Lecture 7

Derivation of the dispersion relation for electron plasma waves (Langmuir waves) from fluid equations

We neglect collisions in this problem. Equations of two fluid hydrodynamics have the form (see Lecture 6

$$\frac{\partial n_\alpha}{\partial t} + div(n_\alpha \vec{u}_\alpha) = 0 \qquad (1)$$

$$m_\alpha n_\alpha \left(\frac{\partial}{\partial t} + (\vec{u}^\alpha \cdot \nabla) \right) \vec{u}_\alpha = q_\alpha n_\alpha \vec{E} - \frac{\partial P_\alpha}{\partial \vec{r}} \qquad (2)$$

Here $\vec{E}$ is the wave electric field.

We found before that electron plasma waves are so called longitudinal waves, since the wave electric field is parallel to the direction in which the charge separation takes place. This is just direction of the wave propagation. The wave magnetic field in such wavesis equal zero, so these waves are electrostatic waves. For such waves $\vec{E} = -\nabla \varphi$, φ is the wave electric potential. Electric field $\vec{E}$ is determined selfconsistently by equation

$$\nabla \cdot \vec{E} = \sum_\alpha 4\pi q_\alpha n_\alpha \qquad (3)$$

Very important simplification: We consider here only small amplitude (linear) oscillations (waves). The main procedure in this case is *linearization* of equations (1), (2). We assume that without oscillations plasma is stationary, homogeneous and has no average drift velocity $\vec{u}_\alpha = 0$. If a wave with electric field $\vec{E}$ propagates in plasma, it creates perturbations of plasma parameters

$$\begin{aligned} n_\alpha(t,\vec{r}) &= n_0 + \delta n_\alpha(t,\vec{r}) \\ \vec{u}_\alpha(t,\vec{r}) &= \delta\vec{u}_\alpha(t,\vec{r}) \end{aligned} \qquad (4)$$

Then

$$\frac{\partial P_\alpha}{\partial \vec{r}} = \gamma_\alpha P_\alpha \frac{1}{n_\alpha} \frac{\partial \delta n_\alpha}{\partial \vec{r}} \qquad (5)$$

Linearization means that in equations (1), (2) we will omit all terms nonlinear in wave amplitude. We are looking for solution in the form of a plane wave:

$$\delta n_\alpha, \delta\vec{u}_\alpha, \vec{E} \sim \exp(i\vec{k}\vec{r} - i\omega t)$$

As a result, we obtain from Equations (1)-(3) the following system of linear equations:

$$-i\omega\delta n_\alpha + i\vec{k}\cdot\delta\vec{u}_\alpha = 0 \quad (1')$$

$$-i\omega m_\alpha n_0 \delta\vec{u}_\alpha = e_\alpha n_0 \vec{E} - \gamma_\alpha k_B T_\alpha i\vec{k}\cdot\delta n_\alpha \quad (2')$$

$$i\vec{k}\cdot\vec{E} = -4\pi e(\delta n_e - \delta n_i) \quad (3')$$

From Equation (1'):

$$\vec{k}\cdot\delta\vec{u}_\alpha = \omega\frac{\delta n_\alpha}{n_0} \quad (6)$$

By multiplying (2') by $\vec{k}\,\cdot$ and substituting (6), we obtain

$$\delta n_\alpha\left(1 + \gamma_\alpha \frac{k_B T_\alpha}{m_\alpha}\frac{k^2}{\omega^2}\right) = \frac{e_\alpha n_0}{m_\alpha \omega^2}\vec{k}\vec{E} \quad (7)$$

From (3') and (7) we obtain

$$\vec{k}\vec{E}\left(1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \gamma_\alpha k^2\, k_B T_\alpha / m_\alpha}\right) = 0 \quad (8)$$

Condition when $\vec{k}\cdot\vec{E} \neq 0$, i.e. the wave electric field of electrostatic wave is not equal zero has the form

$$1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \gamma_\alpha k^2\, k_B T_\alpha / m_\alpha} = 0 \quad (9)$$

This is dispersion relation of electron plasma waves.

Electron plasma waves occur in phase velocity range

$$\frac{\omega}{k} \geq 3v_{Te} >> v_{Ti} \quad (10)$$

necessary to avoid strong Landau damping.

From (9) we have

$$\omega^2 = \omega_{pe}^2\left(1 + \gamma_\alpha\, k^2 v_{Te}^2 / \omega^2\right) + \omega_{pi}^2 \quad (11)$$

Comments:

1. The last term in (11) is very small and can be neglected. That means that ions contribute negligible amount because they can't respond effectively at high frequency ($\omega \approx \omega_{pe}$).
2. Dispersiion relation (11) can be written in form of equation for mass attached to an ideal spring $\delta\ddot{x} = -(K/m)\delta x$ where δx is displacement of the mass.

2a ω_{pe}^2 represents an electric contribution to spring constant

2b. $\gamma_\alpha k^2 v_{Te}^2$ represents electron pressure contribution to spring constant

Determination of γ_e

Since $\omega >> kv_{Te}$, the compression is adiabatic

$$\left|\frac{\partial \delta T}{\partial t}\right| >> v_{Te}\left|\frac{\partial \delta T}{\partial x}\right|$$

and $\gamma = (f+2)/f$.

Since there is no collisions to produce equipartition between the three velocity components, $f = 1 \Rightarrow \gamma = 3$.

As a result,

$$\omega^2 = \omega_{pe}^2\left(1 + 3k^2 v_{Te}^2/\omega_{pe}^2\right) \Rightarrow \omega = \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_{De}^2\right) \qquad (12)$$

Dispersion relation for ion-sound (ion-acoustic) waves

As in any continuous media, the forces of gas kinetic pressure play role of elastic force for ion-sound oscillations (waves) in plasma. We can estimate the velocity of ion-sound waves by generalizing the expression for sound speed in gas and applying it to two-component plasma:

$$c_s^2 = \frac{\partial P}{\partial \rho} = \frac{\partial (P_e + P_i)}{\partial \rho}$$

For adiabatic derivative

$$c_s^2 = \gamma \frac{T_e + T_i)}{m_i} \tag{13}$$

Anomalous peculiarity of plasma appear for small wave length. Sound waves in gas exist when their wavelength is much larger than mean free path. Usual sound damps very strongly when λ approaches $l_{m.f.p.}$ and doesn't exist when $\lambda << l_{m.f.p.}$.

However, ion-sound waves can exist in collisionless plasma as well. In plasma, particles interact through selfconsistent fields. We have seen that such an electric field leads to high frequency Langmuir oscillations of electrons. Similarly, there is a possibility of existence of electrostatic sound-type oscillations in which ions participate as well, if the oscillation frequency is sufficiently small. Thus, the elastic coupling between electrons and ions is achieved due to selfconsistent electric field.

We will assume that ion-sound waves occur in phase velocity range

$$3v_{Ti} \leq \frac{\omega}{k} << v_{Te} \Rightarrow T_e >> T_i \tag{14}$$

The first condition is necessary to avoid strong Landau damping on ions. The second condition is necessary to avoid Landau damping on electrons. The last relationship between electron and ion temperatures means that ion-sound waves can propagate only in so called non-isothermal plasma.

Condition $\omega << kv_{Te}$ in (14) means that perturbation created by the wave is quasi-static for electrons and they are described by Boltzman distribution

$$n_e = n_0 \exp\left(e\varphi / k_B T\right) \tag{15}$$

φ is the wave electric potential, $\vec{E} = -\nabla\varphi$.

Equations for ions

$$\frac{\partial n_i}{\partial t} + div(n_i \vec{u}_i) = 0 \tag{16}$$

$$m_i n_i \left(\frac{\partial}{\partial t} + (\vec{u}_i \cdot \nabla)\right)\vec{u}_i = -en_i \nabla\varphi - \nabla P_i \tag{17}$$

Once more we will use here the linearization procedure that means

$$\delta n << n_0;\ e\varphi << k_B T_e, m_e \frac{\omega^2}{k^2};\ \delta u << \frac{\omega}{k};\ \xi << \lambda \sim \frac{1}{k} \tag{18}$$

Here ξ is the ion displacement.

As a result, we have from Equations (15)-(17):

$$\delta n_e = n_0 \frac{e\varphi}{k_B T}$$

$$m_i \frac{\partial \delta \vec{u}_i}{\partial t} = -e\nabla\varphi - \gamma_i \frac{k_B T_i}{n_0} \nabla \delta n_i \tag{19}$$

$$\frac{\partial \delta n_i}{\partial t} + n_0 \frac{\partial \delta \vec{u}_i}{\partial \vec{r}} = 0$$

We will neglect the last term in the second equation in (19) assuming that $\omega/k >> v_{Ti}$.

For $\delta n, \delta \vec{u}_i, \varphi \sim \exp(i\vec{k}\vec{r} - i\omega t)$ we have from (19)

$$\delta \vec{u}_i = \frac{e}{m_i \omega} \vec{k} \varphi$$

$$\delta n_i = n_0 \frac{\vec{k} \cdot \delta \vec{u}_i}{\omega} = \frac{e n_0}{m_i \omega^2} k^2 \varphi \tag{20}$$

We should use Poissoin equation for the wave potential φ to obtain dispersion relation

$$\nabla^2 \varphi = -4\pi e(\delta n_i - \delta n_e) \tag{21}$$

Consider long wavelength oscillations $k\lambda_{De} << 1$.

It is easy to see that the left hand side term in (20) is much smaller than each of two terms in the right hand side. So, these oscillations can be considered as quasineutral

$$\delta n_e = \delta n_i = \delta n \tag{22}$$

From first equation in (19) and equations (20) and (22), we obtain the dispersion relation for long wavelength ion-sound waves:

$$\omega^2 = k^2 \frac{k_B T_e}{m_i}, \quad \Rightarrow \quad \frac{\omega}{k} = c_s, \quad c_s = \sqrt{\frac{k_B T_e}{m_i}} \tag{23}$$

Important: *Quasineutrality condition (22) doesn't mean that l. h. side of equation (21) is equal zero! It means that the term* $k^2\varphi$ *is much smaller than each of terms in the r.h. side of (21)* $4\pi e\delta n_\alpha$.

Landau Damping on ions:

Condition $\frac{\omega}{k} \geq 3v_{Ti}$ in (14) is equivalent to $T_e \geq 10T_i$ and ensures a weak Landau damping on ions.

Landau Damping on electrons:

Ratio of the phase speed of ion-sound waves to electron thermal velocity is small

$$\frac{\omega}{kv_{Te}} \approx \left(\frac{m_e}{m_i}\right)^{1/2} << 1$$

providing a weak Landau damping on electrons, since in this case the number of resonant electrons with velocities smaller than the wave speed and that of resonant electrons with velocities larger than the wave phase speed are approximately equal (see Figure 1).

$$\left.\frac{\partial f_0^e}{\partial v_x}\right|_{v_x=\omega/k} \approx \frac{\omega}{kv_{Te}^2} f_0^e \Rightarrow \frac{\gamma}{\omega} \approx \left(\frac{m_e}{m_i}\right)^{1/2} << 1 \qquad (24)$$

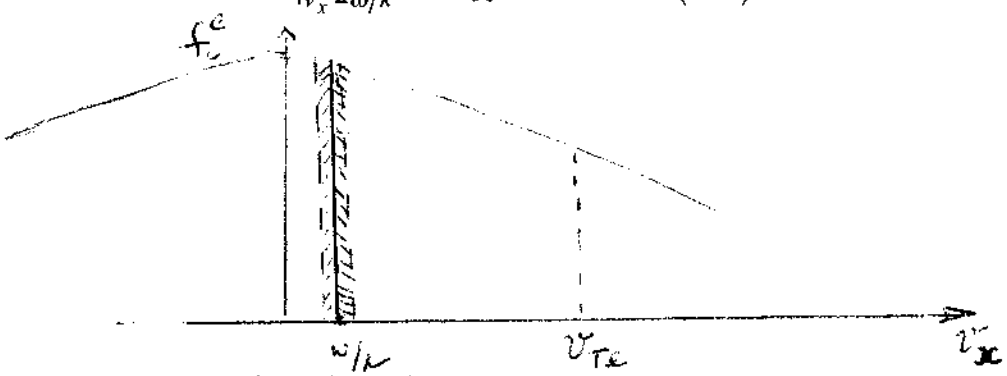

Fig. 1

Dispersion relation of ion-sound waves in general case can be obtained from (9) using condition (14).

Determinations of γ_e and γ_i in (9).

For ions $\omega >> kv_{Ti}$ => $|\partial T_i/\partial t| >> v_{Ti}|\partial T_i/\partial x|$, so compression is adiabatic $\gamma_i = (f+2)/f = 3$.

For electrons $\omega << kv_{Te}$ => $|\partial T_e/\partial t| << v_{Te}|\partial T_e/\partial x|$

Temperature of electrons is equalized very rapidly, processes of compression and decompression are isothermal.

$\gamma_e = 1$

The dispersion relation of ion-sound waves has the following form:

$$1 + \frac{\omega_{pe}^2}{k^2 v_{Te}^2} - \frac{\omega_{pi}^2}{\omega^2} = 0 \qquad (25)$$

or

$$\omega^2 = \frac{\omega_{pi}^2}{1 + \frac{1}{k^2 \lambda_{De}^2}} = \frac{k^2 c_s^2}{1 + k^2 \lambda_{De}^2} \qquad (26)$$

From (26) we obtain dispersion relation in two limit cases

$$\omega = \begin{cases} kc_s & \text{for} \quad k^2 \lambda_{De}^2 << 1 \\ \omega_{pi} & \phantom{\text{for}} \quad k^2 \lambda_{De}^2 >> 1 \end{cases} \qquad (27)$$

So now we know all three branches of waves that can propagate in homogeneous plasma without magnetic field:

(i) high frequency electrostatic electron plasma waves

(ii) high frequency electromagnetic waves

(iii) low frequency electrostatic ion-sound waves

Branches of oscillations in homogeneous isotropic plasma are shown in Figure 2.

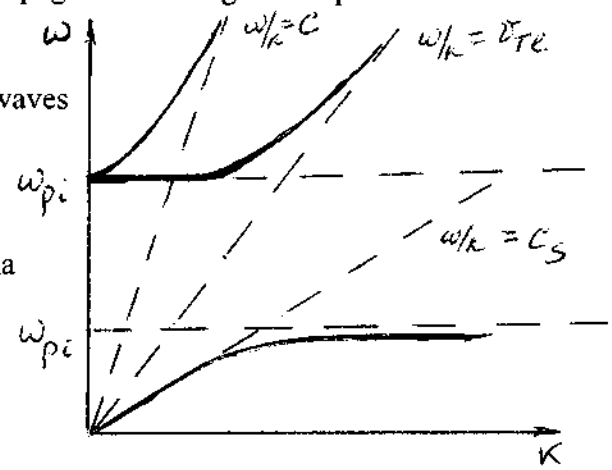

Fig. 2

Physical interpretation of ion-sound mode

1. $k\lambda_{De} >> 1$ => $\omega \approx \omega_{pi}$. Electron pressure force is so large that electron fluid cannot be compressed. Ions oscillate independently in negative background charge (ion plasma wave).

2. $k\lambda_{De} << 1$ => $\omega \approx kc_s$. Pressure forces are smaller than charge separation forces. Electrons and ions move together in compression of the wave

$c_s = \sqrt{T_e / m_i}$.