Lecture 5

Kinetic equation for plasma

$f^\alpha(t,\vec{r},\vec{v})$ is a distribution function for particles of type α (electrons, ions, $\alpha = e,i$).

$f(t,\vec{r},\vec{v})$ is a number density of particles in 6-d phase space.

$\int f^\alpha(t,\vec{r},\vec{v})d\vec{v} = n^\alpha(t,\vec{r})$ - the number density of the particles of type α.

$n(t,\vec{r})dxdydz \equiv n(t,\vec{r})dV$ - an average number of particles in $dV = dxdydz$.

Two types of fields are acting on charged particles:

(i) regular electric and magnetic fields created by external sources or by charge separation or by currents in plasma in volumes with sizes larger than λ_D.

(ii) Collisions

If mean free time for collisions ($\tau_{ei},\tau_{ee},\tau_{ii}$) is longer than characteristic time for processes under consideration, the plasma can be considered as collisionless. Dynamics of the f^α is described by the kinetic equation

$$\frac{\partial f^\alpha}{\partial t} + \vec{v}\frac{\partial f^\alpha}{\partial \vec{r}} + \frac{e^\alpha}{m_\alpha}(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})\frac{\partial f^\alpha}{\partial \vec{v}} = 0 \qquad (1)$$

the kinetic equation is a consequence of Liouville theorem: Every elementary volume $d\Gamma$ in phase space occupied by some particles is constant.

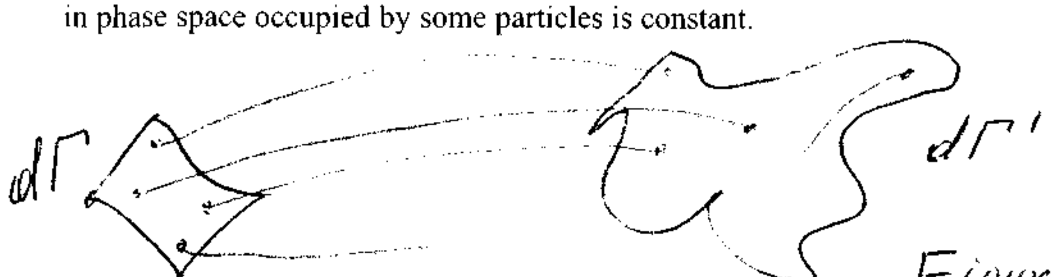

Figure 1

This is true because all points in phase space move under continuous action of macroscopic fields.

$$d\Gamma = d\Gamma' \qquad (2)$$

The number of particles in the phase volume is constant $dN = fd\Gamma = const$. Therefore $f^\alpha(t,\vec{r},\vec{v}) = const$, i.e.

$$\frac{df^\alpha}{dt} = \frac{\partial f^\alpha}{\partial t} + \vec{v}\frac{\partial f^\alpha}{\partial \vec{r}} + \frac{e^\alpha}{m_\alpha}(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B})\frac{\partial f^\alpha}{\partial \vec{v}} = 0 \tag{3}$$

Electric and magnetic fields are created not only by external sources but by space charges and currents of electrons and ions as well. For example,

$$\begin{aligned}
&\nabla \cdot \vec{E} = 4\pi\rho = 4\pi e\left(\int f^i d\vec{v} + \int f^e d\vec{v})\right) \\
&\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\
&\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi e}{c}\left(\int \vec{v} f^i d\vec{v} + \int \vec{v} f^e d\vec{v})\right)
\end{aligned} \tag{4}$$

$\vec{E}$ and $\vec{B}$ fields are so called self-consistent fields.

Kinetic equation (4) where $\vec{E}$ and $\vec{B}$ are self-consistent or created by external sources is Vlasov equation. The self-consistent fields are superposition of micro-fields of separate particles that are acting coherently on the distances comparable with the wavelength. They describe collective interaction of particles. In ideal gas, each atom moves on its independent trajectory and collides with other atoms. In that case forces are externally imposed:

$$\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{r}} + \frac{\vec{F}}{m}\frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll} \tag{5}$$

This is Boltzman equation.

Ideal gas is an individualistic society.

Plasma is a medium where collective interactions of big ensemble of particles drives self-consistent $\vec{E}$ and $\vec{B}$ fields.

$\vec{E}$ and $\vec{B}$ are determined from Maxwell's equations where charge density and current density are calculated using particle distribution functions

$$\rho = \sum_\alpha e^\alpha \int f^\alpha d\vec{v}$$

$$\vec{j} = \sum_{\alpha} e^{\alpha} \int \vec{v} f^{\alpha} d\vec{v} \tag{6}$$

In case when not only macroscopic fields ($\lambda > \lambda_D$) but also microfields of separate particles play significant role, equation (1) is not correct. Due to Coulomb collisions particles can jump from one region of the phase space to another. Because of this, the number of particles in an element of the phase volume is not constant.

Taking into account Coulomb collisions we obtain

$$\frac{df^{\alpha}}{dt} = \frac{\partial f^{\alpha}}{\partial t} + \vec{v}\frac{\partial f^{\alpha}}{\partial \vec{r}} + \frac{e^{\alpha}}{m_{\alpha}}(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B})\frac{\partial f^{\alpha}}{\partial \vec{v}} = \left(\frac{df}{dt}\right)_{coll} \tag{7}$$

$\left(\frac{df}{dt}\right)_{coll}$ is a collision operator – the change of distribution function due to collisions in unit time.

We will use a simplified form of collision operator, so called τ-approximation:

$$\left(\frac{df}{dt}\right)_{coll} = \frac{f_0 - f}{\tau} \tag{8}$$

τ means an average time between two collisions, f_0 is an equilibrium Maxwellian distribution function. The physical meaning of r.h. side of (8) is that as a result of collisions distribution function f approaches exponentially to f_0 (restoration of Maxwellian distribution over velocities).

When we use kinetic equation in τ-approximation, we ought to be careful. We found before that τ is different for different types of collisions in plasma. Different processes in plasma are characterized by different τ. For electrical conductivity $\tau = \tau_{ei}$. For thermal conductivity both $\tau = \tau_{ee}$ and $\tau = \tau_{ei}$ are important.

Electrical conductivity of plasma in electric field

In homogeneous plasma in stationary case:

$$-\frac{e}{m_e}\vec{E}\frac{\partial f^{e}}{\partial \vec{v}} = \frac{f_0^{e} - f^{e}}{\tau_{ei}} \tag{9}$$

Let $f^e = f_0^e + f_1^e$. For sufficiently small electric field $f_1^e << f_0^e$ and we neglect the term $\sim \vec{E}\frac{\partial f_1^e}{\partial \vec{v}}$. Then (9) has the form:

$$-\frac{e}{m_e}\vec{E}\frac{\partial f_0^e}{\partial \vec{v}} = -\frac{f_1^e}{\tau_{ei}} \tag{10}$$

The current density is defined by:

$$\vec{j} = -e\int \vec{v} f^e d\vec{v} = -e\int \vec{v} f_1^e d\vec{v} = \frac{n_0 e^2}{m_e}\tau_{ei}\vec{E} \tag{11}$$

The relation (11) that is the same as Equation (3.3) is just Ohm's law for plasma.

The physical meaning of approximation of a weak electric field:

$$\frac{f_1^e}{f_0^e} \approx \frac{eE}{m_e}\tau_{ei}\frac{\partial f_0^e}{\partial v}\Big/ f_0^e \approx \frac{eE\tau_{ei}}{m_e v_{Te}} << 1 \tag{12}$$

The relation (12) can be written in the form

$$\frac{E}{E_{Dreicer}} << 1 \tag{12'}$$

where $E_{Dreicer}$ is described by formula (3.10').

Once more, we obtained the sane result: the Ohm's law for plasma (11) is applicable if electric field is much smaller than the Dreicer field. This is equivalent to the statement that momentum acquired by electron between two collisions should be much smaller them the thermal momentum.

Thermal conductivity

For electron thermal conductivity, we should take into account e-i and e-e collisions. For ion thermal conductivity we should take into account i-I collisions. Ions are not scattered by electrons.

To find the thermal conductivity, we use kinetic equation, assuming that plasma in stationary state, $E = 0$, and $T = T(x)$ where x is direction of non-homogeneity:

$$v\frac{\partial f}{\partial x} = \frac{f_0 - f}{\tau} \tag{13}$$

Again let $f = f_0 + f_1$ and $f_1 << f_0$.

Then from Equation (13), assuming that

$$f_0 = n_0\left(\frac{m}{2\pi T}\right)^{1/2}\exp\left(-\frac{mv^2}{2T(x)}\right)$$

we obtain

$$f_1 = -\tau v\frac{\partial f_0}{\partial x} = -n_0\tau v\left(\frac{m}{2\pi T}\right)^{1/2}\left[\frac{mv^2}{2T}-\frac{1}{2}\right]\frac{1}{T}\frac{dT}{dx}\exp\left(-\frac{mv^2}{2T}\right) \tag{14}$$

The heat flux is equal

$$\begin{aligned} q &= \int_{-\infty}^{\infty} v\frac{mv^2}{2}f(v)dv = \int_{-\infty}^{\infty} v\frac{mv^2}{2}f_1(v)dv \\ &= -\frac{n_0\tau}{2}\left(\frac{m}{2\pi T}\right)^{1/2}\frac{1}{T}\frac{dT}{dx}\int_{-\infty}^{\infty} v^2\frac{mv^2}{2}\left(\frac{mv^2}{T}-1\right)\exp\left(-\frac{mv^2}{2T}\right)dv \end{aligned} \tag{15}$$

Introducing the new variable $\frac{mv^2}{2T} = \xi$ we obtain

$$\begin{aligned} q &= -\frac{n_0\tau}{2}\left(\frac{m}{2\pi T}\right)^{1/2}\frac{1}{T}\frac{dT}{dx}\left(\frac{2T}{m}\right)^{3/2}T\int_0^{\infty} e^{-\xi}\xi^{3/2}(2\xi-1)d\xi \\ &= -\frac{n_0\tau}{m\sqrt{\pi}}T\frac{dT}{dx}\left(2\Gamma(7/2)-\Gamma(5/2)\right) \end{aligned}$$

and finally

$$q = -3n_0\tau\frac{T}{m}\frac{dT}{dx} \tag{16}$$

The total heat flux consists of fluxes transported by electrons and ions. The ion flux is on the order of m_e/m_i in comparison with the electron flux.

The thermal conductivity coefficient

$$\kappa = \frac{3n_0T\tau}{m_e} \tag{17}$$

A strong kinetic theory with Landau collision operator gives

$$\kappa = 3.2\frac{n_0T}{m_e}\tau_{ei} \tag{18}$$

Let us compare (18) with electric conductivity of plasma

$$\sigma = \frac{n_0 e^2}{m_e} \tau_{ei}$$

It follows that the ratio of thermal to electric conductivity is proportional to T – Videman-Franz law for metals.

Landau diffusive form of collision operator

The simplified form of the collision operator in τ-approximation describes formation of Maxwellian distribution. However, it doesn't describe the main peculiarity of Coulomb scattering that leads to the slow (diffusive) change of the particle velocity vector due to multiple scatterings at small angles.

Consider a distribution function integrated over two velocity components $f(v_x)$ and assume that scattering particles are in an equilibrium state with temperature T. Multiple scatterings lead to

(i) friction between plasma particles

(ii) (ii) diffusion in velocity space

The friction force $F = -mv_x \nu$. The frictional part of the collision operator can be written in the form

$$\left(\frac{df}{dt}\right)_{coll}^{F} = \frac{\partial}{\partial v_x}\left(\frac{dv_x}{dt} f\right) = -\frac{\partial}{\partial v_x}\left(\nu v_x f\right) \tag{19}$$

This term is similar to the term $div(\vec{v}n)$ in the continuity equation.

Diffusive part of the collision operator should have the form

$$\left(\frac{df}{dt}\right)_{coll}^{F} = \frac{\partial}{\partial v_x}\left(D(v_x)\frac{\partial f}{\partial v_x}\right) \tag{20}$$

where $D(v_x) = \left\langle (\Delta v_x)^2 \right\rangle \nu$, Δv_x is the velocity change in a collision, the brackets mean averaging over scatterings that can be obtained by exact consideration of multiple scatterings.

We will obtain $D(v_x)$ by using assumption that scattering particles have temperature T. These particles play a role of thermostat: The distribution function of scattered particles should become a Maxwellian distribution with temperature T.

The total collision operator can be written as

$$\left(\frac{df}{dt}\right)_{coll} = \frac{\partial}{\partial v_x}\left(-\nu v_x f + D(v_x)\frac{\partial f}{\partial v_x}\right) \tag{21}$$

For Maxwellian distribution function $f = f_0 = const\exp(-mv_x^2/2T)$, we have from (21):

$$D(v_x) = -\frac{T}{m}\nu \tag{22}$$

and

$$\left(\frac{df}{dt}\right)_{coll} = -\frac{\partial}{\partial v_x}\left(\nu v_x f + \frac{T}{m}\nu\frac{\partial f}{\partial v_x}\right) \tag{23}$$

The 3-d form of (23) is

$$\left(\frac{df}{dt}\right)_{coll} = -\frac{\partial}{\partial v_i}\left[\nu\left(v_i f + \frac{T}{m}\frac{v_i v_j}{v^2}\frac{\partial f}{\partial v_j}\right)\right] \tag{24}$$