Chapter 7

Noether's Theorem

7.1 Continuous Symmetry Implies Conserved Charges

Consider a particle moving in two dimensions under the influence of an external potential $U(r)$. The potential is a function only of the magnitude of the vector $\boldsymbol{r}$. The Lagrangian is then

$$L = T - U = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\phi}^2\right) - U(r)\ , \tag{7.1}$$

where we have chosen generalized coordinates (r, ϕ). The momentum conjugate to ϕ is $p_\phi = m\,r^2\,\dot{\phi}$. The generalized force F_ϕ clearly vanishes, since L does not depend on the coordinate ϕ. (One says that L is 'cyclic' in ϕ.) Thus, although $r = r(t)$ and $\phi = \phi(t)$ will in general be time-dependent, the combination $p_\phi = m\,r^2\,\dot{\phi}$ is constant. This is the conserved angular momentum about the $\hat{\boldsymbol{z}}$ axis.

In general, whenever the system exhibits a *continuous symmetry*, there is an associated *conserved charge*. (The terminology 'charge' is from field theory.) Indeed, this is a rigorous result, known as *Noether's Theorem*. Consider a one-parameter family of transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q, \zeta)\ , \tag{7.2}$$

where ζ is the continuous parameter. Suppose further (without loss of generality) that at $\zeta = 0$ this transformation is the identity, *i.e.* $\tilde{q}_\sigma(q, \zeta) = q_\sigma$. The transformation may be nonlinear in the generalized coordinates. Suppose further that the Lagrangian

L s invariant under the replacement $q \to \tilde{q}$. Then we must have

$$
\begin{aligned}
0 = \frac{d}{d\zeta}\bigg|_{\zeta=0} L(\tilde{q},\dot{\tilde{q}},t) &= \frac{\partial L}{\partial q_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \dot{\tilde{q}}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{d}{dt}\left(\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} . \qquad (7.3)
\end{aligned}
$$

Thus, there is an associated conserved charge

$$
\Lambda = \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} . \qquad (7.4)
$$

7.1.1 Examples of one-parameter families of transformations

Consider the Lagrangian

$$
L = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) - U\big(\sqrt{x^2+y^2}\big) . \qquad (7.5)
$$

In two-dimensional polar coordinates, we have

$$
L = \tfrac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2) - U(r) , \qquad (7.6)
$$

and we may now define

$$
\tilde{r}(\zeta) = r \qquad (7.7)
$$

$$
\tilde{\phi}(\zeta) = \phi + \zeta . \qquad (7.8)
$$

Note that $\tilde{r}(0) = r$ and $\tilde{\phi}(0) = \phi$, *i.e.* the transformation is the identity when $\zeta = 0$. We now have

$$
\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} = \frac{\partial L}{\partial \dot{r}}\,\frac{\partial \tilde{r}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}}\,\frac{\partial \tilde{\phi}}{\partial \zeta}\bigg|_{\zeta=0} = mr^2\dot{\phi} . \qquad (7.9)
$$

Another way to derive the same result which is somewhat instructive is to work out the transformation in Cartesian coordinates. We then have

$$
\tilde{x}(\zeta) = x\,\cos\zeta - y\,\sin\zeta \qquad (7.10)
$$

$$
\tilde{y}(\zeta) = x\,\sin\zeta + y\,\cos\zeta . \qquad (7.11)
$$

Thus,

$$\frac{\partial \tilde{x}}{\partial \zeta} = -y(\zeta) \quad , \quad \frac{\partial \tilde{y}}{\partial \zeta} = x(\zeta) \tag{7.12}$$

and

$$\Lambda = \frac{\partial L}{\partial \dot{x}} \frac{\partial \tilde{x}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{y}} \frac{\partial \tilde{y}}{\partial \zeta}\bigg|_{\zeta=0} = m(x\dot{y} - y\dot{x}) \ . \tag{7.13}$$

But

$$m(x\dot{y} - y\dot{x}) = m\hat{\boldsymbol{z}} \cdot \boldsymbol{r} \times \dot{\boldsymbol{r}} = mr^2\dot{\phi} \ . \tag{7.14}$$

As another example, consider the potential

$$U(\rho, \phi, z) = V(\rho, a\phi + z) \ , \tag{7.15}$$

where (ρ, ϕ, z) are cylindrical coordinates for a particle of mass m, and where a is a constant with dimensions of length. The Lagrangian is

$$\tfrac{1}{2}m\big(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{x}^2\big) - V(\rho, a\phi + z) \ . \tag{7.16}$$

This model possesses a helical symmetry, with a one-parameter family

$$\tilde{\rho}(\zeta) = \rho \tag{7.17}$$

$$\tilde{\phi}(\zeta) = \phi + \zeta \tag{7.18}$$

$$\tilde{z}(\zeta) = z - \zeta a \ . \tag{7.19}$$

Note that

$$a\tilde{\phi} + \tilde{z} = a\phi + z \ , \tag{7.20}$$

so the potential energy, and the Lagrangian as well, is invariant under this one-parameter family of transformations. The conserved charge for this symmetry is

$$\Lambda = \frac{\partial L}{\partial \dot{\rho}} \frac{\partial \tilde{\rho}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}} \frac{\partial \tilde{\phi}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{z}} \frac{\partial \tilde{z}}{\partial \zeta}\bigg|_{\zeta=0} = m\rho^2\dot{\phi} - ma\dot{z} \ . \tag{7.21}$$

We can check explicitly that Λ is conserved, using the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}\big(m\rho^2\dot{\phi}\big) = \frac{\partial L}{\partial \phi} = -a\frac{\partial V}{\partial z} \tag{7.22}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}(m\dot{z}) = \frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial z} \ . \tag{7.23}$$

Thus,

$$\dot{\Lambda} = \frac{d}{dt}\big(m\rho^2\dot{\phi}\big) - a\,\frac{d}{dt}(m\dot{z}) = 0 \ . \tag{7.24}$$

7.2 Conservation of Linear and Angular Momentum

Suppose that the Lagrangian of a mechanical system is invariant under a uniform translation of all particles in the $\hat{\boldsymbol{n}}$ direction. Then our one-parameter family of transformations is given by

$$\tilde{\boldsymbol{x}}_a = \boldsymbol{x}_a + \zeta\,\hat{\boldsymbol{n}} \ , \tag{7.25}$$

and the associated conserved Noether charge is

$$\Lambda = \sum_a \frac{\partial L}{\partial \dot{\boldsymbol{x}}_a} \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{P} \ , \tag{7.26}$$

where $\boldsymbol{P} = \sum_a \boldsymbol{p}_a$ is the *total momentum* of the system.

If the Lagrangian of a mechanical system is invariant under rotations about an axis $\hat{\boldsymbol{n}}$, then

$$\begin{aligned} \tilde{\boldsymbol{x}}_a &= R(\zeta, \hat{\boldsymbol{n}})\,\boldsymbol{x}_a \\ &= \boldsymbol{x}_a + \zeta\,\hat{\boldsymbol{n}} \times \boldsymbol{x}_a + \mathcal{O}(\zeta^2) \ , \end{aligned} \tag{7.27}$$

where we have expanded the rotation matrix $R(\zeta, \hat{\boldsymbol{n}})$ in powers of ζ. The conserved Noether charge associated with this symmetry is

$$\Lambda = \sum_a \frac{\partial L}{\partial \dot{\boldsymbol{x}}_a} \cdot \hat{\boldsymbol{n}} \times \boldsymbol{x}_a = \hat{\boldsymbol{n}} \cdot \sum_a \boldsymbol{x}_a \times \boldsymbol{p}_a = \hat{\boldsymbol{n}} \cdot \boldsymbol{L} \ , \tag{7.28}$$

where $\boldsymbol{L}$ is the *total angular momentum* of the system.

7.3 Advanced discussion : Invariance of L *vs.* invariance of S

Observant readers might object that demanding invariance of L is too strict. We should instead be demanding invariance of the action S[1]. Suppose S is invariant under

$$t \rightarrow \tilde{t}(q, t, \zeta) \tag{7.29}$$

$$q_\sigma(t) \rightarrow \tilde{q}_\sigma(q, t, \zeta) \ . \tag{7.30}$$

[1]Indeed, we should be demanding that S only change by a function of the endpoint values.

Then invariance of S means

$$S = \int_{t_a}^{t_b} dt\, L(q, \dot q, t) = \int_{\tilde t_a}^{\tilde t_b} dt\, L(\tilde q, \dot{\tilde q}, t) \ . \tag{7.31}$$

Note that t is a dummy variable of integration, so it doesn't matter whether we call it t or $\tilde t$. The endpoints of the integral, however, do change under the transformation. Now consider an infinitesimal transformation, for which $\delta t = \tilde t - t$ and $\delta q = \tilde q(\tilde t) - q(t)$ are both small. Invariance of S means

$$S = \int_{t_a}^{t_b} dt\, L(q, \dot q, t) = \int_{t_a+\delta t_a}^{t_b+\delta t_b} dt \left\{ L(q, \dot q, t) + \frac{\partial L}{\partial q_\sigma}\, \bar\delta q_\sigma + \frac{\partial L}{\partial \dot q_\sigma}\, \bar\delta \dot q_\sigma + \ldots \right\} , \tag{7.32}$$

where

$$\begin{aligned} \bar\delta q_\sigma(t) &\equiv \tilde q_\sigma(t) - q_\sigma(t) \\ &= \tilde q_\sigma(\tilde t) - \tilde q_\sigma(\tilde t) + \tilde q_\sigma(t) - q_\sigma(t) \\ &= \delta q_\sigma - \dot q_\sigma\, \delta t + \mathcal{O}(\delta q\, \delta t) \end{aligned} \tag{7.33}$$

Subtracting the top line from the bottom, we obtain

$$\begin{aligned} 0 &= L_b\, \delta t_b - L_a\, \delta t_a + \frac{\partial L}{\partial \dot q_\sigma}\bigg|_b \bar\delta q_{\sigma,b} - \frac{\partial L}{\partial \dot q_\sigma}\bigg|_a \bar\delta q_{\sigma,a} + \int_{t_a+\delta t_a}^{t_b+\delta t_b} dt \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\sigma}\right)\right\} \bar\delta q(t) \\ &= \int_{t_a}^{t_b} dt\, \frac{d}{dt} \left\{ \left(L - \frac{\partial L}{\partial \dot q_\sigma}\, \dot q_\sigma \right) \delta t + \frac{\partial L}{\partial \dot q_\sigma}\, \delta q_\sigma \right\} . \end{aligned} \tag{7.34}$$

Thus, if $\zeta \equiv \delta\zeta$ is infinitesimal, and

$$\delta t = A(q,t)\, \delta\zeta \tag{7.35}$$

$$\delta q_\sigma = B_\sigma(q,t)\, \delta\zeta \ , \tag{7.36}$$

then the conserved charge is

$$\begin{aligned} \Lambda &= \left(L - \frac{\partial L}{\partial \dot q_\sigma}\, \dot q_\sigma \right) A(q,t) + \frac{\partial L}{\partial \dot q_\sigma}\, B_\sigma(q,t) \\ &= -\, H(q,p,t)\, A(q,t) + p_\sigma\, B_\sigma(q,t) \ . \end{aligned} \tag{7.37}$$

Thus, when $A = 0$, we recover our earlier results, obtained by assuming invariance of L. Note that conservation of H follows from time translation invariance: $t \to t + \zeta$, for which $A = 1$ and $B_\sigma = 0$. Here we have written

$$H = p_\sigma \, \dot{q}_\sigma - L \ , \tag{7.38}$$

and expressed it in terms of the momenta p_σ, the coordinates q_sigma, and time t. H is called the *Hamiltonian*.

7.3.1 The Hamiltonian

The Lagrangian is a function of generalized coordinates, velocities, and time. The canonical momentum conjugate to the generalized coordinate q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \ . \tag{7.39}$$

The Hamiltonian is a function of coordinates, *momenta*, and time. It is defined as the Legendre transform of L:

$$H(q,p,t) = \sum_\sigma p_\sigma \, \dot{q}_\sigma - L \ . \tag{7.40}$$

Let's examine the differential of H:

$$\begin{aligned} dH &= \sum_\sigma \left(\dot{q}_\sigma \, dp_\sigma + p_\sigma \, d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} \, dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} \, d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} \, dt \\ &= \sum_\sigma \left(\dot{q}_\sigma \, dp_\sigma - \frac{\partial L}{\partial q_\sigma} \, dq_\sigma \right) - \frac{\partial L}{\partial t} \, dt \ , \end{aligned} \tag{7.41}$$

where we have invoked the definition of p_σ to cancel the coefficients of $d\dot{q}_\sigma$. Since $\dot{p}_\sigma = \partial L / \partial q_\sigma$, we have *Hamilton's equations of motion*,

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \quad , \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \ . \tag{7.42}$$

Thus, we can write

$$dH = \sum_\sigma \left(\dot{q}_\sigma \, dp_\sigma - \dot{p}_\sigma \, dq_\sigma \right) - \frac{\partial L}{\partial t} \, dt \ . \tag{7.43}$$

Dividing by dt, we obtain

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \ , \tag{7.44}$$

which says that the Hamiltonian is *conserved* (*i.e.* it does not change with time) whenever there is no *explicit* time dependence to L.

Example #1 : For a simple $d = 1$ system with $L = \frac{1}{2}m\dot{x}^2 - U(x)$, we have $p = m\dot{x}$ and

$$H = p\,\dot{x} - L = \tfrac{1}{2}m\dot{x}^2 + U(x) = \frac{p^2}{2m} + U(x)\ . \tag{7.45}$$

Example #2 : Consider now the mass point – wedge system analyzed above, with

$$L = \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2}m\,(1+\tan^2\!\alpha)\,\dot{x}^2 - mg\,x\,\tan\alpha\ , \tag{7.46}$$

The canonical momenta are

$$P = \frac{\partial L}{\partial \dot{X}} = (M+m)\,\dot{X} + m\dot{x} \tag{7.47}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{X} + m\,(1+\tan^2\!\alpha)\,\dot{x}\ . \tag{7.48}$$

The Hamiltonian is given by

$$\begin{aligned} H &= P\,\dot{X} + p\,\dot{x} - L \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2}m\,(1+\tan^2\!\alpha)\,\dot{x}^2 + mg\,x\,\tan\alpha\ . \end{aligned} \tag{7.49}$$

However, this is not quite H, since $H = H(X, x, P, p, t)$ must be expressed in terms of the coordinates and the *momenta* and not the coordinates and velocities. So we must eliminate $\dot{X}$ and $\dot{x}$ in favor of P and p. We do this by inverting the relations

$$\begin{pmatrix} P \\ p \end{pmatrix} = \begin{pmatrix} M+m & m \\ m & m\,(1+\tan^2\!\alpha) \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{x} \end{pmatrix} \tag{7.50}$$

to obtain

$$\begin{pmatrix} \dot{X} \\ \dot{x} \end{pmatrix} = \frac{1}{m\,\big(M + (M+m)\tan^2\!\alpha\big)} \begin{pmatrix} m\,(1+\tan^2\!\alpha) & -m \\ -m & M+m \end{pmatrix} \begin{pmatrix} P \\ p \end{pmatrix}\ . \tag{7.51}$$

Substituting into 7.49, we obtain

$$H = \frac{M+m}{2m}\,\frac{P^2\,\cos^2\!\alpha}{M + m\,\sin^2\!\alpha} - \frac{Pp\,\cos^2\!\alpha}{M + m\,\sin^2\!\alpha} + \frac{p^2}{2\,(M + m\,\sin^2\!\alpha)} + mg\,x\,\tan\alpha\ . \tag{7.52}$$

Notice that $\dot{P} = 0$ since $\frac{\partial L}{\partial X} = 0$. P is the total horizontal momentum of the system (wedge plus particle) and it is conserved.

7.3.2 Is $H = T + U$?

The most general form of the kinetic energy is

$$\begin{aligned} T &= T_2 + T_1 + T_0 \\ &= \tfrac{1}{2} T^{(2)}_{\sigma\sigma'}(q,t)\,\dot{q}_\sigma\,\dot{q}_{\sigma'} + T^{(1)}_\sigma(q,t)\,\dot{q}_\sigma + T^{(0)}(q,t) \ , \end{aligned} \tag{7.53}$$

where $T_n(q,\dot{q},t)$ is homogeneous of degree n in the velocities[2]. We assume a potential energy of the form

$$\begin{aligned} U &= U_1 + U_0 \\ &= U^{(1)}_\sigma(q,t)\,\dot{q}_\sigma + U^{(0)}(q,t) \ , \end{aligned} \tag{7.54}$$

which allows for velocity-dependent forces, as we have with charged particles moving in an electromagnetic field. The Lagrangian is then

$$L = T - U = \tfrac{1}{2} T^{(2)}_{\sigma\sigma'}(q,t)\,\dot{q}_\sigma\,\dot{q}_{\sigma'} + T^{(1)}_\sigma(q,t)\,\dot{q}_\sigma + T^{(0)}(q,t) - U^{(1)}_\sigma(q,t)\,\dot{q}_\sigma - U^{(0)}(q,t) \ . \tag{7.55}$$

We have assumed $U(q,t)$ is velocity-independent, but the above form for $L = T - U$ is quite general. (*E.g.* any velocity-dependence in U can be absorbed into the $B_\sigma\,\dot{q}_\sigma$ term.) The canonical momentum conjugate to q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T^{(2)}_{\sigma\sigma'}\,\dot{q}_{\sigma'} + T^{(1)}_\sigma(q,t) - U^{(1)}_\sigma(q,t) \tag{7.56}$$

which is inverted to give

$$\dot{q}_\sigma = {T^{(2)}_{\sigma\sigma'}}^{-1} \left(p_{\sigma'} - T^{(1)}_{\sigma'} + U^{(1)}_{\sigma'} \right) \ . \tag{7.57}$$

The Hamiltonian is then

$$H = p_\sigma\,\dot{q}_\sigma - L$$

$$= \tfrac{1}{2}\,{T^{(2)}_{\sigma\sigma'}}^{-1} \left(p_\sigma - T^{(1)}_\sigma + U^{(1)}_\sigma \right) \left(p_{\sigma'} - T^{(1)}_{\sigma'} + U^{(1)}_{\sigma'} \right) - T_0 + U_0 \tag{7.58}$$

$$= T_2 - T_0 + U_0 \ . \tag{7.59}$$

If T_0, T_1, and U_1 vanish, *i.e.* if $T(q,\dot{q},t)$ is a homogeneous function of degree two in the generalized velocities, and $U(q,t)$ is velocity-independent, then $H = T + U$. But if T_0 or T_1 is nonzero, or the potential is velocity-dependent, then $H \neq T + U$.

[2] A homogeneous function of degree k satisfies $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$. It is then easy to prove *Euler's theorem*, $\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = kf$.

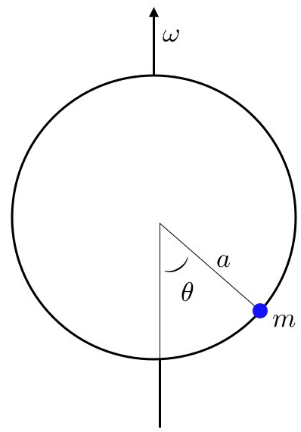

Figure 7.1: A bead of mass m on a rotating hoop of radius a.

7.3.3 Example: A Bead on a Rotating Hoop

Consider a bead of mass m constrained to move along a hoop of radius a. The hoop is further constrained to rotate with angular velocity ω about the $\hat{\boldsymbol{z}}$-axis, as shown in Fig. 7.1.

The most convenient set of generalized coordinates is spherical polar (r, θ, ϕ), in which case

$$\begin{aligned} T &= \tfrac{1}{2} m \big(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2\big) \\ &= \tfrac{1}{2} m a^2 \big(\dot{\theta}^2 + \omega^2 \sin^2\theta\big) \ . \end{aligned} \tag{7.60}$$

Thus, $T_2 = \frac{1}{2} m a^2 \dot{\theta}^2$ and $T_0 = \frac{1}{2} m a^2 \omega^2 \sin^2\theta$. The potential energy is $U(\theta) = mga(1 - \cos\theta)$. The momentum conjugate to θ is $p_\theta = ma^2\dot{\theta}$, and thus

$$\begin{aligned} H(\theta, p) &= T_2 - T_0 + U \\ &= \tfrac{1}{2} m a^2 \dot{\theta}^2 - \tfrac{1}{2} m a^2 \omega^2 \sin^2\theta + mga(1-\cos\theta) \\ &= \frac{p_\theta^2}{2ma^2} - \tfrac{1}{2} m a^2 \omega^2 \sin^2\theta + mga(1-\cos\theta) \ . \end{aligned} \tag{7.61}$$

For this problem, we can define the *effective potential*

$$\begin{aligned} U_{\rm eff}(\theta) \equiv U - T_0 &= mga(1-\cos\theta) - \tfrac{1}{2} m a^2 \omega^2 \sin^2\theta \\ &= mga\bigg(1 - \cos\theta - \frac{\omega^2}{2\omega_0^2}\,\sin^2\theta\bigg) \ , \end{aligned} \tag{7.62}$$

where $\omega_0 \equiv g/a^2$. The Lagrangian may then be written

$$L = \tfrac{1}{2} m a^2 \dot{\theta}^2 - U_{\rm eff}(\theta) \ , \tag{7.63}$$

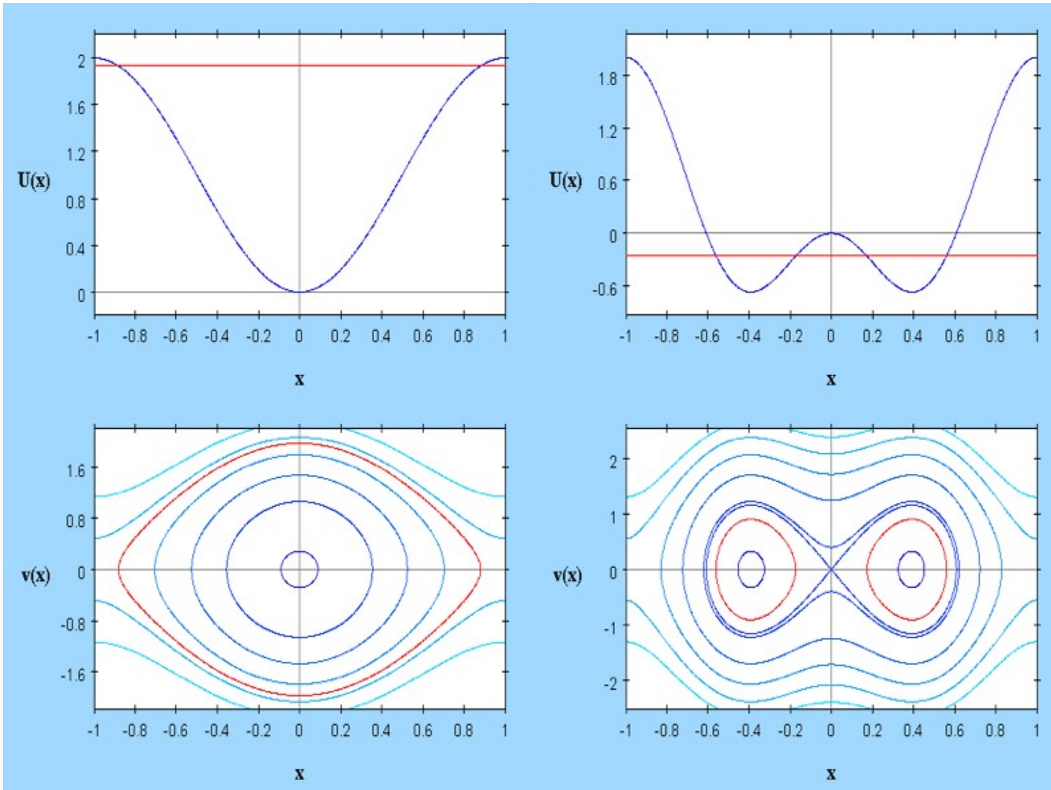

Figure 7.2: The effective potential $U_{\text{eff}}(\theta) = mga\big[1 - \cos\theta - \frac{\omega^2}{2\omega_0^2}\sin^2\theta\big]$. (The dimensionless potential $\tilde{U}_{\text{eff}}(x) = U_{\text{eff}}/mga$ is shown, where $x = \theta/\pi$.) Left panels: $\omega = \frac{1}{2}\sqrt{3}\,\omega_0$. Right panels: $\omega = \sqrt{3}\,\omega_0$.

and thus the equations of motion are

$$ma^2\ddot{\theta} = -\frac{\partial U_{\text{eff}}}{\partial\theta} \ . \tag{7.64}$$

Equilibrium is achieved when $U'_{\text{eff}}(\theta) = 0$, which gives

$$\frac{\partial U_{\text{eff}}}{\partial\theta} = mga\sin\theta\left\{1 - \frac{\omega^2}{\omega_0^2}\cos\theta\right\} = 0 \ , \tag{7.65}$$

i.e. $\theta^* = 0$, $\theta^* = \pi$, or $\theta^* = \pm\cos^{-1}(\omega_0^2/\omega^2)$, where the last pair of equilibria are present only for $\omega^2 > \omega_0^2$. The stability of these equilibria is assessed by examining the sign of $U''_{\text{eff}}(\theta^*)$. We have

$$U''_{\text{eff}}(\theta) = mga\left\{\cos\theta - \frac{\omega^2}{\omega_0^2}\big(2\cos^2\theta - 1\big)\right\} \ . \tag{7.66}$$

Thus,

$$U''_{\rm eff}(\theta^*) = \begin{cases} mga\left(1 - \frac{\omega^2}{\omega_0^2}\right) & \text{at } \theta^* = 0 \\ \\ -mga\left(1 + \frac{\omega^2}{\omega_0^2}\right) & \text{at } \theta^* = \pi \\ \\ mga\left(\frac{\omega^2}{\omega_0^2} - \frac{\omega_0^2}{\omega^2}\right) & \text{at } \theta^* = \pm\cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right) \ . \end{cases} \tag{7.67}$$

Thus, $\theta^* = 0$ is stable for $\omega^2 < \omega_0^2$ but becomes unstable when the rotation frequency ω is sufficiently large, *i.e.* when $\omega^2 > \omega_0^2$. In this regime, there are two new equilibria, at $\theta^* = \pm\cos^{-1}(\omega_0^2/\omega^2)$, which are both stable. The equilibrium at $\theta^* = \pi$ is always unstable, independent of the value of ω. The situation is depicted in Fig. 7.2.

7.4 Charged Particle in a Magnetic Field

Consider next the case of a charged particle moving in the presence of an electromagnetic field. The particle's potential energy is

$$U(\boldsymbol{r}) = q\,\phi(\boldsymbol{r},t) - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\cdot\dot{\boldsymbol{r}} \ , \tag{7.68}$$

which *is* velocity-dependent. The kinetic energy is $T = \frac{1}{2}m\,\dot{\boldsymbol{r}}^2$, as usual. Here $\phi(\boldsymbol{r})$ is the scalar potential and $\boldsymbol{A}(\boldsymbol{r})$ the vector potential. The electric and magnetic fields are given by

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\,\frac{\partial\boldsymbol{A}}{\partial t} \quad , \quad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \ . \tag{7.69}$$

The canonical momentum is

$$\boldsymbol{p} = \frac{\partial L}{\partial\dot{\boldsymbol{r}}} = m\,\dot{\boldsymbol{r}} + \frac{q}{c}\,\boldsymbol{A} \ , \tag{7.70}$$

and hence the Hamiltonian is

$$\begin{aligned} H(\boldsymbol{r},\boldsymbol{p},t) &= \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - L \\ &= m\dot{\boldsymbol{r}}^2 + \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} - \tfrac{1}{2}m\,\dot{\boldsymbol{r}}^2 - \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} + q\,\phi \\ &= \tfrac{1}{2}m\,\dot{\boldsymbol{r}}^2 + q\,\phi \\ &= \frac{1}{2m}\Big(\boldsymbol{p} - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\Big)^2 + q\,\phi(\boldsymbol{r},t) \ . \end{aligned} \tag{7.71}$$

If $\boldsymbol{A}$ and ϕ are time-independent, then $H(\boldsymbol{r},\boldsymbol{p})$ is conserved.

Let's work out the equations of motion. We have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}} \tag{7.72}$$

which gives

$$m\,\ddot{\boldsymbol{r}} + \frac{q}{c}\,\frac{d\boldsymbol{A}}{dt} = -q\,\boldsymbol{\nabla}\phi + \frac{q}{c}\,\boldsymbol{\nabla}(\boldsymbol{A}\cdot\dot{\boldsymbol{r}}) \ , \tag{7.73}$$

or, in component notation,

$$m\,\ddot{x}_i + \frac{q}{c}\,\frac{\partial A_i}{\partial x_j}\,\dot{x}_j + \frac{q}{c}\,\frac{\partial A_i}{\partial t} = -q\,\frac{\partial \phi}{\partial x_i} + \frac{q}{c}\,\frac{\partial A_j}{\partial x_i}\,\dot{x}_j \ , \tag{7.74}$$

which is to say

$$m\,\ddot{x}_i = -q\,\frac{\partial \phi}{\partial x_i} - \frac{q}{c}\,\frac{\partial A_i}{\partial t} + \frac{q}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\dot{x}_j \ . \tag{7.75}$$

It is convenient to express the cross product in terms of the completely antisymmetric tensor of rank three, ϵ_{ijk}:

$$B_i = \epsilon_{ijk}\,\frac{\partial A_k}{\partial x_j} \ , \tag{7.76}$$

and using the result

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km} \ , \tag{7.77}$$

we have $\epsilon_{ijk}\,B_i = \partial_j\,A_k - \partial_k\,A_j$, and

$$m\,\ddot{x}_i = -q\,\frac{\partial \phi}{\partial x_i} - \frac{q}{c}\,\frac{\partial A_i}{\partial t} + \frac{q}{c}\,\epsilon_{ijk}\,\dot{x}_j\,B_k \ , \tag{7.78}$$

or, in vector notation,

$$\begin{aligned} m\,\ddot{\boldsymbol{r}} &= -q\,\boldsymbol{\nabla}\phi - \frac{q}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} + \frac{q}{c}\,\dot{\boldsymbol{r}}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) \\ &= q\,\boldsymbol{E} + \frac{q}{c}\,\dot{\boldsymbol{r}}\times\boldsymbol{B} \ , \end{aligned} \tag{7.79}$$

which is, of course, the Lorentz force law.

7.5 Field Theory: Systems with Several Independent Variables

Suppose $\phi_a(\boldsymbol{x})$ depends on several independent variables: $\{x^1, x^2, \ldots, x^n\}$. Furthermore, suppose

$$S\big[\{\phi_a(\boldsymbol{x})\big] = \int\limits_{\Omega} d\boldsymbol{x}\,\mathcal{L}(\phi_a\,\partial_\mu\phi_a, \boldsymbol{x}) \ , \tag{7.80}$$

i.e. the *Lagrangian density* $\mathcal{L}$ is a function of the fields ϕ_a and their partial derivatives $\partial\phi_a/\partial x^\mu$. Here Ω is a region in R^K. Then the first variation of S is

$$\delta S = \int\limits_{\Omega} d\boldsymbol{x} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}\,\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\frac{\partial\,\delta\phi_a}{\partial x^\mu} \right\}$$
$$= \oint\limits_{\partial\Omega} d\Sigma\, n^\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a - \int\limits_{\Omega} d\boldsymbol{x} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right\}\delta\phi_a \ , \tag{7.81}$$

where $\partial\Omega$ is the $(n-1)$-dimensional boundary of Ω, $d\Sigma$ is the differential surface area, and n^μ is the unit normal. If we demand $\partial\mathcal{L}/\partial(\partial_\mu\phi_a)\big|_{\partial\Omega} = 0$ of $\delta\phi_a\big|_{\partial\Omega} = 0$, the surface term vanishes, and we conclude

$$\frac{\delta S}{\delta\phi_a(\boldsymbol{x})} = \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) \ . \tag{7.82}$$

As an example, consider the case of a stretched string of linear mass density μ and tension τ. The action is a functional of the height $y(x,t)$, where the coordinate along the string, x, and time, t, are the two independent variables. The Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 \ , \tag{7.83}$$

whence the Euler-Lagrange equations are

$$0 = \frac{\delta S}{\delta y(x,t)} = -\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial y'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right)$$
$$= \tau\,\frac{\partial^2 y}{\partial x^2} - \mu\,\frac{\partial^2 y}{\partial t^2} \ , \tag{7.84}$$

where $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$. Thus, $\mu\ddot{y} = \tau y''$, which is the Helmholtz equation. We've assumed boundary conditions where $\delta y(x_a,t) = \delta y(x_b,t) = \delta y(x,t_a) = \delta y(x,t_b) = 0$.

The Lagrangian density for an electromagnetic field with sources is

$$\mathcal{L} = -\tfrac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} - J_\mu\,A^\mu \ . \tag{7.85}$$

The equations of motion are then

$$\frac{\partial\mathcal{L}}{\partial A^\nu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) = 0 \quad\Rightarrow\quad \partial_\mu\,F^{\mu\nu} = 4\pi J^\nu \ , \tag{7.86}$$

which are Maxwell's equations.

Recall the result of Noether's theorem for mechanical systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} = 0\ , \tag{7.87}$$

where $\tilde{q}_\sigma = \tilde{q}_\sigma(q,\zeta)$ is a one-parameter (ζ) family of transformations of the generalized coordinates which leaves L invariant. We generalize to field theory by replacing

$$q_\sigma(t) \longrightarrow \phi_a(\boldsymbol{x},t)\ , \tag{7.88}$$

where $\{\phi_a(\boldsymbol{x},t)\}$ are a set of fields, which are functions of the independent variables $\{x,y,z,t\}$. We will adopt covariant relativistic notation and write for four-vector $x^\mu = (ct,x,y,z)$. The generalization of $d\Lambda/dt = 0$ is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\right)_{\zeta=0} = 0\ , \tag{7.89}$$

where there is an implied sum on both μ and a. We can write this as $\partial_\mu\, J^\mu = 0$, where

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0}\ . \tag{7.90}$$

We call $\Lambda = J^0/c$ the *total charge*. If we assume $\boldsymbol{J} = 0$ at the spatial boundaries of our system, then integrating the conservation law $\partial_\mu\, J^\mu$ over the spatial region Ω gives

$$\frac{d\Lambda}{dt} = \int\limits_\Omega d^3\!x\, \partial_0\, J^0 = -\int\limits_\Omega d^3\!x\, \boldsymbol{\nabla}\cdot\boldsymbol{J} = -\oint\limits_{\partial\Omega} d\Sigma\, \hat{\boldsymbol{n}}\cdot\boldsymbol{J} = 0\ , \tag{7.91}$$

assuming $\boldsymbol{J} = 0$ at the boundary $\partial\Omega$.

As an example, consider the case of a complex scalar field, with Lagrangian density[3]

$$\mathcal{L}(\psi,,\psi^*,\partial_\mu\psi,\partial_\mu\psi^*) = \tfrac{1}{2}K\,(\partial_\mu\psi^*)(\partial^\mu\psi) - U\big(\psi^*\psi\big)\ . \tag{7.92}$$

This is invariant under the transformation $\psi \to e^{i\zeta}\,\psi$, $\psi^* \to e^{-i\zeta}\,\psi^*$. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = i\,e^{i\zeta}\,\psi \qquad , \qquad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -i\,e^{-i\zeta}\,\psi^*\ , \tag{7.93}$$

and, summing over both ψ and ψ^* fields, we have

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\cdot(i\psi) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi^*\right)}\cdot(-i\psi^*) \\ &= \frac{K}{2i}\big(\psi^*\partial^\mu\psi - \psi\,\partial^\mu\psi^*\big)\ . \end{aligned} \tag{7.94}$$

The potential, which depends on $|\psi|^2$, is independent of ζ. Hence, this form of conserved 4-current is valid for an entire class of potentials.

[3]We raise and lower indices using the Minkowski metric $g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$.

7.5.1 Gross-Pitaevskii Model

As one final example of a field theory, consider the Gross-Pitaevskii model, with

$$\mathcal{L} = i\hbar\, \psi^* \frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\, \boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi - g\left(|\psi|^2 - n_0\right)^2 . \tag{7.95}$$

This describes a Bose fluid with repulsive short-ranged interactions. Here $\psi(\boldsymbol{x}, t)$ is again a complex scalar field, and ψ^* is its complex conjugate. Using the Leibniz rule, we have

$$\begin{aligned}
\delta S[\psi^*, \psi] &= S[\psi^* + \delta\psi^*, \psi + \delta\psi] \\
&= \int\!dt\!\int\!d^d\!x \left\{ i\hbar\, \psi^* \frac{\partial\delta\psi}{\partial t} + i\hbar\, \delta\psi^* \frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\, \boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\delta\psi - \frac{\hbar^2}{2m}\, \boldsymbol{\nabla}\delta\psi^* \cdot \boldsymbol{\nabla}\psi \right. \\
&\qquad\qquad \left. - \, 2g\left(|\psi|^2 - n_0\right)\left(\psi^*\delta\psi + \psi\delta\psi^*\right) \right\} \\
&= \int\!dt\!\int\!d^d\!x \left\{ \left[-\, i\hbar\, \frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\, \nabla^2\psi^* - 2g\left(|\psi|^2 - n_0\right)\psi^* \right] \delta\psi \right. \\
&\qquad\qquad \left. + \left[i\hbar\, \frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\, \nabla^2\psi - 2g\left(|\psi|^2 - n_0\right)\psi \right] \delta\psi^* \right\} ,
\end{aligned} \tag{7.96}$$

where we have integrated by parts where necessary and discarded the boundary terms. Extremizing $S[\psi^*, \psi]$ therefore results in the *nonlinear Schrödinger equation* (NLSE),

$$i\hbar\, \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\, \nabla^2\psi + 2g\left(|\psi|^2 - n_0\right)\psi \tag{7.97}$$

as well as its complex conjugate,

$$-i\hbar\, \frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\, \nabla^2\psi^* + 2g\left(|\psi|^2 - n_0\right)\psi^* . \tag{7.98}$$

Note that these equations are indeed the Euler-Lagrange equations:

$$\frac{\delta S}{\delta\psi} = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\, \partial_\mu\psi}\right) \tag{7.99}$$

$$\frac{\delta S}{\delta\psi^*} = \frac{\partial\mathcal{L}}{\partial\psi^*} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\, \partial_\mu\psi^*}\right) , \tag{7.100}$$

with $x^\mu = (t, \boldsymbol{x})$[4] Plugging in

$$\frac{\partial\mathcal{L}}{\partial\psi} = -2g\left(|\psi|^2 - n_0\right)\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\, \partial_t\psi} = i\hbar\, \psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\, \boldsymbol{\nabla}\psi} = -\frac{\hbar^2}{2m}\, \boldsymbol{\nabla}\psi^* \tag{7.101}$$

[4] In the nonrelativistic case, there is no utility in defining $x^0 = ct$, so we simply define $x^0 = t$.

and

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = i\hbar\,\psi - 2g\left(|\psi|^2 - n_0\right)\psi \quad , \quad \frac{\partial \mathcal{L}}{\partial\,\partial_t \psi^*} = 0 \quad , \quad \frac{\partial \mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi \ , \tag{7.102}$$

we recover the NLSE and its conjugate.

The Gross-Pitaevskii model also possesses a U(1) invariance, under

$$\psi(\boldsymbol{x},t) \to \tilde{\psi}(\boldsymbol{x},t) = e^{i\zeta}\,\psi(\boldsymbol{x},t) \quad , \quad \psi^*(\boldsymbol{x},t) \to \tilde{\psi}^*(\boldsymbol{x},t) = e^{-i\zeta}\,\psi^*(\boldsymbol{x},t) \ . \tag{7.103}$$

Thus, the conserved Noether current is then

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\,\partial_\mu \psi}\,\frac{\partial \tilde{\psi}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial \mathcal{L}}{\partial\,\partial_\mu \psi^*}\,\frac{\partial \tilde{\psi}^*}{\partial \zeta}\bigg|_{\zeta=0}$$

$$J^0 = -\hbar\,|\psi|^2 \tag{7.104}$$

$$\boldsymbol{J} = -\frac{\hbar^2}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) \ . \tag{7.105}$$

Dividing out by $\hbar$, taking $J^0 \equiv -\hbar\rho$ and $\boldsymbol{J} \equiv -\hbar\boldsymbol{j}$, we obtain the continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0 \ , \tag{7.106}$$

where

$$\rho = |\psi|^2 \quad , \quad \boldsymbol{j} = \frac{\hbar}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) \ . \tag{7.107}$$

are the particle density and the particle current, respectively.