Physics 110A: Midterm Exam #2

[1] Two masses are configured as shown on the diagram below. Mass m_1 moves frictionlessly along a stationary wedge of opening angle α; it is connected to a point at the base of the wedge by a spring of spring constant K. From the mass m_1 hangs a massless rigid rod of length ℓ, at the bottom of which is the mass m_2. The unstretched length of the spring is a, so the potential energy of the spring is $U_{\text{spring}} = \frac{1}{2}K(r-a)^2$.

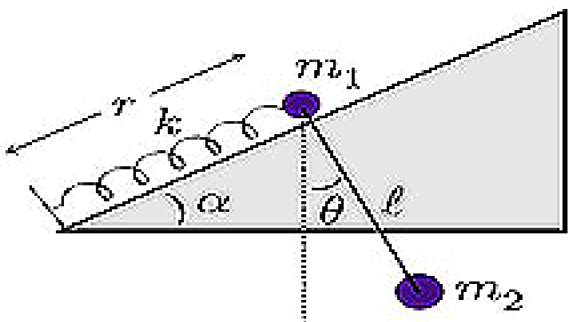
(a) Choose as generalized coordinates the distance r from the base of the wedge to the mass m_1 (*i.e.* r is the length of the spring) and the angle θ the rod makes with respect to the vertical. Find the Lagrangian $L(r, \dot{r}, \theta, \dot{\theta}, t)$.
`[15 Points]`

(b) Write down the equations of motion.
`[15 Points]`

(c) Write down all conserved quantities.
`[10 Points]`

(d) Linearize the equations of motion. That is, assume the deviations $\delta r = r - r_{\text{eq}}$ and $\delta\theta = \theta - \theta_{\text{eq}}$ are small and drop all terms in the equations of motion which are higher order than $\mathcal{O}(\delta r, \delta\theta)$. *Note:* While the equilibrium angle $\theta_{\text{eq}} = 0$, the equilibrium length of the spring is not a. To find the equilibrium length of the spring, set the forces to zero.
`[10 Points]`

(e) Write down an algebraic equation $P(\omega) = 0$ which determines the two normal frequencies of oscillation. *Hint:* your equation should be quadratic in the quantity ω^2.
`[10 Points]`



[2] The functional

$$\mathcal{F}[y(x)] = \int_0^1 dx \left[\exp(y) \left(\frac{dy}{dx} \right)^2 + y^2 \right]$$

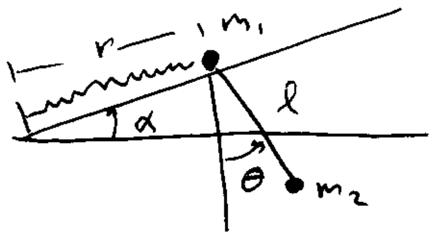
is to be extremized, subject to the constraint

$$\int_0^1 dx \left(\frac{dy}{dx} \right)^2 = C \, .$$

and the boundary conditions $y(0) = a$ and $y(1) = b$. Write down all equations which must be solved. (But don't try to solve them!)
`[40 Points]`

PHYSICS 110A SOLUTION: MIDTERM 2

1.



$x_1 = r\cos\alpha \qquad x_2 = r\cos\alpha + l\sin\theta$

$y_1 = r\sin\alpha \qquad y_2 = r\sin\alpha - l\cos\theta$

$\dot{x}_2 = \dot{r}\cos\alpha + l\dot{\theta}\cos\theta$

$\dot{y}_2 = \dot{r}\sin\alpha + l\dot{\theta}\sin\theta$

$$T = \frac{1}{2}m_1\dot{r}^2 + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_2 l^2\dot{\theta}^2 + m_2 l\dot{r}\dot{\theta}\cos(\theta-\alpha)$$

$$U = \frac{1}{2}k(r-a)^2 + m_1 g y_1 + m_2 g y_2$$

$$= \frac{1}{2}k(r-a)^2 + (m_1+m_2)gr\sin\alpha - m_2 g l\cos\theta$$

(a) $$L = T - U = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_2 l^2\dot{\theta}^2 + m_2 l\dot{r}\dot{\theta}\cos(\theta-\alpha) - \frac{1}{2}k(r-a)^2 - (m_1+m_2)gr\sin\alpha + m_2 g l\cos\theta$$

(b) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

(r) : $$(m_1+m_2)\ddot{r} + m_2 l\ddot{\theta}\cos(\theta-\alpha) - m_2 l\dot{\theta}^2\sin(\theta-\alpha) + k(r-a) + (m_1+m_2)g\sin\alpha = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

(θ) : $$(m_2 l^2\ddot{\theta} + m_2 l\ddot{r}\cos(\theta-\alpha) + m_2 g l\sin\theta = 0$$

(d) SET $\dot{r} = \dot{\theta} = 0 = \ddot{r} = \ddot{\theta}$

(θ) : $m_2 g l \sin\theta_{eq} = 0 \Rightarrow \underline{\theta_{eq} = 0}$

(r) : $k(r_{eq} - a) + (m_1 + m_2) g \sin\alpha = 0 \Rightarrow \underline{r_{eq} = a - (m_1 + m_2)\frac{g}{k}\sin\alpha}$

LINEARIZED EQUATIONS

$$(m_1 + m_2)\delta\ddot{r} + \cos\alpha\, m_2 l\, \delta\ddot{\theta} + k\,\delta r = 0$$
$$\delta\ddot{\theta} + \frac{\cos\alpha}{l}\delta\ddot{r} + \frac{g}{l}\delta\theta = 0$$

(e) SET $\frac{d}{dt} = i\omega$

$$\Rightarrow \underbrace{\begin{pmatrix} k - (m_1 + m_2)\omega^2 & -\cos\alpha\, m_2 l \omega^2 \\ -\frac{\cos\alpha}{l}\omega^2 & \frac{g}{l} - \omega^2 \end{pmatrix}}_{M} \begin{pmatrix} \delta r \\ \delta\theta \end{pmatrix} = 0$$

$$\|M\| = 0 \Rightarrow \underline{\left(k - (m_1 + m_2)\omega^2\right)\left(\frac{g}{l} - \omega^2\right) - \cos^2\alpha\, m_2 \omega^4 = 0}$$

(c) SINCE L IS A FUNCTION OF $r, \theta \Rightarrow$ NEITHER p_r, p_θ CONSERVED.

BUT, $\frac{\partial L}{\partial t} = 0 \Rightarrow$ ENERGY IS CONSERVED.

2. $\mathcal{F}[y(x)] = \int_0^1 dx \left[e^y \left(\frac{dy}{dx}\right)^2 + y^2 \right]$

$\mathcal{G}[y(x)] = \int_0^1 dx \left(\frac{dy}{dx}\right)^2 - C = 0$

$\mathcal{F}^*[y(x), \lambda] = \mathcal{F} + \lambda \mathcal{G} = \int_0^1 dx\, \mathcal{L}^*(y, y', x) - \lambda C$

$\therefore \mathcal{L}^* = (\lambda + e^y) y'^2 + y^2$

$\frac{d}{dx}\left(\frac{\partial L^*}{\partial y'}\right) = \frac{\partial L^*}{\partial y}$

$\Rightarrow 0 = -2\lambda y'' - 2e^y y'' - 2e^y y'^2 + e^y y'^2 + 2y$

$\therefore$ EQUATIONS TO SOLVE ARE

- $(e^y + \lambda) \frac{d^2y}{dx^2} + \frac{1}{2} e^y \left(\frac{dy}{dx}\right)^2 - y = 0$
- $\int_0^1 dx \left(\frac{dy}{dx}\right)^2 = C$
- $y(0) = a$
- $y(1) = b$