Physics 110A: Midterm Exam #1

[1] A particle of mass m moves in one dimension under the influence of the potential

$$U(x) = \frac{U_0}{a^6}\, x^2\, (x^2 - a^2)^2 \ ,$$

with $U_0 > 0$.

(a) Sketch $U(x)$, and find the location of all minima and maxima.

(b) Sketch the phase curves for a particle with energy $E_1 = 0$, $E_2 = \frac{1}{8}U_0$, $E_3 = \frac{1}{4}U_0$.

(c) Sketch the separatrix and determine its associated energy.

(d) Compute the frequency of oscillation when x is close to a.

[2] A damped oscillator obeys the usual second order homogeneous differential equation

$$\ddot{x} + 2\beta\,\dot{x} + \omega_0^2\, x = 0 \ ,$$

with $\beta = 5\,\text{Hz}$ and $\omega_0 = 4\,\text{Hz}$.

(a) Write down the solution $x(t)$ given initial conditions $x(0) = 0$ and $\dot{x}(0) = -1\,\text{m/s}$.

Next assume the oscillator is forced:

$$\ddot{x} + 2\beta\,\dot{x} + \omega_0^2\, x = f(t) \ ,$$

with $f(t) = f_0\,\cos(\Omega t)$. The forcing amplitude is $f_0 = \sqrt{10}\,\text{m/s}^2$, and the forcing frequency is $\Omega = 6\,\text{Hz}$. Assume that a long time has passed so that any transient homogeneous oscillations have damped away.

(b) What is the time delay Δt between the maximum of the force $f(t)$ and the maximum of the response $x(t)$?

(c) What is the amplitude of the forced oscillations?

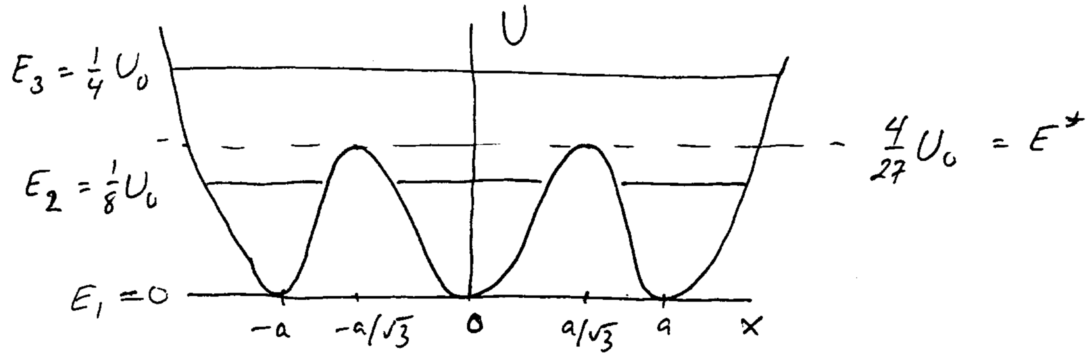
PHYSICS 110A MIDTERM #1 SOLUTIONS

[1] $U(x) = \frac{U_0}{a^6}\, x^2 (x^2 - a^2)^2$

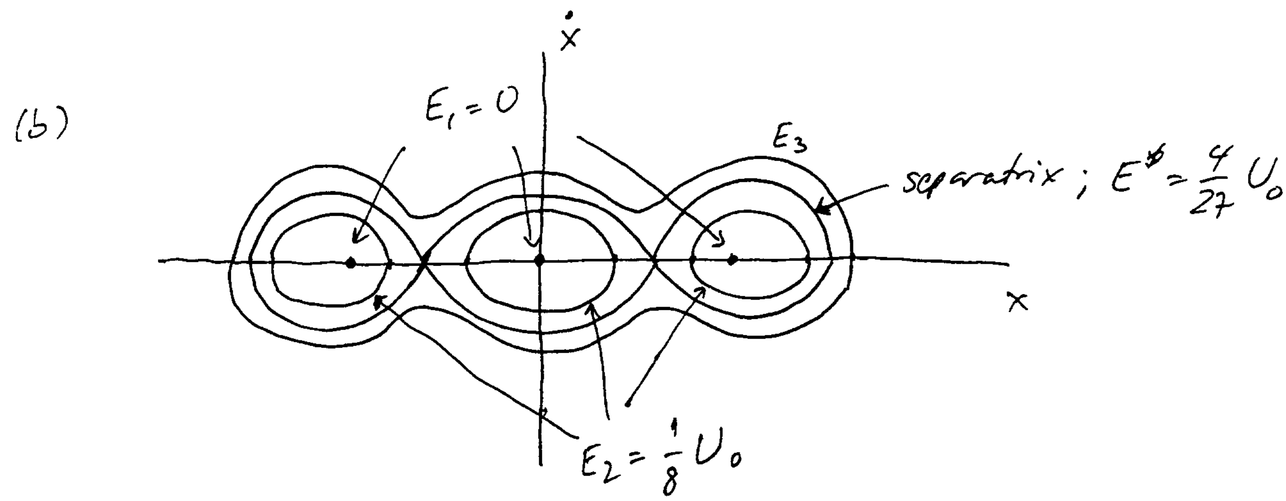
(a) Note $U(x) \geq 0$ and $U(0) = U(\pm a) = 0$. Differentiating gives $U'(x) = U_0 \cdot \left(\frac{3x^4}{a^4} - \frac{4x^2}{a^2} + 1 \right) \cdot \frac{2x}{a}$, so

$U'(x) = 0 \Rightarrow x = 0,\ (x/a)^2 = \frac{4 \pm \sqrt{16-12}}{6} = 1, \frac{1}{3}$ i.e. $\frac{x}{a} = \pm\frac{1}{\sqrt{3}}, \pm 1$.

$x = \pm \frac{a}{\sqrt{3}}$ are then local maxima. Note $U(\pm \frac{a}{\sqrt{3}}) = \frac{4}{27} U_0$. Thus,



(b)



(c) $E^* = \frac{4}{27} U_0 = U(\pm \frac{a}{\sqrt{3}})$.

(d) $U(x) = U_0 \cdot \left\{ \frac{x^6}{a^6} - \frac{2x^4}{a^4} + \frac{x^2}{a^2} \right\} \Rightarrow U''(x) = \frac{U_0}{a^2} \cdot \left\{ \frac{30x^4}{a^4} - \frac{24x^2}{a^2} + 2 \right\}$

$U''(a) = \frac{8U_0}{a^2} \Rightarrow \omega_{osc} = \sqrt{\frac{8U_0}{ma^2}}$

[2] (a) $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$; $\beta = 5\,Hz$, $\omega_0 = 4\,Hz$

General sol$^{\underline{n}}$: $x_h = C_+ e^{-i\omega_+ t} + C_- e^{-i\omega_- t}$

with $\omega_\pm = -i\beta \pm \sqrt{\omega_0^2 - \beta^2} = -5i\,Hz \pm \sqrt{16-25}\,Hz$

$= -2i\,Hz,\ -8i\,Hz$

Hence, with t measured in seconds,

$x(t) = C_+ e^{-2t} + C_- e^{-8t}$

$\dot{x}(t) = -2C_+ e^{-2t} - 8C_- e^{-8t}$

Match initial conditions :

$x(0) = 0 = C_+ + C_- \Rightarrow C_- = -C_+$

$\dot{x}(0) = -1 = -2C_+ - 8C_- = +6C_+$

Solution : $C_+ = -\frac{1}{6}$, $C_- = +\frac{1}{6}$.

$$\boxed{x(t) = -\frac{1}{6}e^{-2t} + \frac{1}{6}e^{-8t}}$$ x in meters, t in seconds

(b) $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = f_0 \cos(\Omega t)$; $f_0 = \sqrt{10}\,\frac{m}{s^2}$, $\Omega = 6\,Hz$

$\Rightarrow x(t) = x_h(t) + A(\Omega) f_0 \cos(\Omega t - \delta(\Omega))$

$A(\Omega) = \left[(\omega_0^2 - \Omega^2)^2 + 4\beta^2\Omega^2\right]^{-1/2}$; $\delta(\Omega) = \tan^{-1}\left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\right)$

Compute the phase lag $\delta = \tan^{-1}\left(\frac{2\cdot 5\cdot 6}{4^2 - 6^2}\right) = \tan^{-1}(-3) \simeq 1.893$

To compute Δt, set $\Omega \cdot \Delta t - \delta(\Omega) = 0$, so

$$\boxed{\Delta t = \frac{\delta(\Omega)}{\Omega} = \tan^{-1}(-3)\cdot\frac{1}{6\,Hz} = 0.315\ sec}$$

(c) $A = \left[(16-36)^2 + 4\cdot 5^2 \cdot 6^2\right]^{-1/2} s^2 = \left[400 + 3600\right]^{-1/2} s^2 = (4000)^{-1/2} s^2$

$A = \frac{1}{20\sqrt{10}} s^2 \Rightarrow \boxed{x_{max} = \frac{1}{20\sqrt{10}} s^2 \cdot \sqrt{10}\,\frac{m}{s^2} = \frac{1}{20}\,m = 5\,cm}$