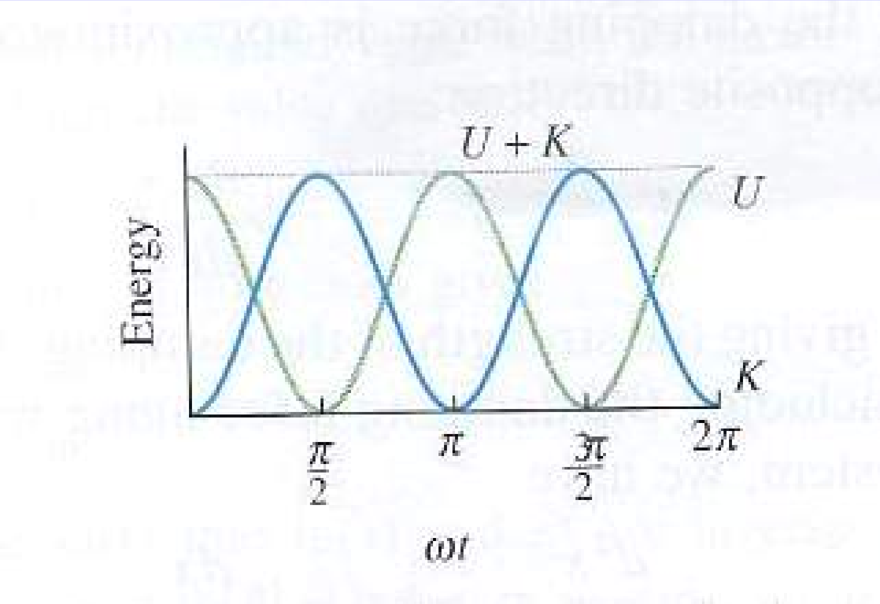


We can think of SHO as the ***x*** component of an object undergoing circular motion with a uniform angular velocity $\boldsymbol{\omega}$. In the figure $\boldsymbol{\theta = \omega(t)}$ and $\boldsymbol{x = A \cos\theta}$.

The tangential velocity is $\boldsymbol{\omega A}$ and the ***x*** component of this velocity is proportional to $\boldsymbol{\sin\theta}$. Also from the figure we see that the ***x*** component of the velocity is pointing in negative direction when $\boldsymbol{\sin\theta}$ is positive. Hence, $\boldsymbol{v = -\omega A \sin\theta}$.

This should help you to understand why we used $\boldsymbol{\omega t}$ as the argument for the solution to the displacement of an object under a linear restoring force.

Energy in Simple Harmonic Oscillations



For the mass spring system the potential energy is ***U(x)* = ½ *k x*²**, where ***x*** is the displacement from equilibrium.

The kinetic energy is ***K* = ½ *m v*²**.

Assuming that ***x* = *A* cos(*ωt*)** we find:

$$U = \frac{1}{2} kA^2 \cos^2 \omega t$$

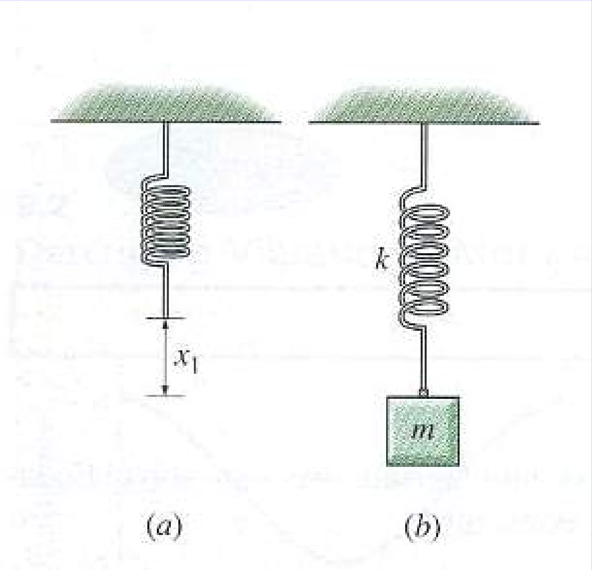
$$K = \frac{1}{2} m\omega^2 A^2 \sin^2 \omega t$$

However ***ω*² = *k*/*m***. Hence the total energy is:

$$E = U + K = \frac{1}{2} kA^2 \cos^2 \omega t + \frac{1}{2} kA^2 \sin^2 \omega t = \frac{1}{2} kA^2$$

The kinetic and potential energy are out of phase so that when one is a minimum the other is a maximum and vice versa. Their total is a constant!

Example: Vertical Mass Spring



There are now two forces acting on the mass m. The force of gravity and that due to the spring. The resulting differential equation from Newton's 2[nd] is:

$$mg - kx = m\frac{d^2x}{dt^2} \rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = g$$

The solution ***x_h = Acos($\omega t+\phi$)*** satisfies:

$$\frac{d^2x_h}{dt^2} + \frac{k}{m}x_h = 0, \quad \omega^2 = k/m$$

Simply adding ***x_1 = mg/k*** to the *homogeneous solution* yields the full solution:

$$x = x_h + x_1 = A\cos(\omega t + \phi) + mg/k$$

The mass continues to oscillate at the same frequency as before. It simply oscillates about a new equilibrium position, ***x_1 = mg/k***.

Example: Torsional Oscillator

For this example it is a good approximation to assume that the restoring torque is proportional to the angular displacement, $\boldsymbol{\tau = -\kappa\theta}$. The resulting differential equation, an analog of Newton's 2nd, is:

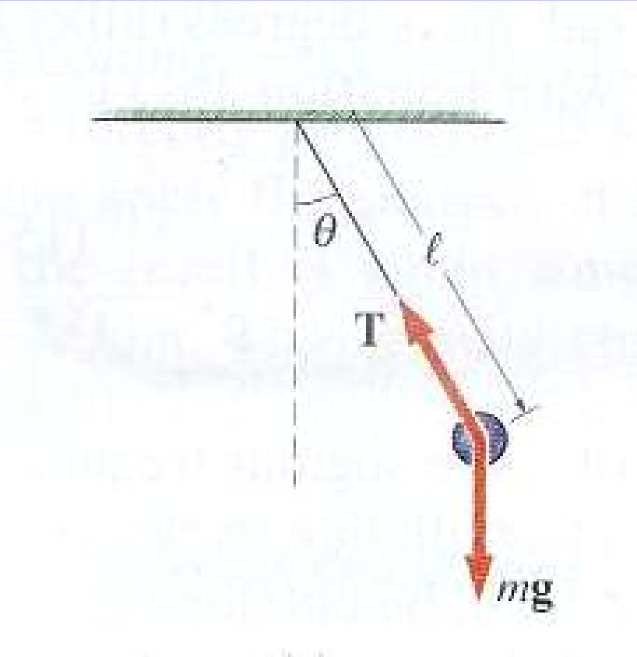
$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} = -\kappa\theta$$

By analogy to the differential equation for a spring, the solution to this differential equation is:

$$\theta(t) = A\cos(\omega t + \theta_o) \;\text{ where }\; \omega = \sqrt{\kappa/I}\,.$$

For this example $\boldsymbol{\kappa}$ is analogous to the spring constant while the moment of inertia, $\boldsymbol{I}$, is analogous to the mass of the oscillating object. Be careful here as $\boldsymbol{\omega}$ is not the same thing as **d*θ*/dt** which is obtained by taking the time derivative of the solution. The torsional oscillator provides an accurate way to measure $\boldsymbol{I}$.

Example: Simple Pendulum



Consider the pendulum as shown in the figure. There are two forces acting on the pendulum bob, the tension in the string and gravity. If we take torques about the pivot point where the string is attached to the ceiling then the tension makes no contribution (nor do any forces supporting the string). The result is:

$$\tau = I\alpha \rightarrow -\ell mg\sin(\pi - \theta) = -\ell mg\sin\theta = I\frac{d^2\theta}{dt^2}$$

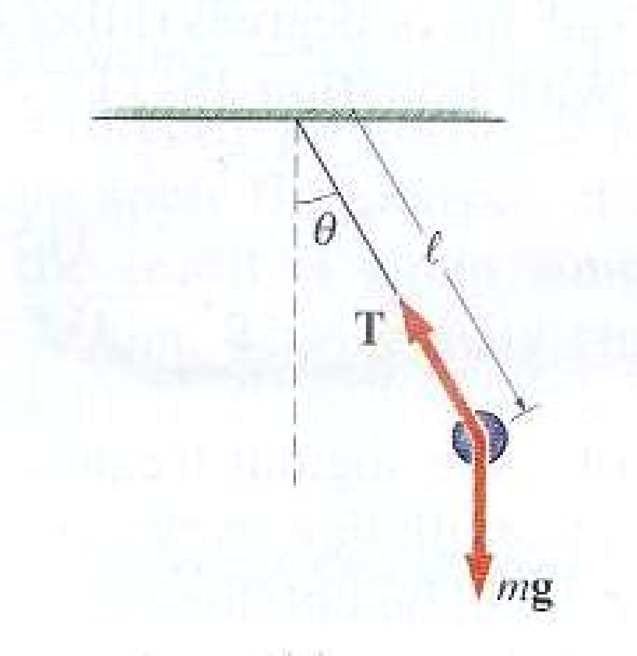
Simplifying this differential equation yields:

$$\frac{d^2\theta}{dt^2} + \frac{mg\ell}{I}\sin\theta = 0$$

Treating the bob as a point particle and substituting for the moment of inertia, ***ml²***, we find:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$

Example: Simple Pendulum



Consider the pendulum as shown in the figure.

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$

This has a somewhat different form than the equations describing SHO. It is in the limit of small $\boldsymbol{\theta}$ when $\mathbf{sin}\boldsymbol{\theta} = \boldsymbol{\theta}$ that the motion is described by SHO. Then we find:

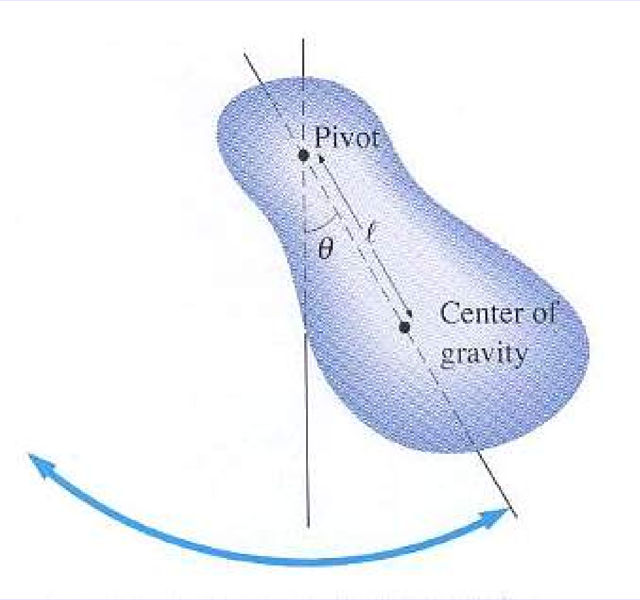
$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$$

The solution to this differential equation is:

$$\theta(t) = A\cos(\omega t + \phi) \quad \text{with} \quad \omega^2 = g/\ell$$

Hence the simple pendulum executes SHO only when its amplitude $\boldsymbol{\theta} \ll \mathbf{1}$.

Example: Physical Pendulum



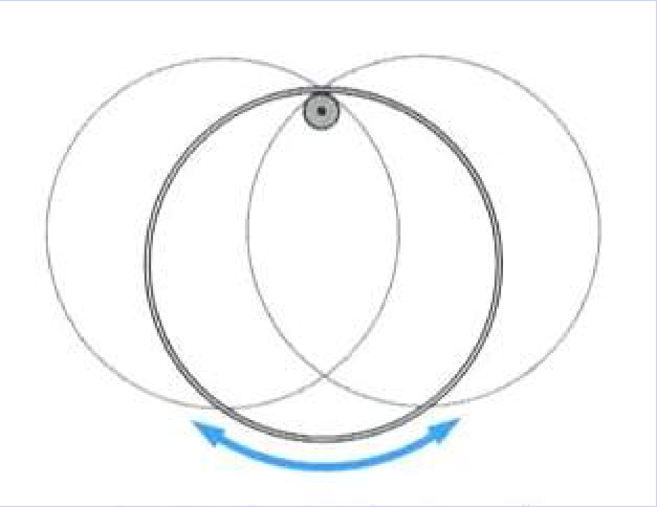
Consider the physical pendulum as shown in the figure. The center of gravity is a distance ***l*** from the pivot point. The equation of motion obtained from finding the torques about the pivot point is:

$$\frac{d^2\theta}{dt^2} + \frac{mg\ell}{I}\sin\theta = 0$$

This is the same as we obtained earlier, only now the momentum of inertia does not necessarily have a simple form. In the limit $\boldsymbol{\theta \ll 1}$ the solution is:

$$\theta(t) = A\cos(\omega t + \phi) \quad \text{with} \quad \omega^2 = mg\ell/I$$

Example: Physical Pendulum - Hoop



Consider a hoop of mass ***m*** and radius ***R***. It is oscillating about a thin horizontal rod as shown. Find its period of oscillations.

The moment of inertia of a hoop about an axis through its center is $I_{cm} = m R^2$. From the parallel axis theorem if it is rotating about an axis on its circumference, $I = I_{cm} + mR^2 = 2mR^2$.

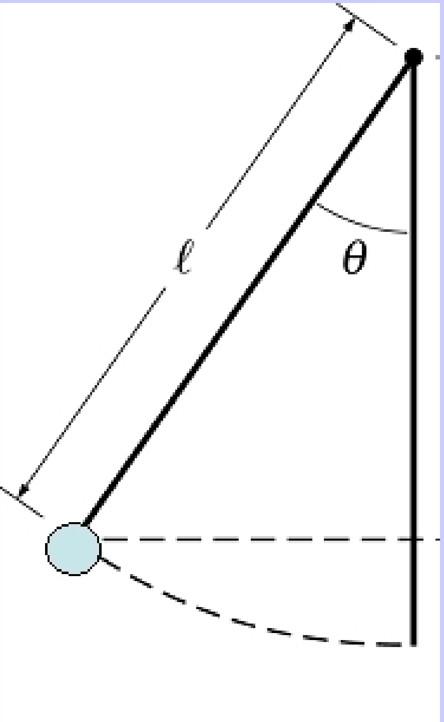
The angular frequency for small amplitude oscillations is:

$$\omega = \sqrt{mg\ell/I} = \sqrt{mgR/2mR^2} = \sqrt{g/2R}$$

The period is:

$$T = 2\pi/\omega = 2\pi\sqrt{2R/g}$$

Example: Physical Pendulum



Consider a pendulum with a uniform rod of length ℓ and mass ***M***. The pendulum bob has a mass ***m***. Find its period of oscillations.

For this pendulum there are two torques. One due to the weight of the bob and the other due to the weight of the pendulum rod. These torques are:

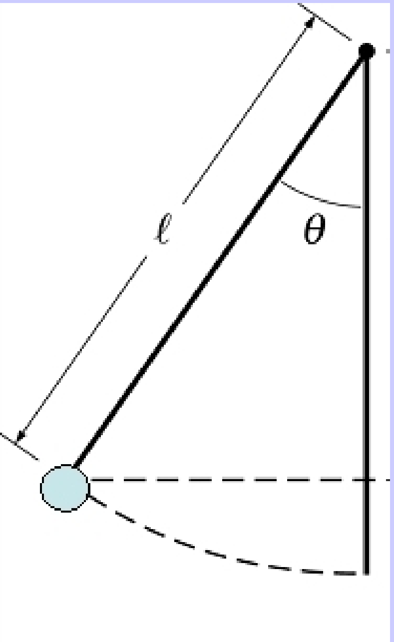
$$\tau = -mg\ell\sin\theta - Mg\frac{\ell}{2}\sin\theta$$

The moment of inertia for the pendulum is: $I = m\ell^2 + \frac{1}{3}M\ell^2$

For small angular displacements the EOM is:

$$I\frac{d^2\theta}{dt^2} = \tau \rightarrow \left(m\ell^2 + \frac{1}{3}M\ell^2\right)\frac{d^2\theta}{dt^2} = -\left(mg\ell + Mg\frac{\ell}{2}\right)\theta$$

Example: Physical Pendulum



Consider a pendulum with a uniform rod of length ℓ and mass ***M***. The pendulum bob has a mass ***m***. Find its period of oscillations.

For small angular displacements the EOM is:

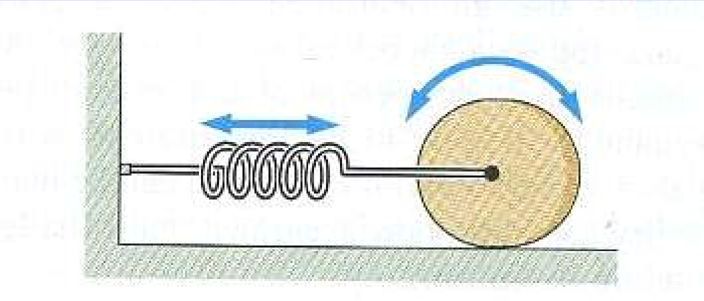
$$\left(m\ell^2 + \frac{1}{3}M\ell^2\right)\frac{d^2\theta}{dt^2} = -\left(mg\ell + Mg\frac{\ell}{2}\right)\theta$$

Note that the torque is the result of the gravitational force acting at the CM which is the CG for a uniform gravitational field. The angular frequency is:

$$\omega^2 = \frac{mg\ell + Mg\ell/2}{m\ell^2 + M\ell^2/3} = \frac{m + M/2}{m + M/3}\,\frac{g}{\ell}$$

The angular frequency is increased compared to a massless rod. Note that if we let the bob mass go to zero, ***m* = 0**, we have the frequency for a swinging rod.

Example: Rolling Disk



A uniform cylinder of mass ***M*** and radius ***R*** is mounted on axle through its center. The axle is attached to a spring of spring constant k and the cylinder rolls back and forth without slipping. Write down the conservation of energy and deduce the frequency of the motion.

The kinetic energy is:

$$T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} M \dot{x}^2$$

For the no slip condition for a solid cylinder this expression becomes:

$$T = \frac{1}{2}\frac{3}{2} M \dot{x}^2$$

The total energy for this system is:

$$E = T + U = \frac{1}{2}\frac{3}{2} M \dot{x}^2 + \frac{1}{2} k x^2$$

By analogy with the mass-spring system the frequency is $\omega^2 = 2k/3M$.