PHYSICS 110A : CLASSICAL MECHANICS
MIDTERM EXAM #2

[**1**] A point mass m slides frictionlessly, under the influence of gravity, along a massive ring of radius a and mass M. The ring is affixed by horizontal springs to two fixed vertical surfaces, as depicted in fig. 1. All motion is within the plane of the figure.

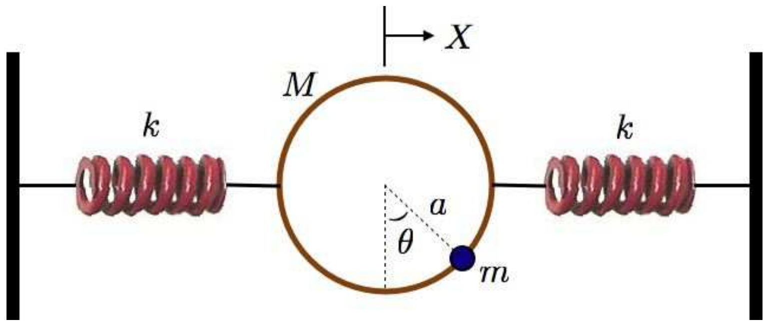

Figure 1: A point mass m slides frictionlessly along a massive ring of radius a and mass M, which is affixed by horizontal springs to two fixed vertical surfaces.

(a) Choose as generalized coordinates the horizontal displacement X of the center of the ring with respect to equilibrium, and the angle θ a radius to the mass m makes with respect to the vertical (see fig. 1). You may assume that at $X = 0$ the springs are both unstretched. Find the Lagrangian $L(X, \theta, \dot{X}, \dot{\theta}, t)$.
[15 points]

The coordinates of the mass point are

$$x = X + a\sin\theta \quad , \quad y = -a\cos\theta \ .$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\big(\dot{X} + a\cos\theta\,\dot{\theta}\big)^2 + \tfrac{1}{2}ma^2\sin^2\!\theta\,\dot{\theta}^2 \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}ma^2\dot{\theta}^2 + ma\cos\theta\,\dot{X}\,\dot{\theta} \ . \end{aligned}$$

The potential energy is

$$U = kX^2 - mga\cos\theta \ .$$

Thus, the Lagrangian is

$$L = \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}ma^2\dot{\theta}^2 + ma\cos\theta\,\dot{X} - kX^2 + mga\cos\theta \ .$$

(b) Find the generalized momenta p_X and p_θ, and the generalized forces F_X and F_θ
[10 points]

We have

$$p_X = \frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + ma\cos\theta\,\dot{\theta} \quad , \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} + ma\cos\theta\,\dot{X} \ .$$

For the forces,

$$F_X = \frac{\partial L}{\partial X} = -2kX \quad , \quad F_\theta = \frac{\partial L}{\partial \theta} = -ma\sin\theta\,\dot{X}\,\dot{\theta} - mga\sin\theta \ .$$

(c) Derive the equations of motion.
[15 points]

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial L}{\partial q_\sigma} \ ,$$

for each generalized coordinate q_σ. For X we have

$$(M+m)\ddot{X} + ma\cos\theta\,\ddot{\theta} - ma\sin\theta\,\dot{\theta}^2 = -2kX \ .$$

For θ,

$$ma^2\,\ddot{\theta} + ma\cos\theta\ddot{X} = -mga\sin\theta \ .$$

(d) Find expressions for all conserved quantities.
[10 points]

Horizontal and vertical translational symmetries are broken by the springs and by gravity, respectively. The remaining symmetry is that of time translation. From $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$, we have that $H = \sum_\sigma p_\sigma\,\dot{q}_\sigma - L$ is conserved. For this problem, the kinetic energy is a homogeneous function of degree 2 in the generalized velocities, and the potential is velocity-independent. Thus,

$$H = T + U = \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}ma^2\dot{\theta}^2 + ma\cos\theta\,\dot{X}\,\dot{\theta} + kX^2 - mga\cos\theta \ .$$

[2] A point particle of mass m moves in three dimensions in a helical potential

$$U(\rho,\phi,z) = U_0\,\rho\,\cos\!\left(\phi - \frac{2\pi z}{b}\right) .$$

We call b the pitch of the helix.

(a) Write down the Lagrangian, choosing (ρ,ϕ,z) as generalized coordinates.
`[10 points]`

The Lagrangian is

$$L = \tfrac{1}{2}m\big(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\big) - U_0\,\rho\,\cos\!\left(\phi - \frac{2\pi z}{b}\right)$$

(b) Find the equations of motion.
`[20 points]`

Clearly

$$p_\rho = m\dot\rho \quad , \quad p_\phi = m\rho^2\,\dot\phi \quad , \quad p_z = m\dot z \ ,$$

and

$$F_\rho = m\rho\,\dot\phi^2 - U_0\,\cos\!\left(\phi - \frac{2\pi z}{b}\right) \quad , \quad F_\phi = U_0\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) \quad , \quad F_z = -\frac{2\pi U_0}{b}\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) .$$

Thus, the equation of motion are

$$\begin{aligned} m\ddot\rho &= m\rho\,\dot\phi^2 - U_0\,\cos\!\left(\phi - \frac{2\pi z}{b}\right) \\ m\rho^2\,\ddot\phi + 2m\rho\,\dot\rho\,\dot\phi &= U_0\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) \\ m\ddot z &= -\frac{2\pi U_0}{b}\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) . \end{aligned}$$

(c) Show that there exists a continuous one-parameter family of coordinate transformations which leaves L invariant. Find the associated conserved quantity, Λ. Is anything else conserved?
`[20 points]`

Due to the helical symmetry, we have that

$$\phi \to \phi + \zeta \quad , \quad z \to z + \frac{b}{2\pi}\,\zeta$$

is such a continuous one-parameter family of coordinate transformations. Since it leaves the combination $\phi - \frac{2\pi z}{b}$ unchanged, we have that $\frac{dL}{d\zeta} = 0$, and

$$\begin{aligned}
\Lambda &= p_\rho \, \frac{\partial \rho}{\partial \zeta}\bigg|_{\zeta=0} + p_\phi \, \frac{\partial \phi}{\partial \zeta}\bigg|_{\zeta=0} + p_z \, \frac{\partial z}{\partial \zeta}\bigg|_{\zeta=0} \\
&= p_\phi + \frac{b}{2\pi} \, p_z \\
&= m\rho^2 \, \dot\phi + \frac{mb}{2\pi} \, \dot z
\end{aligned}$$

is the conserved Noether 'charge'. The other conserved quantity is the Hamiltonian,

$$H = \tfrac{1}{2} m \big(\dot\rho^2 + \rho^2 \dot\phi^2 + \dot z^2 \big) + U_0 \, \rho \, \cos\left(\phi - \frac{2\pi z}{b} \right) .$$

Note that $H = T + U$, because T is homogeneous of degree 2 and U is homogeneous of degree 0 in the generalized velocities.