Physics 214 UCSD/225a UCSB

Lecture 5

- Symmetries & QCD
  - Greiner QM part 2: Symmetries
  - Halzen & Martin
    - “Low energy” QCD is close to impossible to calculate. Accordingly, symmetries, even approximate ones, play and important role in QCD. E.g.:

      $\Rightarrow$ Meson & Baryon spectroscopy

      $\Rightarrow$ Hadronic decays

      $\Rightarrow$ Pion-nuclean scattering

      $\Rightarrow$ …

Disclaimer

- Don’t expect mathematical rigor !!!
- E.g.:
  - I call semi-simple Lie groups simply Lie groups, and have probably made a few other simplifications that I’m not aware off.
  - If you find one, feel free to point them out, teaching me something in the process.

Origin of Symmetry in QCD

- QCD is flavor blind, i.e. independent of the quark flavor.
- Flavor symmetry of interactions as long as quark masses can be considered the same.
  - m_u =, m_d = => SU(2) symmetry called "Isospin".
  - m_s = => SU(3) flavor symmetry of u,d,s
  - m_c = m_b = => HQET

Symmetries in QM

- a reminder -

- Time evolution of a state ψ is fully described by:

$$ih\frac{\partial}{\partial t}\psi = H\psi$$

- It is obviously of great practical value to underdstand the complete set of symmetry operations S: S ψ = ψ' for which ψ' has the same time evolution as ψ.
- This implies: $S\ H\ S^{-1} = H$ or $[H,S]=0$

Utility of Group Theory

The set of $\{S_1,\ldots,S_n\}$ is called a group if:

1. A product "x" is defined such that
   $S_m \times S_k = S_l$ with $S_l \in G \;\; \forall\, m,k$
2. an element $S_0 \in G$ exists for which
   $S_0 \times S_k = S_k \quad \forall\, k$
3. $\forall$ m there is a k such that $S_m \times S_k = S_0$
4. Multiplication is associative

If you know the symmetry for a hamiltonian you obviously Save yourself a lot of unnecessary calculations.

Lie groups are especially useful

- L is a group for which all elements are infinitely differentiable functions of some set of parameters. E.g.:

$$S(\alpha_i, \ldots, \alpha_n) = e^{-i\sum_{j=1}^{n} \alpha_j L_j}$$

$$\left.\frac{\partial S}{\partial \alpha_j}\right|_{\alpha_j = 0} = -iL_j$$

- L_j are called the generators of the group L.
  Sort of like basis vectors to span S.

- It is obviously useful to know the complete Lie group L of H because it allows straightforward construction of all states ψ' that have the same time evolution as ψ .
- In the following, we will go through some characteristics of Lie groups without proof.
- For more details, see Greiner chapters 1-4.

- Generators of L form an orthogonal set.
- Lie group is fully characterized by the commutator relationship among its generators:

  $[L_k, L_l] = c_{klm} L_m$

  This equation is thus called the "Lie algebra" of L.
- Theorem of Racah:

  For every Lie group L of rank k there is a set of exactly k "Casimir" operators $C_1, \ldots, C_k$ that commute with every operator in L, including themselves.
- A hamiltonian H that has the symmetry L will have exactly 2k good quantum numbers, in addition to E.
- It can be shown that any operator A that commutes with all operators in L (i.e. commutes with the generators) must be a function of the Casimir operators. This implies that $E = E(C_1, \ldots, C_k)$.

Importance to Physics

- The Hilbert Space of all states ψ that satisfy the Schroedinger equation is divided into "multiplets" characterized by the value for the set of k Casimir operator eigenvalues.
- Transitions between multiplets do not happen.
- All states within a given multiplet have the same energy.
- Out of the N generators of the Lie group, a set of k (generally $k < N$ except for abelian groups for which $k=N$) can be diagonalized with H simultaneously, thus providing the second set of k good quantum numbers.

Summary

- Let L be the N dimensional Lie group of Rank k for the Hamiltonian H.
- Then we have the following set of operators that mutually commute:

$$H, C_1,\ldots,C_k,L_1,\ldots,L_k$$

- Any state is thus characterized by 2k quantum numbers.
- The energy E is given as some function of the $C_1,\ldots,C_k$.

Examples

- Translation Group
- Rotation Group
- Flavor SU(3)

Translation group

- Translations commute with each other.
- The generators of the translation group thus commute.
- All generators are thus Casimir operators of the group.
- The generators of the group are the momentum operators p_x, p_y, p_z

Group of Rotations in 3-space

- Generators: J_x, J_y,J_z
- Lie algebra: $[J_k, J_l] = i\ \varepsilon_{klm}\ J_m$
- Rank = 1
- Casimir Operator: J^2
- Multiplets are classified by their total angular momentum J
- States are classified by J and J_z, the latter being one of the three generators.

Group Representations

- A group of NxN matrices is called an N-dimensional representation of a Lie group if there is a one-to-one map: L_k <-> M_k such that $[M_k,M_l] = c_{klm}\, M_m$ with c_{klm} being the structure constants of the Lie group.

Back to rotation group

- SO(3) = group of orthonormal 3x3 matrices with determinant = 1.
- SU(2) = group of all 2x2 unitary traceless matrices.

- Aside: SU(2) is sometimes used as name for the more general rotation group, not just the 2x2 unitary traceless matrices. I will do that from now on because it’s shorter than writing “rotation group”.

SU(2) and Spin

- We know that spin 1/2 is the fundamental representation of spin because all other spin states can be constructed by angular momentum addition of spin 1/2 !!!

$\Rightarrow$ Fundamental representation of the rotation group.

$\Rightarrow$ Fundamental representation of “SU(2)”

Reminder: Generators of SU(2)

$$K_i = \frac{1}{2}\sigma_i$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Arbitrary rotation in spin space:

$$\begin{pmatrix} |u'\rangle \\ |d'\rangle \end{pmatrix} = e^{i\sum_{k=1}^{3}\alpha_k\sigma_k}\begin{pmatrix} |u\rangle \\ |d\rangle \end{pmatrix}$$

And we combine spins the same way we have always combined them, using Glebsch-Gordon coefficients.

Language Comparison

- Group theory:

  $2 \otimes 2 = 1 \oplus 3$

- Spin in QM:

  $1/2 \otimes 1/2 = 0 \oplus 1$

1-dim Matrix → Spin 0

3-dim matrix → Spin 1

3-dim because m=-1,0,1

Reducible vs Irreducible Representation

$$\begin{pmatrix} (1x1) & 0 \\ 0 & \begin{pmatrix} 3 & x & 3 \end{pmatrix} \end{pmatrix}$$

This describes:
$2 \otimes 2 = 1 \oplus 3$

Or in spin language: This 4x4 matrix describes a system with spin 0 and spin 1 as irreducible subspaces.

In contrast to $1 \oplus 3$: Spin 3/2

$$S_z = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

This describes the z-component of a spin in the sense that:

$$\left| j = 3/2; m = 3/2 \right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The 4x4 matrices are thus used here to describe the dim-4 multiplet of a spin 3/2 particle.
Arbitrary rotations in this space are thus implemented via the generators of the 4x4 representation of the rotation group.

Commonality of all representations of the rotation group.

- Rank = 1

$\Rightarrow$ there are only two good quantum numbers !!!

$\Rightarrow |J;m\rangle$ the eigenvalues of J^2 and J_z

What if the physics has more conserved quantum numbers?

Then there must be a higher rank symmetry group that describes the physical system !!!

Examples of higher rank symmetry groups

SU(n) => Rank = n-1

=> # of generators = n^2-1

Applications of Flavor SU(2)

- Spectroscopy
- Scattering
- Partial decay widths

Spectroscopy of Isospin = 1/2

- Quarks:

$$|u\rangle = \left|\frac{1}{2};T_3 = +\frac{1}{2}\right\rangle$$ m = 1.5 to 3 MeV

$$|d\rangle = \left|\frac{1}{2};T_3 = -\frac{1}{2}\right\rangle$$ m = 3 to 7 MeV

- Mesons:

$$\left|K^{(*)+}\right\rangle = \left|\frac{1}{2};T_3 = +\frac{1}{2}\right\rangle$$ m = 493 (892)MeV

$$\left|K^{(*)0}\right\rangle = \left|\frac{1}{2};T_3 = -\frac{1}{2}\right\rangle$$ m = 497 (896)MeV

- Baryons:

$$|p\rangle = \left|\frac{1}{2};T_3 = +\frac{1}{2}\right\rangle$$ m = 938.2 MeV

$$|n\rangle = \left|\frac{1}{2};T_3 = -\frac{1}{2}\right\rangle$$ m = 939.5 MeV

Spectroscopy of Isospin = 1

- $2 \otimes 2 = 1 \oplus 3$ => a singlet and a triplet.
- Pseudoscalar Mesons:

$$|\pi^+\rangle = |1;+1\rangle$$ m = 139 MeV

$$|\pi^0\rangle = |1;0\rangle$$ m = 135 MeV

$$|\pi^-\rangle = |1;-1\rangle$$ m = 139 MeV

$$|\eta_0\rangle = |0;0\rangle$$

Same repeats for vector mesons
ρ^+ ρ^0 ρ^- and ω
… and so forth …

Scattering

$$\frac{\sigma(pp \to \pi^+ d)}{\sigma(np \to \pi^0 d)} = \frac{\left|\left\langle \pi^+ d \middle| S \middle| pp \right\rangle\right|^2}{\left|\left\langle \pi^0 d \middle| S \middle| np \right\rangle\right|^2} \bullet PhSpR = \frac{\left|\langle 1 \| S \| 1 \rangle\right|^2}{\left|\langle 1 \| S \| 1 \rangle + \langle 1 \| S \| 0 \rangle\right|^2 \bullet \frac{1}{2}}$$

$$\left| pp \right\rangle = \left| \frac{1}{2}; + \frac{1}{2} \right\rangle \left| \frac{1}{2}; + \frac{1}{2} \right\rangle = \left| 1;1 \right\rangle_{NN}$$

$$\left| np \right\rangle = \left| \frac{1}{2}; - \frac{1}{2} \right\rangle \left| \frac{1}{2}; + \frac{1}{2} \right\rangle = \frac{1}{\sqrt{2}} \left(\left| 1;0 \right\rangle_{NN} + \left| 0;0 \right\rangle_{NN} \right)$$

$$\left| \pi^+ d \right\rangle = \left| 1;1 \right\rangle_{\pi} \left| 0;0 \right\rangle_{d} = \left| 1;1 \right\rangle_{\pi d}$$

$$\left| \pi^0 d \right\rangle = \left| 1;0 \right\rangle_{\pi} \left| 0;0 \right\rangle_{d} = \left| 1;0 \right\rangle_{\pi d}$$

Partial Decay Width

- K*(892) decays ~100% to Kπ
- The partial decay width is so well determined by isospin that the PDG doesn't even bother writing it down.

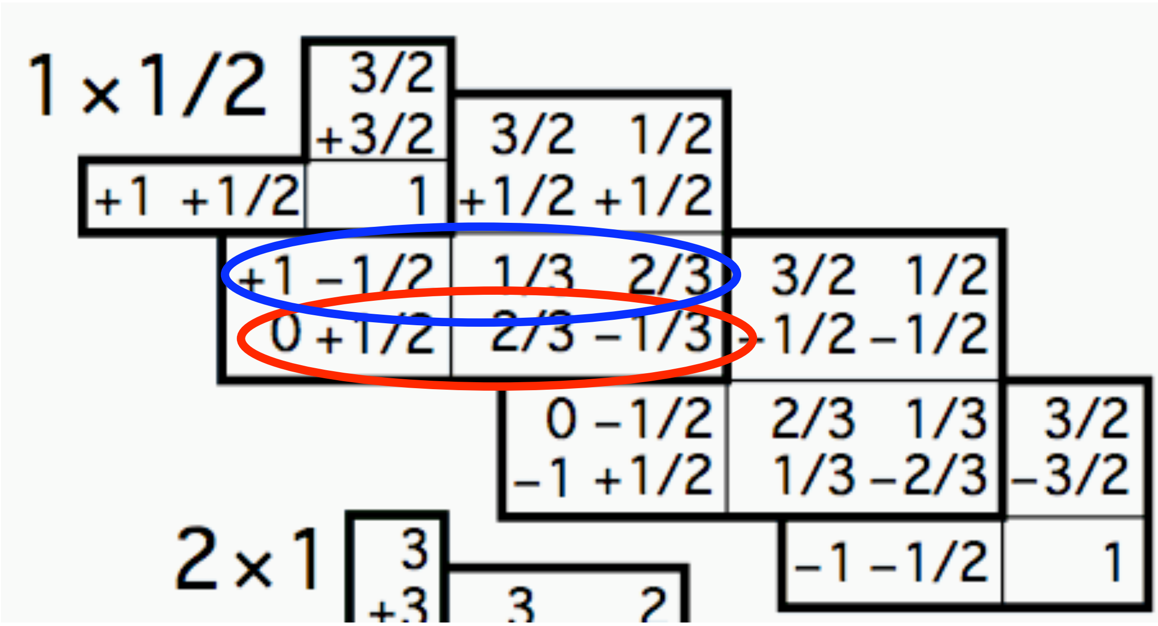
$$\frac{\Gamma(K^{*+} \to K^{+}\pi^{0})}{\Gamma(K^{*+} \to K^{0}\pi^{+})} =$$

$$\begin{pmatrix} K^{(*)+} = u\bar{s} \\ K^{(*)0} = d\bar{s} \end{pmatrix} = \begin{pmatrix} +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$\left|K^{+}\pi^{0}\right\rangle = \left|\frac{1}{2};+\frac{1}{2}\right\rangle_{K} \left|1;0\right\rangle_{\pi} = \sqrt{\frac{2}{3}}\left|\frac{3}{2};+\frac{1}{2}\right\rangle_{K\pi} - \sqrt{\frac{1}{3}}\left|\frac{1}{2};+\frac{1}{2}\right\rangle_{K\pi}$$

$$\left|K^{0}\pi^{+}\right\rangle = \left|\frac{1}{2};-\frac{1}{2}\right\rangle_{K} \left|1;1\right\rangle_{\pi} = \sqrt{\frac{1}{3}}\left|\frac{3}{2};+\frac{1}{2}\right\rangle_{K\pi} + \sqrt{\frac{2}{3}}\left|\frac{1}{2};+\frac{1}{2}\right\rangle_{K\pi}$$

Clebsh-Gordon Reminder



(1;0) + (1/2;+1/2) = sqrt(2/3) (3/2;1/2) - sqrt(1/3) (1/2;1/2)

(1;1) + (1/2;-1/2) = sqrt(1/3) (3/2;1/2) + sqrt(2/3) (1/2;1/2)

Isospin for anti-quarks

- Want to make anti-quark doublet with:
  - charge conserved -> anti-d must have T_3=+1/2 because Q = T_3 + B:
    - Q_u = 1/2 + 1/3 = +2/3 Q_d = -1/2 +1/3 = -1/3
    - Q_{anti-u} = -1/2 - 1/3 = -2/3 Q_{anti-d} = 1/2 -1/3 = +1/3
  - baryon number conserved
  - Same transformation properties as quarks

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow e^{\frac{1}{2} i\theta_y \sigma_y} \begin{pmatrix} u \\ d \end{pmatrix} = \left[\cos\frac{\theta_y}{2} + i\sin\frac{\theta_y}{2}\sigma_y\right] \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_y}{2} & \sin\frac{\theta_y}{2} \\ -\sin\frac{\theta_y}{2} & \cos\frac{\theta_y}{2} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

If you simply bar and flip then you get the wrong sign
In front of the “sin” terms.

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow e^{\frac{1}{2} i\theta_y \sigma_y} \begin{pmatrix} u \\ d \end{pmatrix} = \left[\cos\frac{\theta_y}{2} + i\sin\frac{\theta_y}{2}\sigma_y \right] \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_y}{2} & \sin\frac{\theta_y}{2} \\ -\sin\frac{\theta_y}{2} & \cos\frac{\theta_y}{2} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

$$\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} \rightarrow e^{\frac{1}{2} i\theta_y \sigma_y} \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} = \left[\cos\frac{\theta_y}{2} + i\sin\frac{\theta_y}{2}\sigma_y \right] \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_y}{2} & \sin\frac{\theta_y}{2} \\ -\sin\frac{\theta_y}{2} & \cos\frac{\theta_y}{2} \end{pmatrix} \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$

Note:

The point here is that you can want to be able to derive the rotated doublet either via rotation to the quark doublet followed by Charge conjugation and flip, or by starting with the anti-q doublet and using the same rotation as for q doublet.

Quantum Numbers for Mesons

- J^{PC}

  J = total angular momentum = L+S

  P = parity

  C = charge conjugation

- Only neutral particles can be eigenstates of C, of course.

Generalized Pauli Principle

- The fermion-antifermion wave function must be odd under interchange of all coordinates (space, spin, charge).
  - Space interchange -> $(-1)^{L}$
  - Spin interchange -> $(-1)^{S+1}$
  - Charge interchange -> depends on eigenvalue of C
- Bottom line:

$$(-1)^{L+S+1} C = -1 \quad \Rightarrow \quad C = (-1)^{L+S} ; \quad P = (-1)^{L+1}$$

π^0 : $C = (-1)^{0+0} = 1$; $P = (-1)^{0+1} = -1$ => pseudoscalar meson

ρ^0 : $C = (-1)^{0+1} = -1$; $P = (-1)^{0+1} = -1$ => vector meson

b : $C = (-1)^{1+0} = -1$; $P = (-1)^{1+1} = +1$ => axial vector meson

SU(3) Next Lecture