Today's Lecture

Lecture 13: Chapter 7

Work, Energy, Power

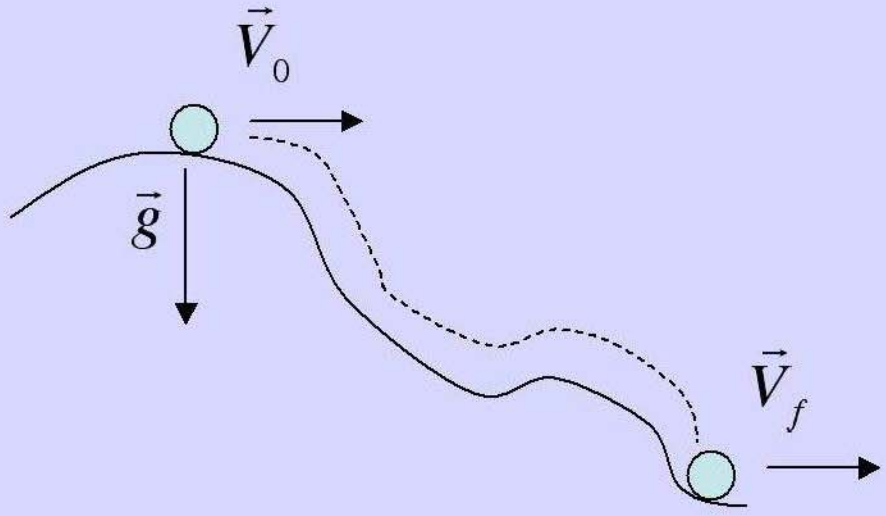
Energy Diagrams

Chapter 8

Conservation of Energy

Concepts of Work, Energy and Power are useful for Solving Complex Motion

Complex trajectories, such as a ball rolling down a bumpy hill, will have a complicated solution in the kinematic trajectory method.



Using Work and Energy allows us to relate the final velocity to the initial velocity without needing to evaluate all of the kinematics in between.

Power and Velocity

When deriving the work-kinetic energy theorem we showed that

$$\frac{dW}{dt} = \vec{F} \bullet \frac{d\vec{r}}{dt} = \vec{F} \bullet \vec{v}$$

The rate of work output is the instantaneous power,

$$\boxed{P = \frac{dW}{dt} = \vec{F} \bullet \vec{v}}$$

Example: Cycling Power

The combined mass of the cyclist and her bike is ***79kg***. (a) What power must she provide to maintain a speed of ***25km/h*** against an aerodynamic drag force of ***30N***?

Since the aerodynamic drag is (always) antiparallel to the velocity, **cos*θ* = 1** and the power requirement is simply

$$P = 30 \frac{25 \times 10^3}{3600} = 208W$$

The SI unit for power is Watt = 1 Joule / sec

Power and Velocity

Cycling Power

The combined mass of the cyclist and her bike is ***79kg***. (b) What power must she provide to maintain a speed of ***25km/h*** if in addition to the aerodynamic drag she climbs a ***5°*** incline?

To climb the hill the cyclist must also overcome the vertical force of gravity. The total power output is now

$$P = 208W + mgv\sin\theta$$

$$P = 208 + 79(9.8)\frac{25\times10^3}{3600}\sin 5^\circ = 664W$$

This is almost one horsepower!

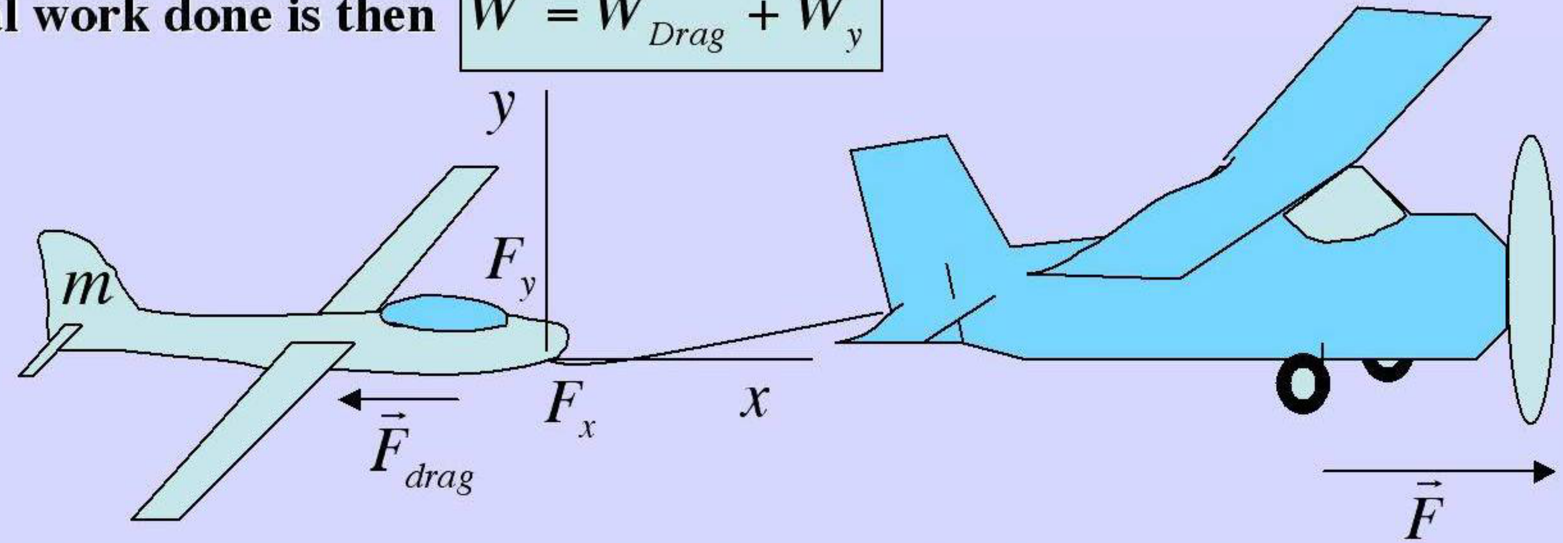
Example: Pulling a Glider

From our work with vector mechanics we know that the x and y directions are separable in force. But, Work is a scalar, and therefore the parts from different directions are simply summed to get the total. ie $\boxed{W = \vec{F} \cdot \Delta\vec{r}}$

The work to raise the glider to an altitude (at a constant F_y) is $W_y = F_y \Delta y$

The work done against drag during this time is $W_{Drag} = F_{drag} \Delta r$

The total work done is then $\boxed{W = W_{Drag} + W_y}$



Work to hold a glider against gravity at a a **constant altitude** is zero, work to pull against the drag force is still nonzero.

Average Power to Lift a Glider

A glider is lifted to a height of ***1600m*** in ***2*** minutes at a constant speed. What is the required power from the engine if the drag coefficient is $C\rho A$ = ***2.4kg/m*** and the average velocity is ***100 mph = 44.7m/s***? The total mass of the glider **+** plane is ***1400kg***.

The average power required from the plane is **Δ*W*/Δ*t*.** Separating this into the power to overcome aerodynamic drag plus gravity gives:

$$\langle P \rangle = \frac{\Delta W}{\Delta t} = \frac{\left\langle \vec{F}_{drag} \right\rangle \cdot \Delta \vec{r} + F_{lift} \Delta y}{\Delta t}$$

$\boldsymbol{F_{drag}}$ is parallel to the displacement (velocity) and $\boldsymbol{F_{lift} = m_{tot} g}$. What about **Δ*r***?

$$\Delta r = \langle v \rangle \Delta t = 44.7 \times 120 = 5364m.$$

Is this path dependent? You Betcha!

Average Power to Lift a Glider

A glider is lifted to a height of ***1600m*** in ***2*** minutes at a constant speed. What is the required power from the engine if the drag coefficient is ***CρA = 2.4kg/m*** and the average velocity is ***100 mph = 44.7m/s***? The total mass of the glider + plane is ***1400kg***.

The average drag force is:

$$\langle F_{drag} \rangle = \frac{1}{2} C\rho A v^2 = \frac{1}{2} 2.4(44.7)^2 = 2398N$$

Substituting in these results on the expression for the average power:

$$\langle P \rangle = \frac{2398\langle v \rangle \Delta t + 1400(9.8)1600}{\Delta t}$$

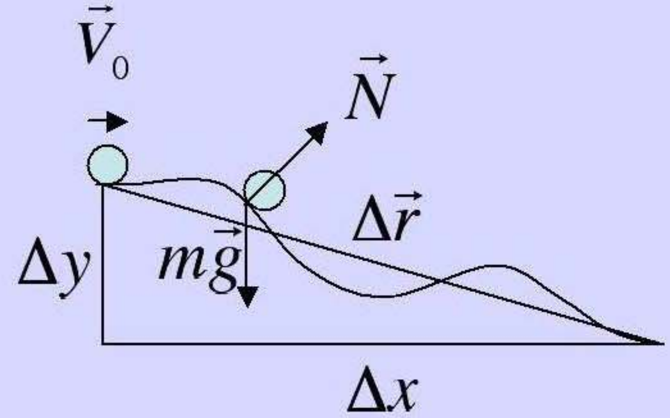
$$\langle P \rangle = 2398\langle v \rangle + 1400(9.8)\langle v_y \rangle = 2.40 \times 10^3(44.7) + 13.7 \times 10^3 \times 13.3$$

$$\langle P \rangle = 107kW + 182kW = 289.5kW$$

Note the dependence on $\boldsymbol{v}^3$. Since ***1hp = .746kW*** this is ***388hp***!

Work done by gravity is independent of path with No Friction or Drag

A ball rolling down a bumpy hill has a normal force and gravitational force acting on it at all points.



However, the normal force does no work because $\vec{N}$ is always $\perp$ to $d\vec{l}$.

$$W_N = \int \vec{N} \cdot d\vec{l} = 0$$

Thus, work done is entirely due to $m\vec{g}$:

$$W_{mg} = \int m\vec{g} \cdot d\vec{l} \neq 0$$

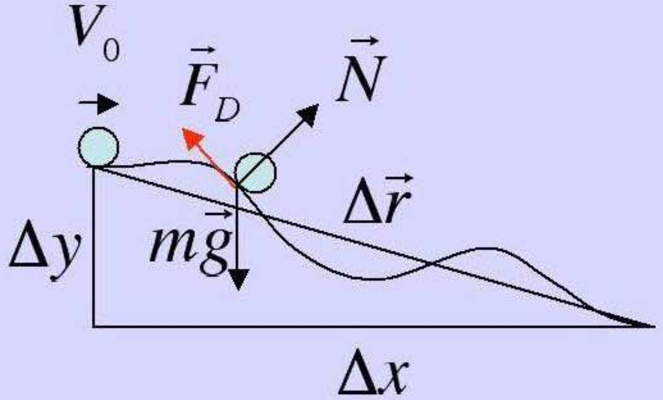
But, this can be written as the component of $d\vec{l}$ along y.

$$W = \int m\vec{g} \cdot d\vec{l} = m\int g dy = \boxed{mg\Delta y}$$

Independent of path taken and Δx

With Friction or Drag, these forces always point along path, causing work

A ball rolling down a bumpy hill with drag force also has a component of work done by this force.



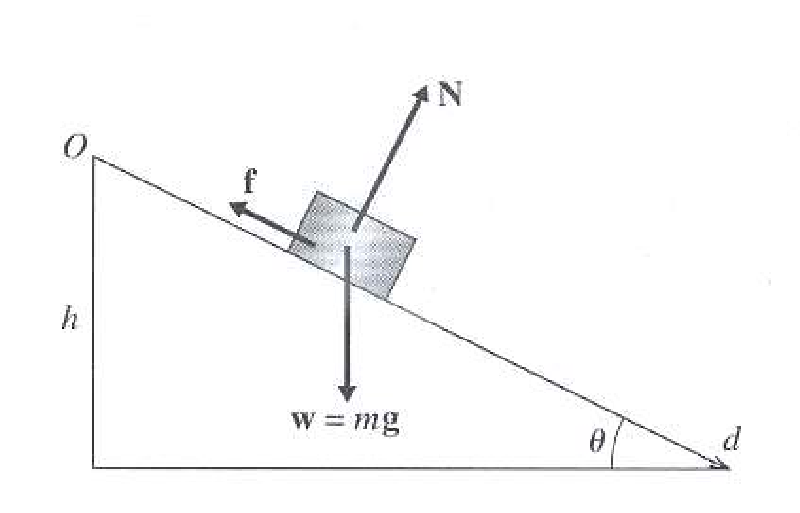
The drag (and/or friction) force is always || to $\vec{dl}$

$$W_D = \int \vec{F}_D \cdot \vec{dl} \neq 0$$

This must be added to the work done by other forces.

Note: the work done on friction and drag WILL in general depend on the path taken.

Incline Plane - Again



Find the velocity of a block sliding down an inclined plane with a coefficient of kinetic friction $\boldsymbol{\mu_k}$.

From the Work-Energy Theorem

$$\Delta K = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{net} \bullet d\vec{r} = W$$

The work done by gravity is: $W_g = mgd\sin\theta = mgh$

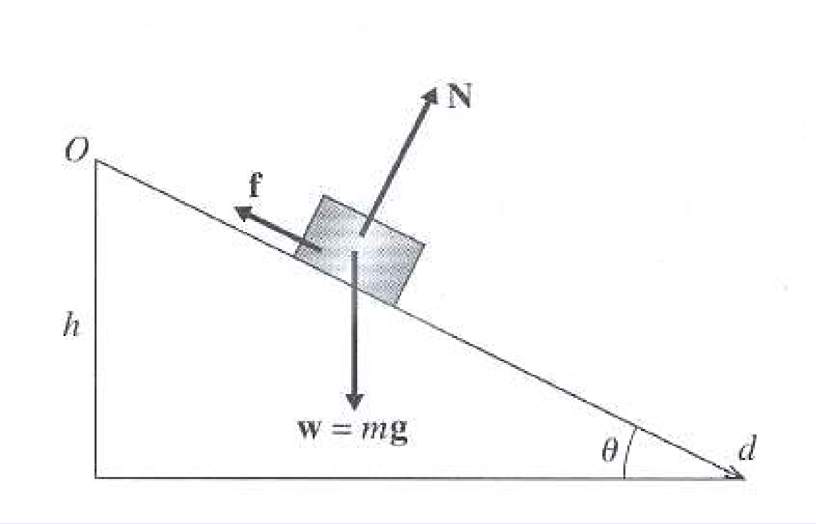
The work done by friction: $W_f = -fd = -\mu_k mgd\cos\theta = -\mu_k mgx$

The change in the kinetic energy:

$$\Delta K = \frac{1}{2}mv^2 = mgd\sin\theta - \mu_k mgd\cos\theta$$

$$v^2 = 2d(\sin\theta - \mu_k\cos\theta)g = 2ad$$

Incline Plane - Again



Find the velocity of a block sliding down an inclined plane with a coefficient of kinetic friction $\boldsymbol{\mu}_{\mathbf{k}}$.

The velocity at the bottom of the incline is what we obtain from kinematics.

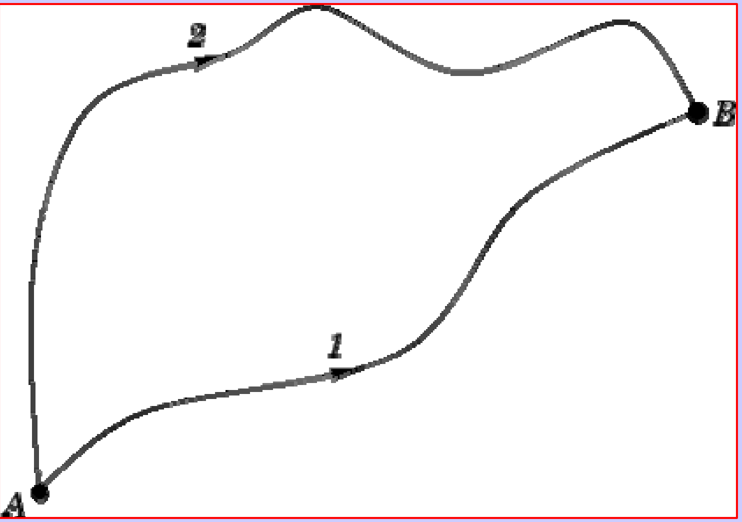
$$v^2 = 2d(\sin\theta - \mu_k \cos\theta)g = 2ad$$

However the work done by gravity depends only on the change in height! The work done by friction depends on the path, in this case the horizontal distance traveled!

$$W_f = -fd = -\mu_k mgd\cos\theta = -\mu_k mgx$$

The work done by friction is path dependent!

Conservative and Nonconservative Forces



If the work done between points **A** and **B** is path independent then we can state:

$$W_{AB} = \int_A^B \left[\vec{F} \cdot d\vec{r}\right]_1 = \int_A^B \left[\vec{F} \cdot d\vec{r}\right]_2$$

The work done by the force ***F*** going from ***A*** to ***B*** and back to ***A***, $\boldsymbol{W_{ABA}}$, is:

$$W_{ABA} = \int_A^B \left[\vec{F} \cdot d\vec{r}\right]_1 - \int_A^B \left[\vec{F} \cdot d\vec{r}\right]_2 = \int_A^B \left[\vec{F} \cdot d\vec{r}\right]_1 + \int_B^A \left[\vec{F} \cdot d\vec{r}\right]_2 = 0$$

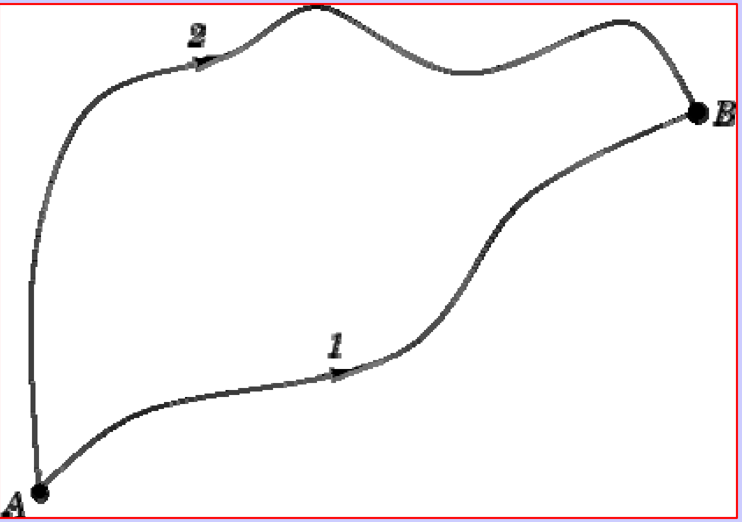
If the total work done by a force over a closed path vanishes,

$$\boxed{W_{ABA} = \oint \vec{F} \cdot d\vec{r} = 0!}$$

the force is said to be **conservative!**

Work done by frictional forces is proportional to ***–dr*** in both directions, ergo – **nonconservative!**

Potential Energy



The negative of the work done by a conservative force along any arbitrary path between two points is defined to be the change in the **potential energy** (associated with that force) between those two points.

$$\Delta U = -\int_A^B \vec{F} \cdot d\vec{r}$$

The difference in potential energy depends only on the location of the endpoints! Also the zero of U is arbitrary. It is usually chosen for convenience.

The change in gravitational potential energy:

$$\Delta U_g = -\int_0^y \left(-mg\hat{j}\right) \cdot d\vec{r} = mg\int_0^y dy = mgy$$

The change in a spring's potential energy:

$$\Delta U_k = -\int_0^x \left(-kx\hat{i}\right) \cdot d\vec{r} = k\int_0^x x dx = \frac{1}{2}kx^2$$

Conservation of Energy

From the Work-Energy Theorem the work done on an object is equal to the change in its kinetic energy:

$$W = m\int \frac{d\vec{v}}{dt} \cdot d\vec{r} = \int \vec{F}_{net} \cdot d\vec{r} = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

If we consider separately the work done by conservative forces, W_c, and non-conservative forces, $\boldsymbol{W_{nc}}$:

$$\Delta K = W_c + W_{nc}$$

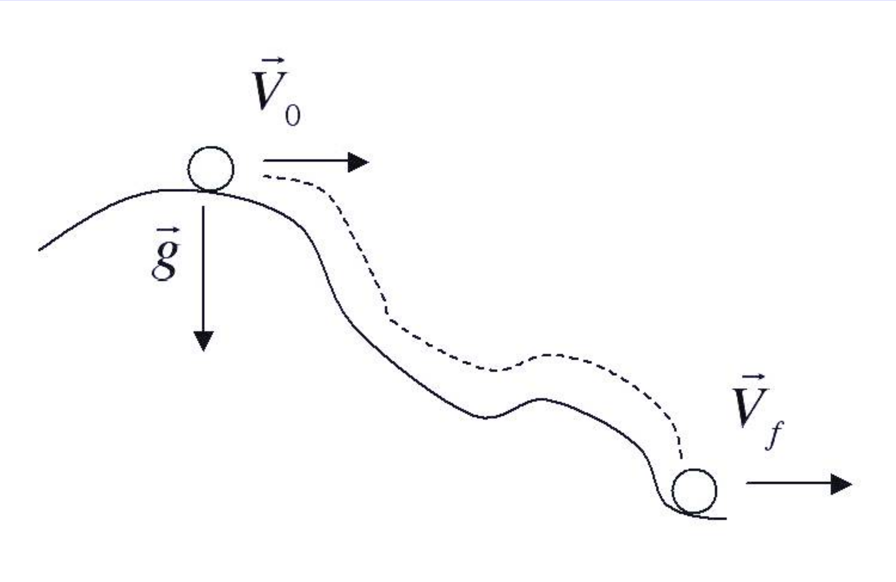
Potential energy was defined as the negative of the work done by conservative forces: $\boldsymbol{\Delta U = - W_c}$. Hence:

$$\Delta K + \Delta U = W_{nc}$$

In the absence of non-conservative forces, the total mechanical energy is conserved!

$$\boxed{\Delta K + \Delta U = 0 \rightarrow \frac{1}{2}mv_i^2 + U_i = \frac{1}{2}mv_f^2 + U_f = E}$$

Conservation of Energy in a Gravitational Field



A ball rolls (slides) down a bumpy but frictionless hill of height ***10m*** with an initial velocity of ***v = 1m/s***. Determine its final velocity.

Since gravity is a conservative force:

$$\Delta K + \Delta U = 0$$

$$K_i + U_i = K_f + U_f$$

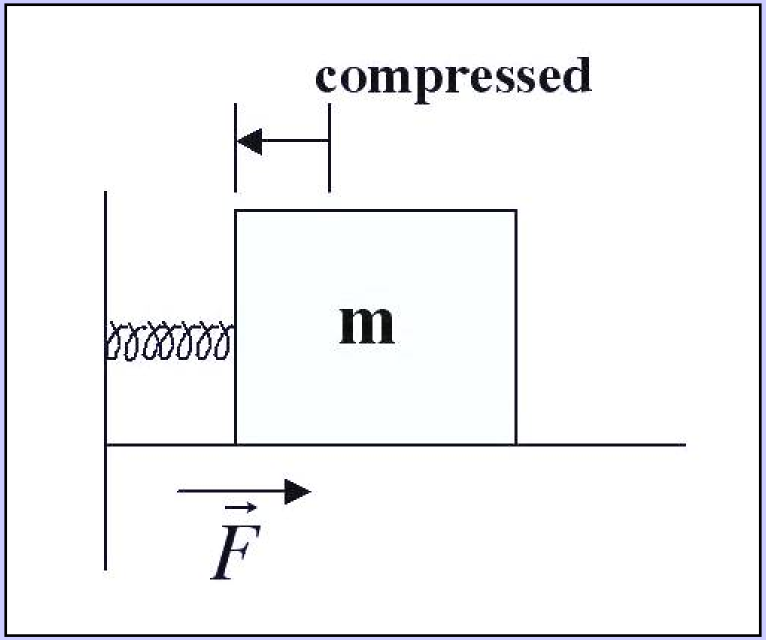
$$\frac{1}{2}mv_i^2 + mgh = \frac{1}{2}mv_f^2 + 0$$

Solving for $\boldsymbol{v_f}$:

$$v_f = \sqrt{v_i^2 + 2gh} = \sqrt{1 + 2(9.8)10} = 14m/s$$

It is important to note that this result did NOT depend on the path, just the change in height!

Conservation of Energy with a Spring



A ***1kg*** mass is compressed ***.2m*** against a spring on a flat frictionless surface. The spring constant is ***k = 15N/m***. Determine the velocity of the mass after it is released from the spring.

Since the force of the spring is a conservative force:

$$K_i + U_i = K_f + U_f$$

$$0 + \frac{1}{2}kx^2 = \frac{1}{2}mv^2 + 0$$

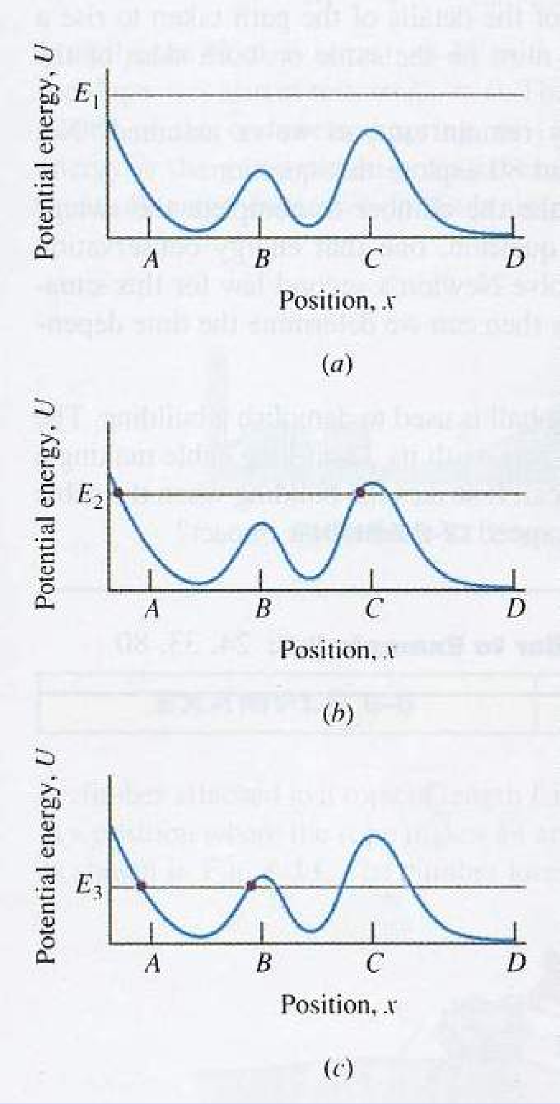
$$v = \sqrt{\frac{k}{m}}\,x$$

The velocity is:

$$v = \sqrt{\frac{15}{1}}\,.2 = .77m/s$$

Note that this is the same result that we obtained from the work-energy theorem. In either case this result easily obtained even though the force was function of distance!

Potential Energy Curves



For a conservative system the energy is conserved. In one-dimension this is expressed as:

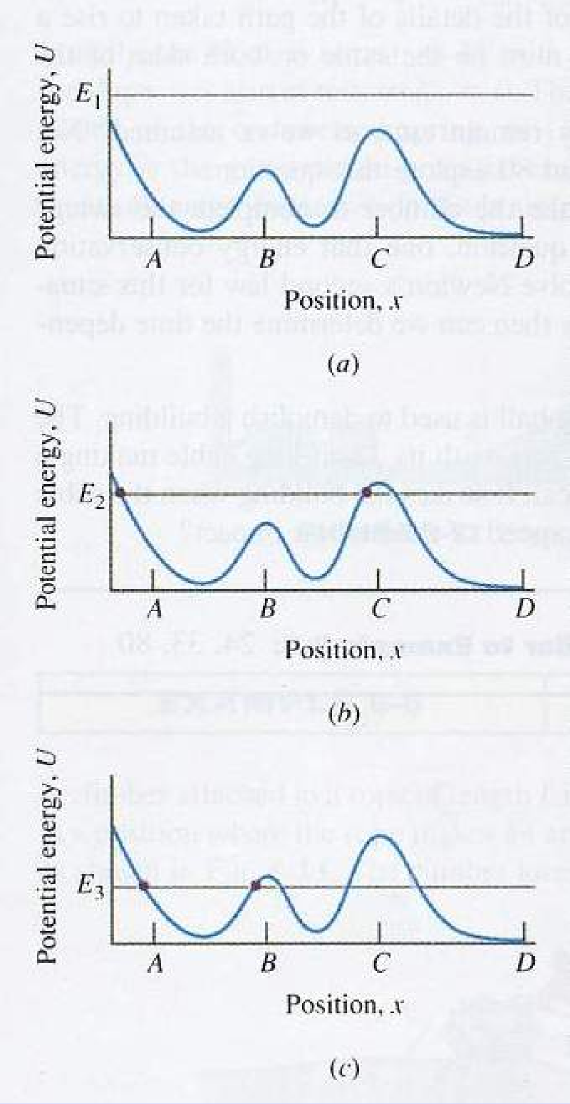
$$E = \frac{1}{2}mv^2 + U(x)$$

Since the kinetic energy is positive definite the regions where $\boldsymbol{E < U(x)}$ are forbidden. This means for a particle with energy $\boldsymbol{E_1}$ (figure (a)) there are no forbidden regions on the plot shown.

For a particle with energy $\boldsymbol{E_2}$, the particle is trapped between the two points shown in figure (b). If the particle is in the region beyond the peak at C then it can never reach the region between the two points in (b).

A particle with energy $\boldsymbol{E_3}$ is trapped between A and B. It may be trapped between the peaks at B and C or again beyond peak C.

Potential Energy Curves



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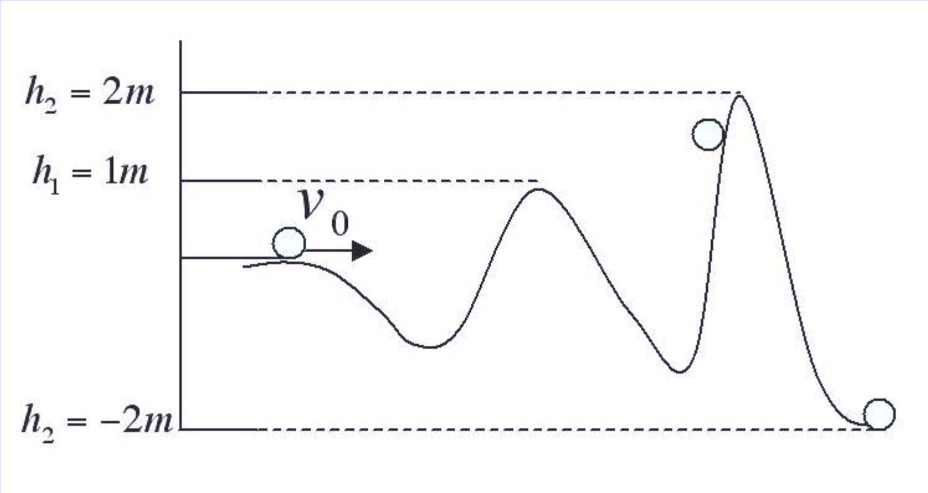
The figures shown can be thought of as a roller coaster track. The analogy is not an accident. For a roller coaster ***U(x)*** is ***mgh*** where ***h*** is the height above ground. Hence the graph of **U(x)** has the same shape as a graph of ***h*** versus ***x***, which is just a picture of the track!

In figures (b) and (c) the bold points are turning points. The object has lost all kinetic energy at those points and begins to fall back "downhill".

Example: Potential Energy Curves



A ***1kg*** ball with an initial velocity of ***6m/s*** rolls (slides) down the hilly path shown. Does the ball make it to the end of the path?

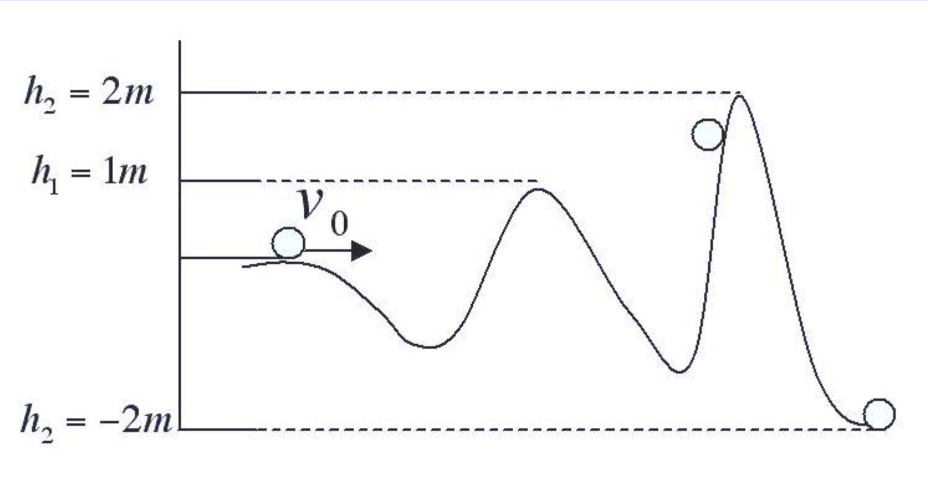
The question that we have to ask is whether or not the initial kinetic energy is greater than the potential energy required to crest the highest hill.

$$E = \frac{1}{2}mv_o^2 \overset{?}{>} mgh_2 \rightarrow v_o^2 \overset{?}{>} 2gh_2$$

$$v_o^2 = \boxed{36J/kg}, \quad 2gh_2 = 2(9.8)2 = \boxed{39.2J/kg}$$

The ball doesn't quite make it!

Example: Potential Energy Curves



A ***1kg*** ball with an initial velocity of ***7m/s*** rolls (slides) down the hilly path shown. (a) What is its velocity at the first crest? (b) At the second crest? (c) At the bottom?

From the conservation of energy:

$$E = \frac{1}{2}mv_o^2 = \frac{1}{2}mv^2 + mgh$$

(a) $v_1^2 = v_o^2 - 2gh_1 \rightarrow v_1 = 5.4m/s$

(b) $v_2^2 = v_o^2 - 2gh_2 \rightarrow v_2 = 3.1m/s$

(c) $v_{bot}^2 = v_o^2 - 2gh_{bot} \rightarrow v_{bot} = 9.4m/s$

Clearly this approach is convenient!

Force and Potential Energy

Consider a force pushing a body along the x axis. The work being done by the force is:

$$W = -\Delta U$$

We also know that

$$W = F_x \Delta x$$

Combining these two we can write:

$$F_x = -\frac{\Delta U}{\Delta x}$$

This applies to 3D motion in general:

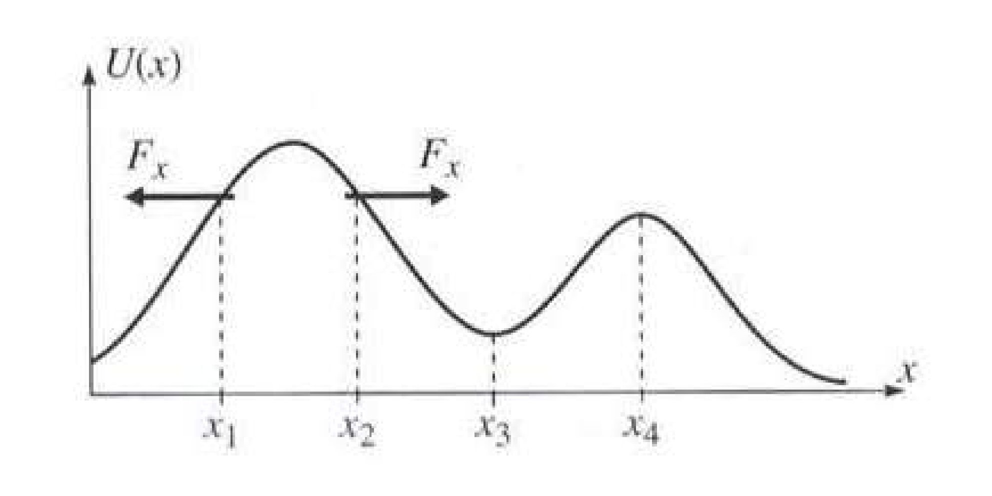
$$\vec{F} = -\left(\frac{dU}{dx}\hat{i} + \frac{dU}{dy}\hat{j} + \frac{dU}{dz}\hat{k}\right)$$

Or, in gradient notation:

$$\vec{F} = -\vec{\nabla} U$$

Here, $\vec{\nabla}$ is a vector differential operator.

Force and Potential Energy



Again think of the potential energy plot as a picture of a roller coaster. The force

$$F_x = -\frac{dU}{dx}$$

tends to push the object downhill as shown in the plot at $\boldsymbol{x=x_1}$ and $\boldsymbol{x=x_2}$.

Note that at the points $\boldsymbol{x_3}$ and $\boldsymbol{x_4}$ where $\boldsymbol{dU/dx = 0}$, $\boldsymbol{U}$ is a minimum or a maximum. The object is in equilibrium as the net force vanishes at those points. However $\boldsymbol{x_3}$ is a point of stable equilibrium (why?) and $\boldsymbol{x_4}$ is a point of unstable equilibrium (why?).

For example consider the potential energy for a spring:

$$U(x) = \frac{1}{2}kx^2 \;\rightarrow\; F = -\frac{dU}{dx} = -kx$$