Lecture 8 Kinetic theory of waves in plasma (part I)

Electrostatic waves in collisionless plasma

$$\frac{\partial f^\alpha}{\partial t} + \vec{v}\frac{\partial f^\alpha}{\partial \vec{r}} - \frac{e_\alpha}{m_\alpha}\frac{\partial \varphi}{\partial \vec{r}}\cdot\frac{\partial f^\alpha}{\partial \vec{v}} = 0 \qquad (1)$$

$$\nabla^2 \varphi = -\sum_\alpha 4\pi e_\alpha \int f^\alpha d\vec{v} \qquad (2)$$

Linearization in kinetic equation: $f^\alpha = f_0^\alpha(\vec{v}) + \delta f_\alpha(\vec{r}, t, \vec{v})$.
Poisson's equation is linear.

$$\left(\frac{\partial}{\partial t} + \vec{v}\frac{\partial}{\partial \vec{r}}\right)\delta f_\alpha = \frac{e_\alpha}{m_\alpha}\frac{\partial \varphi}{\partial \vec{r}}\cdot\frac{\partial f_{0\alpha}}{\partial \vec{v}} \qquad (3)$$

Substitute φ in form of a plane wave

$$\varphi \sim e^{i\vec{k}\vec{r} - i\omega t}$$

$$\left(\frac{\partial \delta f_\alpha}{\partial t} + \vec{v}\frac{\partial}{\partial \vec{r}}\right)\delta f_\alpha = i\,\frac{e_\alpha \varphi}{m_\alpha}\,\vec{k}\,\frac{\partial f_{0\alpha}}{\partial \vec{v}}\, e^{i(\vec{k}\vec{r} - \omega t)}$$

General solution = Particular solution of non-homogeneous eq-n + general solution of homogeneous eq-n

$$\delta f_\alpha = \frac{e_\alpha}{m_\alpha}\,\frac{\varphi}{\vec{k}\cdot\vec{v} - \omega}\,\vec{k}\cdot\frac{\partial f_{0\alpha}}{\partial \vec{v}}\, e^{i(\vec{k}\vec{r} - \omega t)} + g_\alpha(\vec{r} - \vec{v}t) \qquad (4)$$

g_α can be found from initial conditions

Assuming initial condition

$$\delta f_\alpha(t=0, \vec{r}, \vec{v}) = 0$$

we obtain

$$\delta f_\alpha(t, \vec{r}, \vec{v}) = \frac{e_\alpha}{m_\alpha} \frac{1}{\vec{k}\vec{v}-\omega}\Big[\varphi\, e^{i(\vec{k}\vec{r}-\omega t)} - \qquad (5)$$

$$- \varphi\, e^{i\vec{k}(\vec{r}-\vec{v}t)}\Big]\vec{k}\cdot\frac{\partial f_{0\alpha}}{\partial \vec{v}}$$

To obtain dispersion relation we will substitute $f^\alpha = f_{0\alpha} + \delta f_\alpha$ into (2).

$$-k^2\varphi = -\sum_\alpha 4\pi e_\alpha \int \delta f_\alpha \, d\vec{v} \qquad (6)$$

$$k^2\varphi = \sum_\alpha \frac{4\pi e_\alpha^2}{m_\alpha} k \Bigg[\int \frac{\frac{dF_{0\alpha}}{dv_\parallel}}{kv_\parallel-\omega}\, dv_\parallel\, \varphi\, e^{i\vec{k}\vec{r}-i\omega t} - \qquad (7)$$

$$-\varphi\int \frac{\frac{dF_{0\alpha}}{dv_\parallel}}{kv_\parallel-\omega}\, e^{i\vec{k}\vec{r}-ik_\parallel v_\parallel t}\, dv_\parallel \Bigg]$$

$$F_{0\alpha}(v_\parallel) = \int d\vec{v}_\perp\, f_{\alpha 0}(\vec{v}); \quad v_\parallel = \frac{\vec{k}\cdot\vec{v}}{k}; \quad \vec{v}_\perp\cdot\vec{k} = 0$$

Second term in r.h. side of (7) decays in time due to phase mixing of exponents corresponding to different velocities.
Characteristic time scale is

$$t_1 = \frac{1}{k\Delta v} \tag{8}$$

t_1 - time of the phase memory. For $t \gg t_1$ all information about initial conditions is lost in wave electric potential (field), but not in δf!
Thus we can use asymptotic dispersion relation describing the dynamics of waves for times $t \sim \frac{1}{\gamma}$, $\gamma = \mathrm{Im}\,\omega$.

$$1 - \sum_\alpha \frac{4\pi e_\alpha^2}{mk} \int \frac{\frac{dF_{0\alpha}}{dv_\parallel}}{kv_\parallel - \omega}\, dv_\parallel = 0 \tag{9}$$

This relation doesn't depend on initial conditions if $\frac{1}{\gamma} \gg t_1 \approx \frac{1}{k\Delta v}$ or

$$\gamma \ll k\Delta v \tag{10}$$

Example:

Find behavior of the integral

$$I=\left(\frac{m}{2\pi T}\right)^{1/2}\int_{-\infty}^{\infty} dv_{\parallel}\, e^{-\frac{m v_{\parallel}^2}{2T}}\, e^{-i k v_{\parallel} t}$$

as function of time t.

$$I=\left(\frac{m}{2\pi T}\right)^{1/2}\int_{-\infty}^{\infty} dv_{\parallel}\, \exp\left\{-\left(\frac{v_{\parallel}}{\sqrt{2}\,v_T}+i\,\frac{k v_T t}{\sqrt{2}}\right)^2-\frac{k^2 v_T^2}{2}t^2\right\}$$

$$= e^{-\frac{k^2 v_T^2}{2}t^2}\,\frac{1}{\sqrt{2\pi}\,v_T}\int_{-\infty}^{\infty} dv_{\parallel}\, \exp\left[-\left(\frac{v_{\parallel}}{\sqrt{2}\,v_T}+i\,\frac{k v_T t}{\sqrt{2}}\right)^2\right] =$$

$$= e^{-\frac{k^2 v_T^2 t^2}{2}};$$ I decreases with increase of t.

For $t \gg t_1 = \frac{\sqrt{2}}{k v_T}$ $\quad I \to 0$

Vlasov theory of electron plasma waves

Vlasov did not see that there is a pole in integral in equation (9). What he did is he assumed that $\frac{\omega}{k} \gg v_{Te} \gg v_{Ti}$ and integrated $\int_{-\infty}^{\infty} dv_{\parallel} \cdots$ Because main part of particles has velocities $v_{\parallel} \lesssim v_{Te}$ it is possible to use Taylor series for $1/(\omega - k v_{\parallel})$

$$\frac{1}{\omega - kv_\parallel} = \frac{1}{\omega}\left(1 + \frac{kv_\parallel}{\omega} + \left(\frac{kv_\parallel}{\omega}\right)^2 + \left(\frac{kv_\parallel}{\omega}\right)^3 + \cdots\right) \qquad (11)$$

$$\int_{-\infty}^{\infty} \frac{\frac{\partial F_{0\alpha}}{\partial v_\parallel}}{kv_\parallel - \omega}\, dv_\parallel = -\frac{1}{\omega}\int_{-\infty}^{\infty} \frac{\partial F_{0\alpha}}{\partial v_\parallel}\left(1 + \frac{kv_\parallel}{\omega} + \left(\frac{kv_\parallel}{\omega}\right)^2 + \frac{(kv_\parallel)^3}{\omega^3} + \cdots\right) dv_\parallel$$

$$F_{0\alpha} = \frac{1}{(2\pi)^{1/2} v_{T\alpha}}\, e^{-\frac{v_\parallel^2}{2v_{T\alpha}^2}}$$

$$1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\left(1 + 3k^2\frac{v_{T\alpha}^2}{\omega^2}\right) = 0 \qquad (12)$$

Neglect input of ions

$$\omega^2 \approx \omega_{pe}^2\left(1 + 3k^2\lambda_{De}^2\right) \qquad \lambda_{De}^2 = \frac{v_{Te}^2}{\omega_{pe}^2}$$

$$\omega = \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_{De}^2\right) \qquad (13)$$

Compare with (7.11), (7.12)

Ion - sound waves

Assume $3v_{Ti} \lesssim \frac{\omega}{k} \ll v_{Te}$

Ions: $$\int \frac{dv_\parallel \frac{\partial F_{0i}}{\partial v_\parallel}}{kv_\parallel - \omega} = \frac{nk}{\omega^2} \qquad (14)$$

Electrons: $\omega \ll k v_{Te}$

$$\int \frac{dv_\parallel \frac{\partial F_{oe}}{\partial v_\parallel}}{k v_\parallel - \omega} = -\int dv_\parallel \frac{\frac{v_\parallel}{v_{Te}^2} F_{oe}}{k v_\parallel - \underset{\text{neglect}}{\cancel{\omega}}} \approx - \frac{n}{k v_{Te}^2} \qquad (15)$$

Substitute (14) and (15) in (9):

$$1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{k^2 v_{Te}^2} = 0 \qquad (16)$$

$$\omega^2 = \frac{\omega_{pi}^2 k^2 \lambda_{De}^2}{1 + k^2 \lambda_{De}^2} = \frac{k^2 c_s^2}{1 + k^2 \lambda_{De}^2} \qquad (17)$$

Compare with (7.26)

Landau damping

The most non-trivial issue in dispersion relation (9) is presence of singularity in the integrand at the resonance point

$$v_\parallel = \omega / k \qquad (18)$$

(Cherenkov resonance condition).

Resonant phenomena are important and widespread in plasma physics.

We have already discussed one type of resonance (see lecture 4) while considering <u>transformation of electromagnetic waves into electrostatic plasma waves</u>.

We studied a spatial resonance

$$\omega = \omega_p(x) \qquad (19)$$

We found that longitudinal electric field had a singularity at the point of spatial resonance (19).

$$E_z = \frac{B_x(0)\sin\theta}{\mathcal{E}(z)} \qquad (20)$$

$$\mathcal{E}(z) = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{z}{L}\right) = -\frac{z}{L}$$

We introduced small collision frequency to remove the singularity. As a result we obtained

$$E_z = \frac{B_x(0)\sin\theta}{-\frac{z}{L} + i\frac{\nu}{\omega}} = B_x \frac{1}{(2\pi)^{1/2}}\left(\frac{c}{\omega L}\right)^{1/2} \Phi(\tau)\, \frac{1}{-\frac{z}{L} + i\frac{\nu}{\omega}} \qquad (21)$$

We calculated the coefficient of transformation as ratio of the total dissipation power to the energy flux in the incident wave

$$K = \frac{P}{\frac{c}{4\pi}B_x^2} = \frac{1}{\frac{c}{4\pi}B_x^2}\,\nu \int \frac{|E_z|^2}{8\pi}\,dz = \frac{1}{4\pi}\,\frac{1}{\omega L}\,\Phi^2 \int_{-\infty}^{\infty} \frac{\nu}{\frac{z^2}{L^2} + \frac{\nu^2}{\omega^2}}\,dz \qquad (22)$$

$$K = \frac{1}{4\pi}\,\frac{\omega}{L}\,\phi^2 \int_{-\infty}^{\infty} \frac{\nu}{z^2 \frac{\omega^2}{L^2} + \nu^2}\, dz$$

We used that $\lim\limits_{\nu \to 0} \frac{\nu}{z^2 \frac{\omega^2}{L^2} + \nu^2} = \pi \delta\left(\frac{z\omega}{L}\right)$ (23)

As a result we obtained

$$K = \frac{1}{4}\phi^2 \qquad (24)$$

The final result doesn't depend on collision frequency.

High-frequency plasma oscillations

Ions are motionless. Equation (9) has a form:

$$1 = \frac{4\pi e^2}{m_e k^2} \int \frac{\frac{\partial F_{oe}}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}}\, dv_\parallel \qquad (25)$$

Similarly, we introduce very rare collisions to show how to treat singularity due to resonance in velocity space $v_\parallel = \omega/k$.

Let us treat collisions in τ-approximation

$$\frac{df}{dt} = -\frac{f - f_0}{\tau} \equiv -\delta f \nu \Rightarrow \left(-i\omega \delta f \rightarrow (-i\omega + \nu)\delta f\right)$$

$$\Rightarrow (\ \omega \longrightarrow \omega + i\nu)$$

Replacing ω by $\omega + i\nu$ in (25) we obtain :

$$\frac{1}{v_{\parallel} - \omega/k} \longrightarrow \frac{1}{v_{\parallel} - \frac{\omega + i\nu}{k}} \qquad (26)$$

In (26) singularity is in the upper part of complex $v_{\parallel}$-plane. As a result while integrating over real exis $\int_{-\infty}^{\infty} dv_{\parallel} \frac{G(v_{\parallel})}{v_{\parallel} - \frac{\omega + i\nu}{k}}$ we are moving below singularity (Fig. 1)

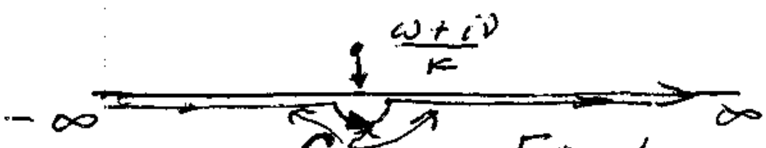


Fig. 1

When $\nu \to 0$, the singularity moves towards the real exis and contour of integration must be deformed in a way shown in the Fig. 1 in order not to destroy analytical properties of integral.

But integral over contour C is sum of a principal value (from which the singularity is excluded) and a half of Cauchy integral over contour surrounding the singularity.

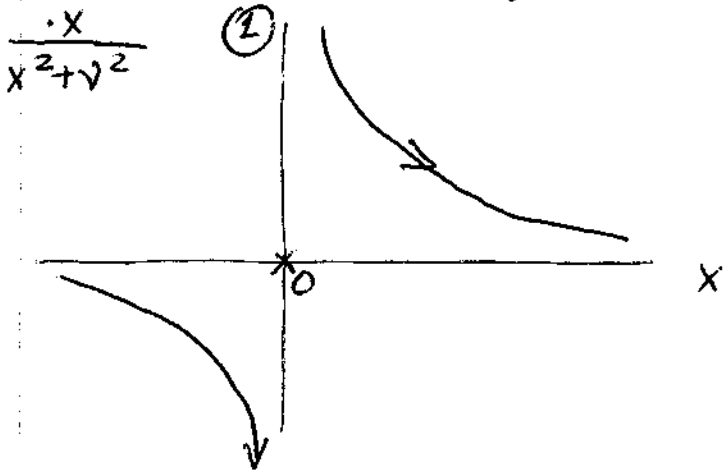
$$\int dv_\parallel \frac{G(v_\parallel)}{v_\parallel - \omega/k} = P.V.\int \frac{dv_\parallel}{v_\parallel-\omega/k}\, G(v_\parallel) + i\pi\, G\left(\frac{\omega}{k}\right) \qquad (27)$$

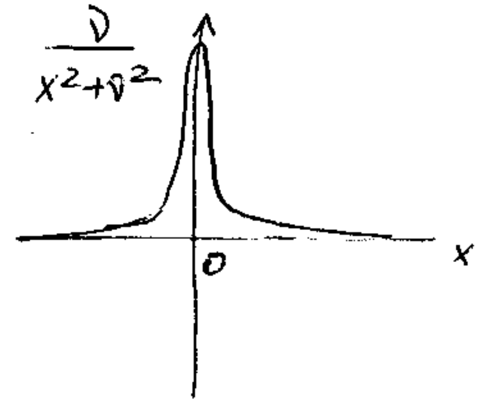
Another way to obtain the same result

$$\frac{1}{kv_\parallel-\omega-i\nu} = \underbrace{\frac{kv_\parallel-\omega}{(kv_\parallel-\omega)^2+\nu^2}}_{(1)} + i\,\underbrace{\frac{\nu}{(kv_\parallel-\omega)^2+\nu^2}}_{(2)}$$

For small ν this term corresponds to principal value of integral

This term for $\nu \to 0$ is equal to $i\pi\,\delta(kv_\parallel-\omega)$





Using relation (27), we obtain instead of (25):

$$1 = \frac{4\pi e^2}{mk^2}\, P.V.\int \frac{dv_\parallel}{v_\parallel-\omega/k}\frac{\partial F_{0e}}{\partial v_\parallel} + i\,\frac{4\pi^2 e^2}{m_e k^2}\frac{\partial F_{0e}}{\partial v_\parallel}\left(\frac{\omega}{k}\right) \qquad (28)$$

For high-frequency electron plasma oscillations, first term in r.h. side of (28) was calculated by Vlasov (see (11), (12)).

As result, we obtain from (28)

$$1 = \frac{\omega_{pe}^2}{\omega^2}\left(1 + 3k^2\frac{T}{m_e\omega^2}\right) + i\,\frac{4\pi^2 e^2}{m_e k^2}\frac{\partial F_{oe}}{\partial v_{\parallel}}\left(\frac{\omega}{k}\right) \quad (29)$$

To satisfy this equation, ω must be complex

$$\omega = \omega_R + i\gamma \quad (30)$$

$$e^{-i\omega t} \sim e^{-i\omega_R t + \gamma t}$$

$\gamma < 0$ – exponential decay

$\gamma > 0$ exponential growth

Assuming $|\gamma| \ll \omega_R$, we have

$$\omega_R = \omega_{pe}\left(1 + 3k^2\lambda_{De}^2\right)$$

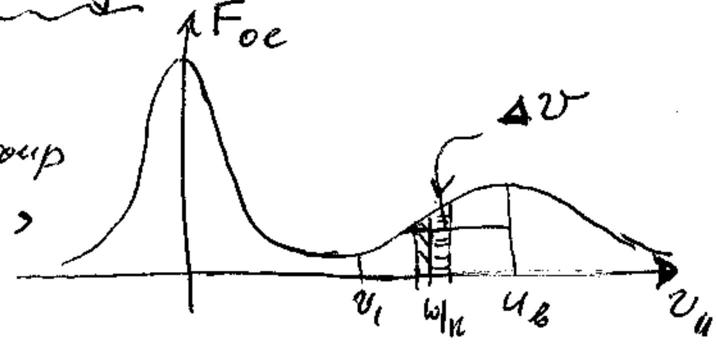
$$\gamma = \frac{2\pi^2 e^2}{m k^2}\,\omega_p\,\frac{\partial F_{oe}}{\partial v_{\parallel}}\left(\frac{\omega}{k}\right) \quad (31)$$

For Maxwellian distribution:

$$\gamma = -\sqrt{\frac{\pi}{8}}\,\omega_p\,\frac{1}{k^3\lambda_{De}^3}\exp\left\{-\frac{1}{2k^2\lambda_{De}^2} - \frac{3}{2}\right\} \quad (32)$$

Bump-on-tail instability

If we have bump-on-tail distribution (plasma with group of energetic particles (beam)), $\gamma > 0$ for $v_1 < \omega/k < u_B$



because $\frac{\partial F_{0e}}{\partial v_{\parallel}} > 0$ in this interval.

Growth instead of damping of the wave amplitude will take place. If n_b is the number density in the beam, We can write for the beam distribution

$$F_0 \approx \frac{n_b}{\Delta v}, \quad n_b \ll n_0 \qquad (33)$$

n_0 - plasma density.

Δv - width of the beam velocity distribution

$$\frac{\partial F_0}{\partial v_{\parallel}} \sim \frac{n_b}{(\Delta v)^2}$$

$$\gamma \approx \frac{\pi}{2} \omega_{pe} \frac{u_b^2}{(\Delta v)^2} \frac{n_b}{n_0} \qquad (34)$$

where u_b - is the beam velocity $\left(u_B \approx \frac{\omega_{pe}}{k}\right)$

Condition of application of acymptotic kinetic description (10) has the form

$$\left(\frac{n_b}{n_0}\right)^{1/3} \ll \frac{\Delta v}{u_b}$$

As discussed before (see lecture 1), Landau damping arises as a result of collisionless energy exchange between wave and resonant particles.