Physics 110A: Final Exam

`Instructions: You are to do four problems. You must do both problems 4 and 5. You may choose any two of problems 1, 2, and 3.`

[**1**] The bletch is a disgusting animal native to the Forest of Jkroo on the planet Barney. The bletch population obeys the dynamical equation

$$\frac{dN}{dt} = aN^2 - bN^3 \ ,$$

where N is the number of bletches, and a and b are positive constants. (Bletches reproduce asexually, but only when another bletch is watching. However, when there are three or more bletches around, they beat the `@!!*$&*` out of each other.)

(a) Sketch the phase flow for N (strange as the bletch is, you still can rule out $N < 0$). Identify all fixed points and classify them according to their stability.

`[5 points]`

(b) The bletch population is now *harvested* (they make nice shoes). To model this, we add an extra term to the dynamics:

$$\frac{dN}{dt} = -hN + aN^2 - bN^3 \ .$$

Show that the phase flow now depends crucially on h, in that there are two qualitatively different flows depending on whether $h < h_c(a,b)$ or $h > h_c(a,b)$. Find the critical value $h_c(a,b)$ and sketch the phase flows for $h < h_c$ and $h > h_c$.

`[10 points]`

(c) In equilibrium, the rate at which bletches are harvested is $R = hN^*$, where N^* is the equilibrium bletch population. Suppose we start with $h = 0$, in which case the equilibrium population is given by the value at the attractive fixed point you found in **(a)**. Now h is increased very slowly from zero. As h is increased, the equilibrium population changes. Sketch R *versus* h. What value of h achieves the biggest bletch harvest? What is the value of $R_{\max}$?

`[10 points]`

[**2**] Consider the forced harmonic oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_\circ^2 x = f_\circ \cos(\Omega t) \ .$$

The value of $f_\circ$ is held constant at $f_\circ = 8\text{m/s}^2$ as the driving frequency Ω is varied. Recall that the amplitude of the inhomogeneous solution, as well as the phase shift, depends on Ω, *viz.*

$$x_{\text{inh}}(t) = A\, f_\circ\, \cos(\Omega t - \delta)$$
$$A(\Omega) = \left[(\Omega^2 - \omega_\circ^2)^2 + 4\beta^2\Omega^2\right]^{-1/2}$$
$$\delta(\Omega) = \tan^{-1}\left(\frac{2\beta\Omega}{\omega_\circ^2 - \Omega^2}\right) \ .$$

In this problem, you are to neglect the homogeneous part of the solution. You are asked to design a system such that the maximum displacement $x_{\rm inh}^{\rm max}$ is $d = 12.5\,{\rm cm}$. This maximum displacement will of course be achieved when the driving frequency Ω hits a particular value.

(a) If $\beta = 8\,{\rm s}^{-1}$, what value of $\omega_\circ$ is needed in order that $x_{\rm inh}^{\rm max} = d$? For what driving frequency Ω^* is this maximum achieved?

`[12 points]`

(b) If $\beta = 4\,{\rm s}^{-1}$, what value of $\omega_\circ$ is needed in order that $x_{\rm inh}^{\rm max} = d$? For what driving frequency Ω^* is this maximum achieved?

`[13 points]`

Hint: At some point, you must contend with the fact that the frequency Ω^* at which the response is maximized is either zero or finite, depending on the values of $\omega_\circ$ and β.

[**3**] A pendulum bob of mass m hangs at the end of a massless rigid rod of length ℓ. The point of support of the pendulum is constrained to move along a horizontally-oriented circle of radius ℓ. The instantaneous plane of motion of the pendulum is always perpendicular to $\hat{\boldsymbol{\phi}}$, where ϕ is the azimuthal angle along the circle. The point of support is itself massless.

(a) Choose as generalized coordinated the azimuthal angle ϕ and the angle θ that the pendulum rod makes with respect to the vertical. Find the Lagrangian L.

`[7 points]`

(b) Write down the Euler-Lagrange equations.

`[7 points]`

(c) What quantities are conserved?

`[6 points]`

(d) Show that θ obeys the equation

$$\ddot{\theta} = -\frac{\partial \mathcal{U}_{\rm eff}}{\partial \theta} \ .$$

Find the (scaled) effective potential $\mathcal{U}_{\rm eff}(\theta)$.

`[5 points]`

[4] Two particles attract each other according to the central potential

$$U(r) = -\frac{k}{r}\left(1 + \frac{a}{2r}\right) ,$$

where $k > 0$ and $a > 0$. (This is gravity with an additional $1/r^2$ potential added.) Denote the reduced mass by μ, and the conserved angular momentum by ℓ, as usual.

(a) Find the radius $r_\circ(\ell)$ of the circular orbit. Show that there is no circular orbit if ℓ is too small.

`[10 points]`

(b) The circular orbit is perturbed slightly. Find the geometric shape of the perturbation, $\delta r(\phi)$. If periapsis comes exactly once every four revolutions (*i.e.* the perturbed orbit closes after $\Delta\phi = 8\pi$), what must ℓ be? Recall the differential equation for the shape of the orbit is

$$\frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = -\frac{\mu r^4}{\ell^2}\,\frac{\partial U_{\text{eff}}}{\partial r} .$$

`[10 points]`

(c) The shape of the orbit can be determined exactly! Set $s = 1/r$ and show that the geometric equation for $s(\phi)$ is that of a harmonic oscillator subjected to a constant force. Solve and find $r(\phi)$.

`[5 points]`

[5] A system of masses and springs is arranged as shown below.

All motion occurs along the x-direction. The masses and spring constants are as assigned in the sketch. Choose as generalized coordinates the displacements η_1 and η_2 from equilibrium.

(a) Find the T and V matrices.

`[7 points]`

(b) Find the normal mode frequencies.

`[7 points]`

(c) Find the (properly normalized) modal matrix A.

`[6 points]`

(d) At time $t = 0$, $\eta_1(0) = d$, $\eta_2(0) = 0$, and $\dot\eta_1(0) = \dot\eta_2(0) = 0$. Find $\eta_1(t)$ and $\eta_2(t)$ for all time.

`[5 points]`