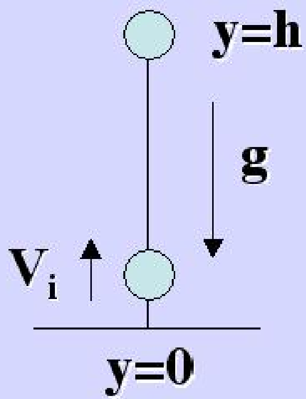
Today’s Lecture

Motion in a Line,
Vectors and Vector Properties
Vector Description of Motion

Review Example: The Velocity Needed to Throw a Stone to Height h

A stone is thrown directly upward, reaches a height h, and falls.

V=0 momentarily



Using equation (2) (without explicit time)

$$V_f^{\,2} - V_i^{\,2} = 2a(x_f - x_i)$$

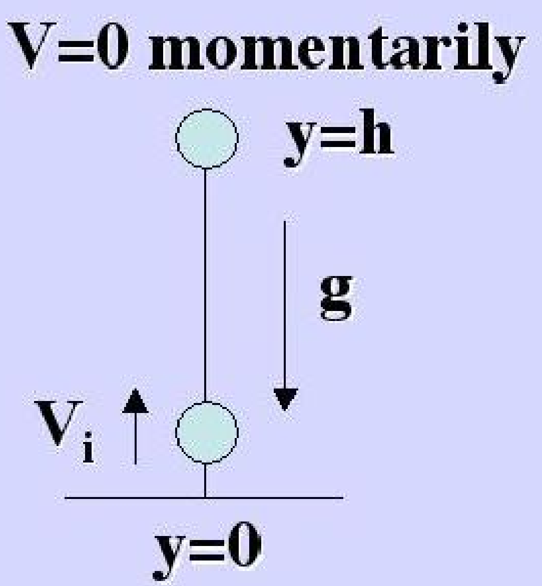
$$0 - V_i^{\,2} = 2(-g)(h - 0)$$

$$V_i = \sqrt{2gh}$$

Example: For h=10m V_i=14m/s

The time to reach 10m: $V_f - V_i = (-g)t$ $\quad t = \dfrac{V_i}{g} = 1.43s$

Example: Velocity Needed to Throw a Stone a Height *h* and the Time to Reach *h*.



Last time we found that the minimum initial velocity required to reach a height *h* was given by:

$$V_i = \sqrt{2gh}$$

And the time to reach a height *h* was given by:

$$t = \frac{V_i}{g}$$

The quadratic equation required to find this time is

$$h = V_i t - \frac{1}{2} g t^2$$

Which has as a solution: $$t = \frac{V_i \pm \sqrt{V_i^2 - 2gh}}{g} \quad \text{or} \quad t = \frac{V_i}{g}$$

The minus sign in front of the sqrt applies on the way up and the plus sign on the way down.

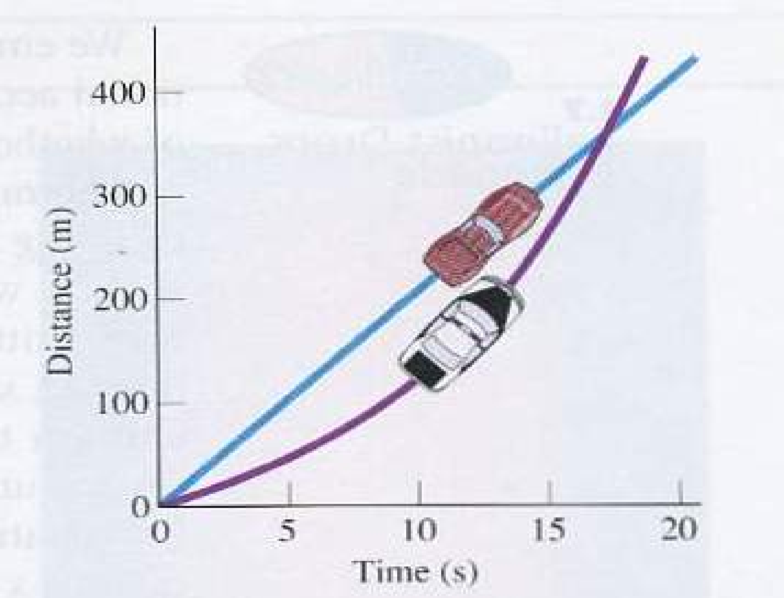
Example – Overtake the Speeder

A speeder passes a stationary police car traveling at a velocity $v = v_s$.The police car pursues from rest with a uniform acceleration of $a = a_p$.

1) When the officer catches the speeder how far have they traveled, and
2) How fast is the police car traveling? (The speeder is traveling $v = v_s$. doh!)

1) When the police car catches the speeder they have traveled the same distance or $x_s = x_p \rightarrow v_s t = \frac{1}{2} a_p t^2 \rightarrow t = 0$ or $t = 2v_s/a_p$

$$\boxed{x = v_s(2v_s/a_p) = 2v_s^2/a_p}$$



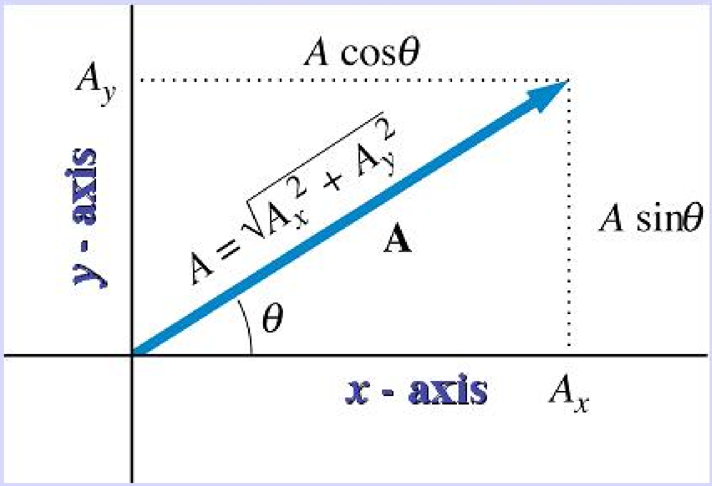
2) The police car is now traveling a velocity given by

$$\boxed{v_p = a_p t = 2v_s}$$

Vectors in 2 Dimensions

A vector is any quantity whose specification requires both size and direction. A scalar quantity has no direction. A scalar multiplying a vector only affects the size of the vector, not the direction.

$\vec{A}$ is defined by two components A_x, A_y



$$A_x = |A|\cos(\theta)$$

$$A_y = |A|\sin(\theta)$$

$$|A| = \sqrt{A_x^{\,2} + A_y^{\,2}}$$

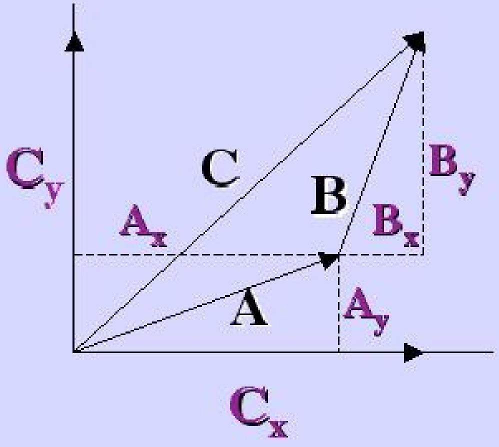
$$\tan(\theta) = \frac{A_y}{A_x}$$

The component of A along any axis is $|A|\cos(\phi)$ where ϕ is the angle A makes with that axis.

The Sum of Two Vectors

The sum of any two vectors is the vector formed by the sum of each set of components of each vector.

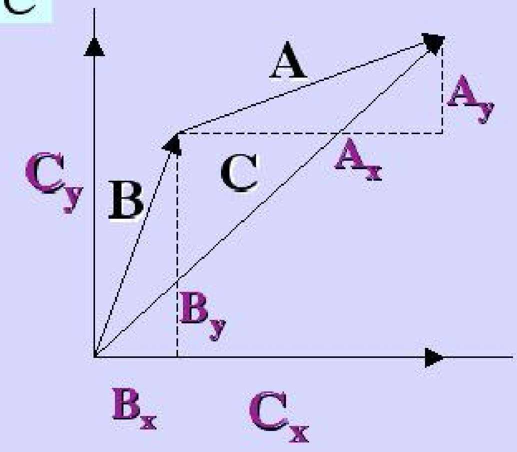
Vector addition is commutative



$$\vec{A} + \vec{B} = \vec{B} + \vec{A} = \vec{C}$$

$$C_x = A_x + B_x$$

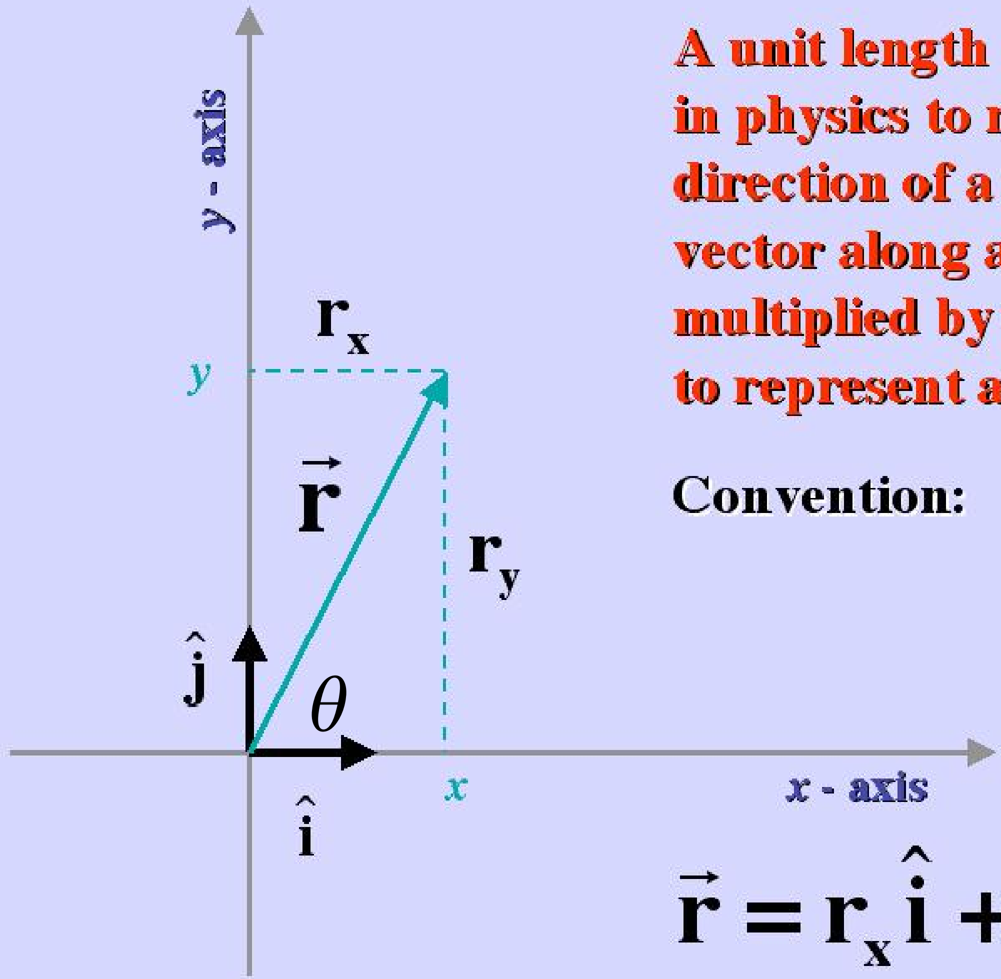
$$C_y = A_y + B_y$$



Subtracting two vectors works the same way as a sum with the second vector's components negated.

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

Unit Vectors: Definition



A unit length vector is often used in physics to represent the direction of a component of a vector along an axis, and is multiplied by a scalar amplitude to represent a vector component.

Convention: $\hat{\mathbf{i}}$ **x-axis direction**

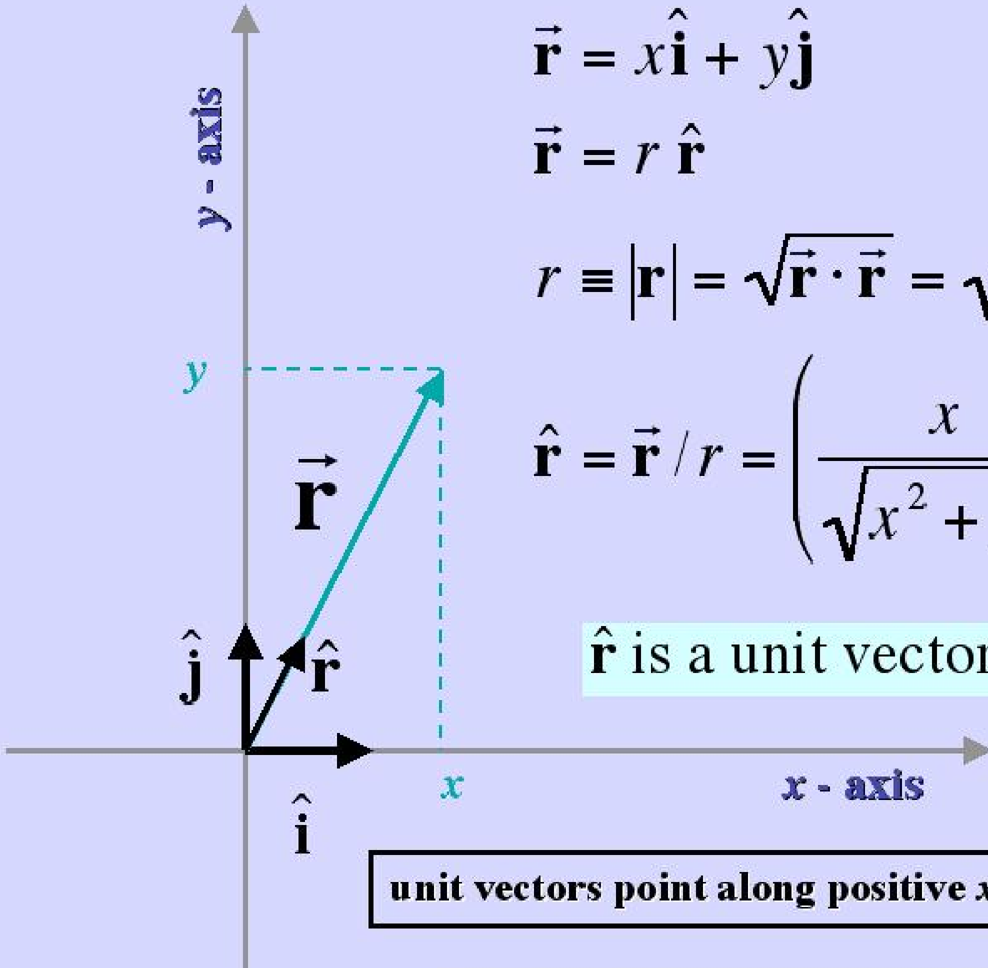
$\hat{\mathbf{j}}$ **y-axis direction**

$\hat{\mathbf{k}}$ **z-axis direction**

$$\vec{\mathbf{r}} = \mathbf{r}_x \hat{\mathbf{i}} + \mathbf{r}_y \hat{\mathbf{j}}$$

where $\mathbf{r}_x = \left|\vec{r}\right| \cos\theta$ and $\mathbf{r}_y = \left|\vec{\mathbf{r}}\right| \sin\theta$

Vectors in 2 Dimensions: Unit Vectors



$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$$

$$r \equiv |\mathbf{r}| = \sqrt{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}} = \sqrt{x^2 + y^2}$$

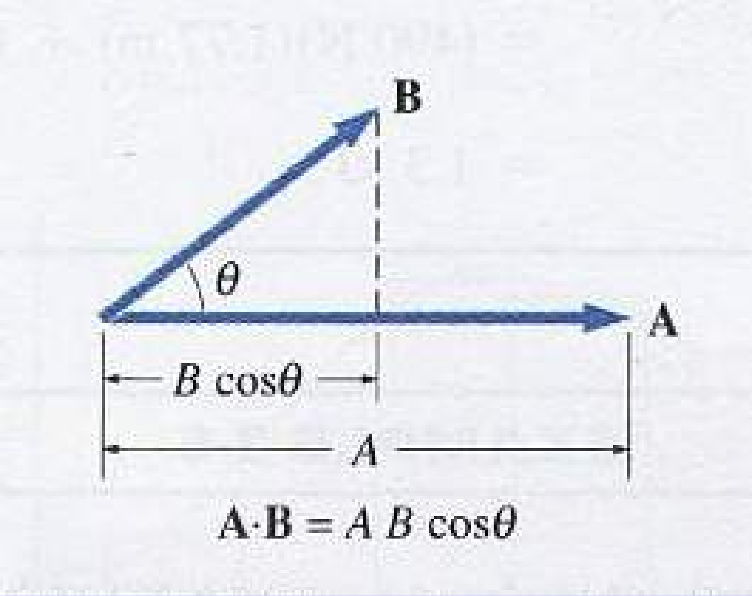
$$\hat{\mathbf{r}} = \vec{\mathbf{r}}/r = \left(\frac{x}{\sqrt{x^2 + y^2}}\right)\hat{\mathbf{i}} + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)\hat{\mathbf{j}}$$

$\hat{\mathbf{r}}$ is a unit vector along $\vec{\mathbf{r}}$, so $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$

unit vectors point along positive *x* and *y* axes

$$\boxed{\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1 \quad \text{and} \quad \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0}$$

Vectors in 2 Dimensions – Scalar Product



From our definitions up to this point we can define a vector $\vec{B}$ such that it points along the *x* axis.

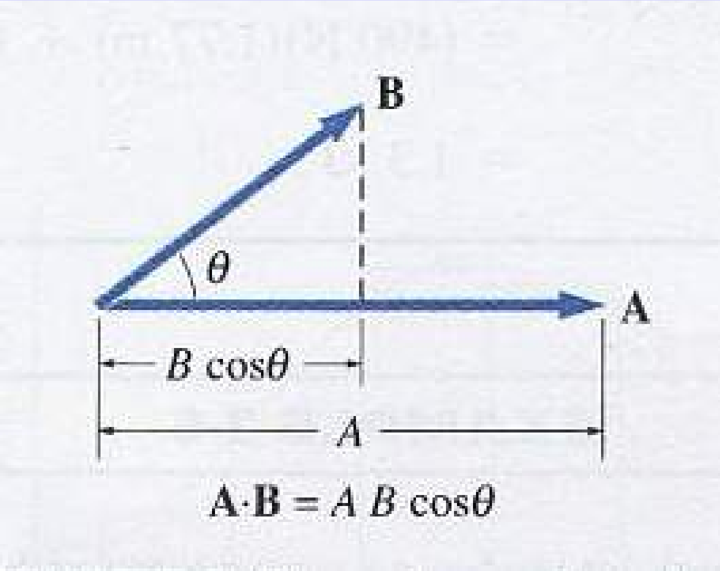
By definition taking the scalar product of this vector with $\vec{A}$ yields the scalar

$$\vec{A} \bullet \vec{B} = AB\cos\theta$$

A and *B* are the magnitudes of their respective vectors and θ is the angle between them.

The axes can be rotated, but their magnitude and the angle between the two vectors remains unchanged. This means that this result is a general **invariant** result for any coordinate system where θ is the angle between the two vectors.

Vectors in 2 Dimensions – Scalar Product



The scalar product is **commutative**.

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A} \text{ as } AB\cos\theta = AB\cos(-\theta)$$

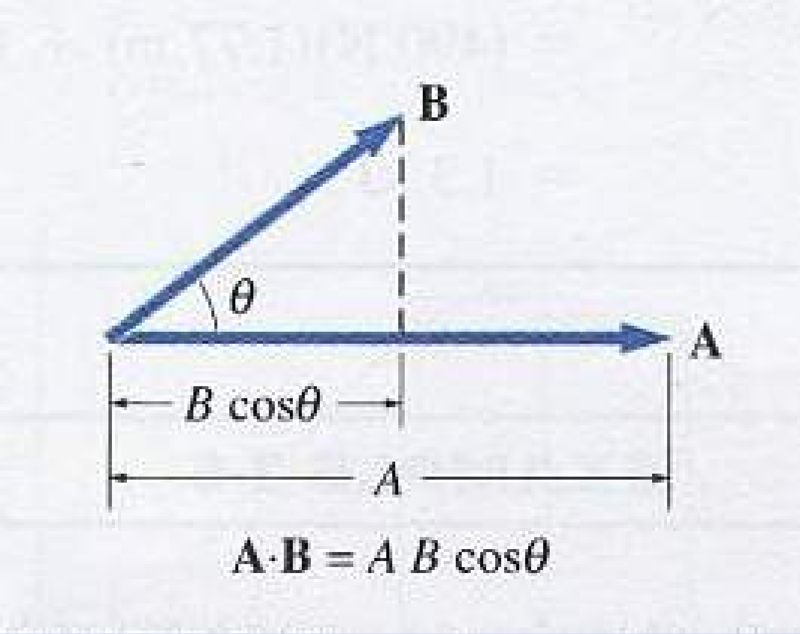
It can easily be shown that the scalar product is **distributive** (exercise for student).

$$\vec{A} \bullet \left(\vec{B} + \vec{C}\right) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$$

For any arbitrary vector we will routinely make use of the notation

$$A^2 = \vec{A} \bullet \vec{A} = A_x^2 + A_y^2 \text{ and } A = \sqrt{A_x^2 + A_y^2}$$

Vectors in 2 Dimensions – Scalar Product



In terms of unit vectors we can write the vector $\vec{A}$ in terms of its components.

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

Using an analogous expansion for $\vec{B}$ the scalar product can be expressed

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y$$

The equivalence of the two different expressions for the scalar product can be seen from

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y$$

$$\vec{A} \bullet \vec{B} = A\cos\alpha(B\cos\beta) + A\sin\alpha(B\sin\beta)$$

$$\vec{A} \bullet \vec{B} = AB\cos(\alpha - \beta) = AB\cos\theta$$

The scalar product is often referred to as the '**dot**' product.

Velocity and Acceleration Vectors

Displacements, Velocities and Accelerations are all Vectors.

For velocity, what was previously discussed as a scalar differential equation is actually a vector differential equation:

$$\vec{\mathbf{V}}(\mathbf{t_a}) = \underset{t_b \to t_a}{Lim} \frac{\vec{x}_b - \vec{x}_a}{t_b - t_a} = \frac{d\vec{x}}{dt}$$

where in unit vector notation

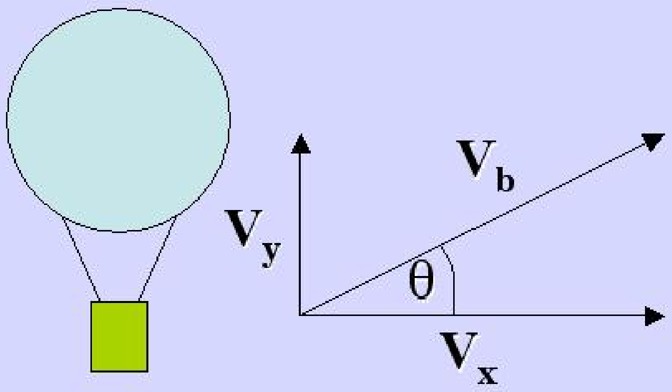
$$\mathbf{v_x}\hat{\mathbf{i}} + \mathbf{v_y}\hat{\mathbf{j}} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}}$$

The acceleration is analogous

$$\vec{\mathbf{a}}(\mathbf{t_a}) = \underset{t_b \to t_a}{Lim} \frac{\vec{V}_b - \vec{V}_a}{t_b - t_a} = \frac{d\vec{V}}{dt} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} = \frac{dv_{\mathbf{x}}}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}}$$

Example: Velocity Vectors

A hot air balloon climbs at 0.5m/s in a west wind of 2m/s. What is the velocity of the balloon?



$$\vec{V}_b = V_{\mathbf{x}}\hat{\mathbf{i}} + V_y\hat{\mathbf{j}}$$

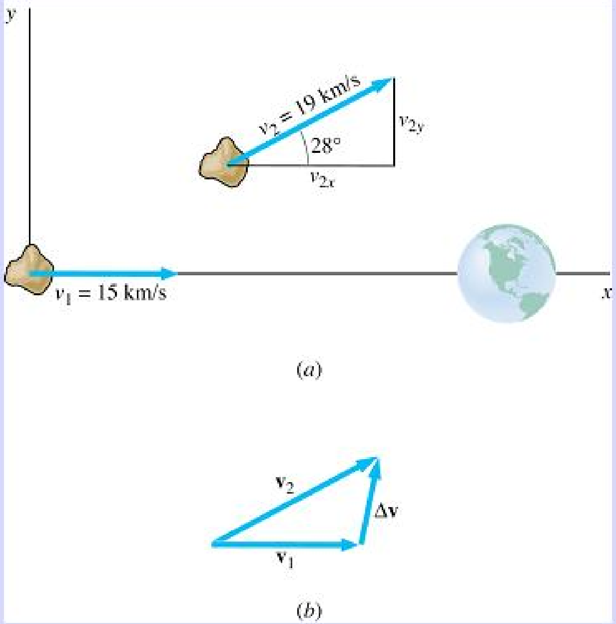
$$\vec{V}_b = (2m/s)\hat{\mathbf{i}} + (0.5m/s)\hat{\mathbf{j}}$$

$$\left|\vec{V}_b\right| = 2.06m/s$$

What angle does the trajectory make with the ground?

$$\theta = \arctan\left(\frac{V_y}{V_x}\right) = 14^o$$

Example: Velocity and Acceleration

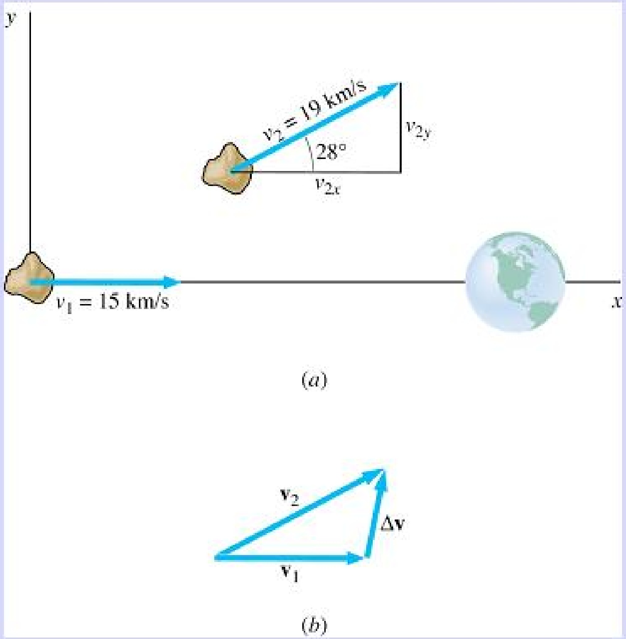


3-4. An asteroid is discovered heading straight toward earth at 15km/s. An international team manages to attach a giant rocket engine to the asteroid. The rocket fires for 10min, after which the asteroid is moving at 28 degrees to its original path, at a speed of 19km/s. Find its average acceleration.

$$V_x = (19km/s)\cos(28^o) = 16.8km/s, V_y = (19km/s)\sin(28^o) = 8.92km/s$$

$$\vec{\mathbf{a}} = \frac{\Delta V}{\Delta t} = \frac{(16.8\hat{\mathbf{i}} + 8.92\hat{\mathbf{j}}) - 15\hat{\mathbf{i}}\ km/s}{10\min(60s/\min)} = 3.0\hat{\mathbf{i}} + 15\hat{\mathbf{j}} m/s^2$$

Example: Velocity and Acceleration



Now find the position vector to the asteroid if it started at 100,000 km away. Will it miss the Earth? (neglect Earth's gravity).

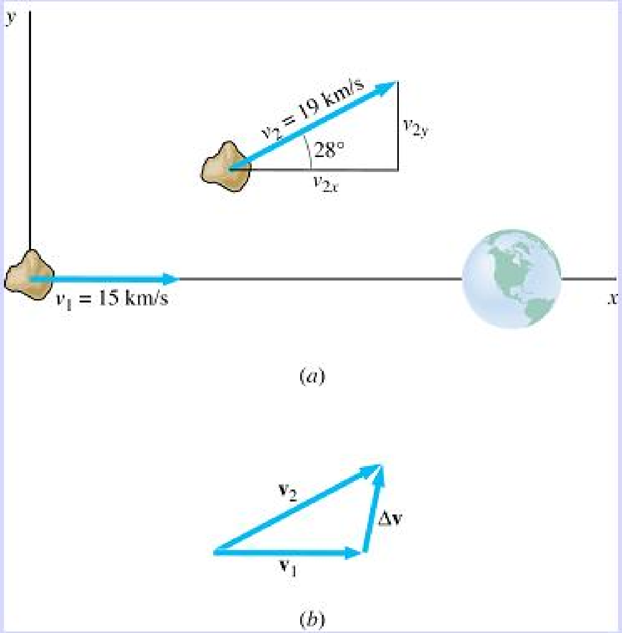
At the end of the rocket burn the y displacement is:

$$y_f = y_i + V_y t + \frac{1}{2} a_y t^2$$

$$y_f = 0 + 0 + \frac{1}{2}(15m/s^2)(600s)^2$$

$$y_f = 2,700,000m$$

Example: Velocity and Acceleration



Now find the position vector to the asteroid if it started at 100,000 km away. Will it miss the Earth? (neglect Earth's gravity).

At the end of the rocket burn the x displacement is:

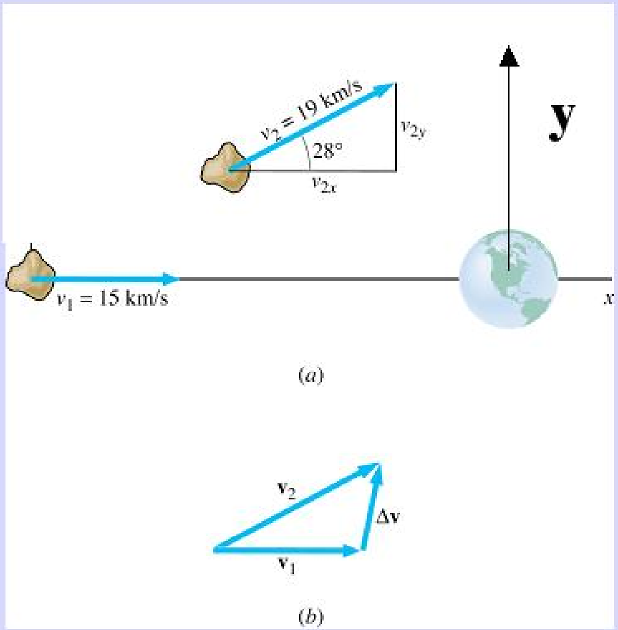
$$x_f = x_i + V_x t + \frac{1}{2} a_x t^2$$

$$x_f = 0m + (15{,}000m/s)(600s) + \frac{1}{2}(3m/s^2)(600s)^2$$

$$x_f = 9{,}540{,}000m$$

x_i=0 because the asteroid's initial location is defines the origin.

Example: Velocity and Acceleration



Now find the position vector to the asteroid if it started at 100,000 km away. Will it miss the Earth? (neglect Earth's gravity).

Alternatively, if we place the origin at Earth:

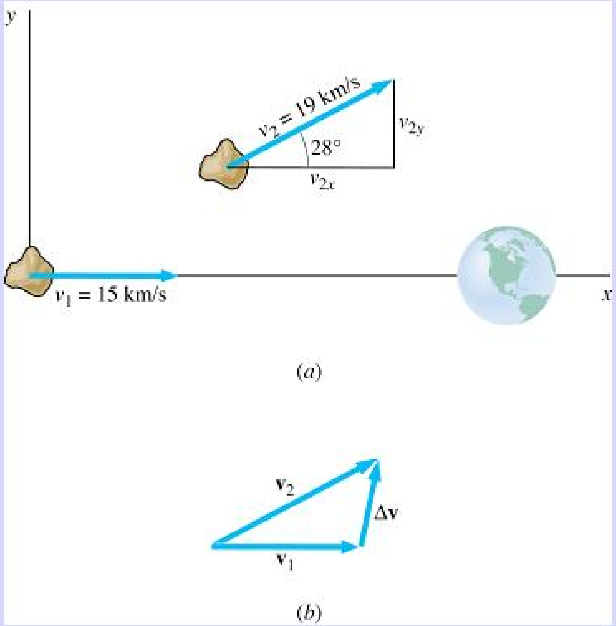
$$x_f = x_i + V_x t + \frac{1}{2} a_x t^2$$

$$x_f = -100{,}000{,}000m + (15{,}000m/s)(600s) + \frac{1}{2}(3m/s^2)(600s)^2$$

$$x_f = -90{,}460{,}000m$$

Note the sign of x_i!

Example: Velocity and Acceleration



Now find the position vector to the asteroid if it started at 100,000 km away. Will it miss the Earth? (neglect Earth's gravity).

The asteroid will then take

$$t = \frac{x_f - x_i}{V_x} = \frac{0 + 90,460,000m}{16,800m/s} = 5384s$$

To reach Earth, at which time the y displacement will be

$$y_f = y_i + V_y t$$

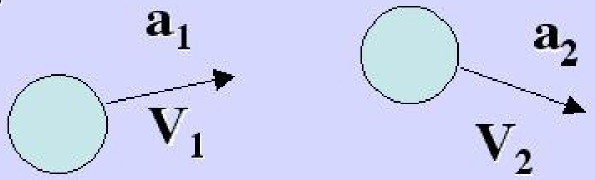
$$y_f = 2,700,000m + (8920m/s)(5384s)$$

$$y_f = 50,725,280m$$

The radius of the Earth is 6,378,100m! Yes! Saved once again by Physics.

Displacement, Velocity and Acceleration are Relative

Just as all displacement is relative, so are the velocity and acceleration. They are measured relative to some coordinate system origin which is assumed to be fixed.



If body 1 is moving with velocity V_1 and body 2 with velocity V_2, then body 1 is moving relative to body 2 with velocity that id the differenec between V_1 and V_2

$$\vec{V}_r = \vec{V}_1 - \vec{V}_2$$

The same holds for acceleration $\vec{a}_r = \vec{a}_1 - \vec{a}_2$

The intertial reference frame is a reference frame with no acceleration.