Forced damped harmonic oscillator

$$\ddot{x} + 2\beta\,\dot{x} + \omega_0^2\,x = f(t)$$

homogeneous solution (valid when $f(t) = 0$):

$$x_{\rm h}(t) = \begin{cases} C\,e^{-\beta t}\,\cos(\nu t + \varphi) & \text{if } \omega_0^2 > \beta^2 \text{ (underdamped)} \\ (C + Dt)\,e^{-\beta t} & \text{if } \omega_0^2 = \beta^2 \text{ (critically damped)} \\ \tilde{C}\,e^{-\beta t}\,\cosh(\tilde{\nu} t + \tilde{\varphi}) & \text{if } \omega_0^2 < \beta^2 \text{ (overdamped)} \end{cases}$$

with $\nu = i\tilde{\nu} = \sqrt{\omega_0^2 - \beta^2}$

harmonic forcing: $f(t) = f_0 \cos(\Omega t) \implies$

$$x(t) = x_{\rm h}(t) + A(\Omega)\,f_0\,\cos\big(\Omega t - \delta(\Omega)\big)$$

amplitude : $$A(\Omega) = \left[(\omega_0^2 - \Omega^2)^2 + 4\beta^2\Omega^2\right]^{-1/2}$$

phase shift : $$\delta(\Omega) = \tan^{-1}\left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\right)$$ $\delta(\omega_0) = \frac{\pi}{2}$

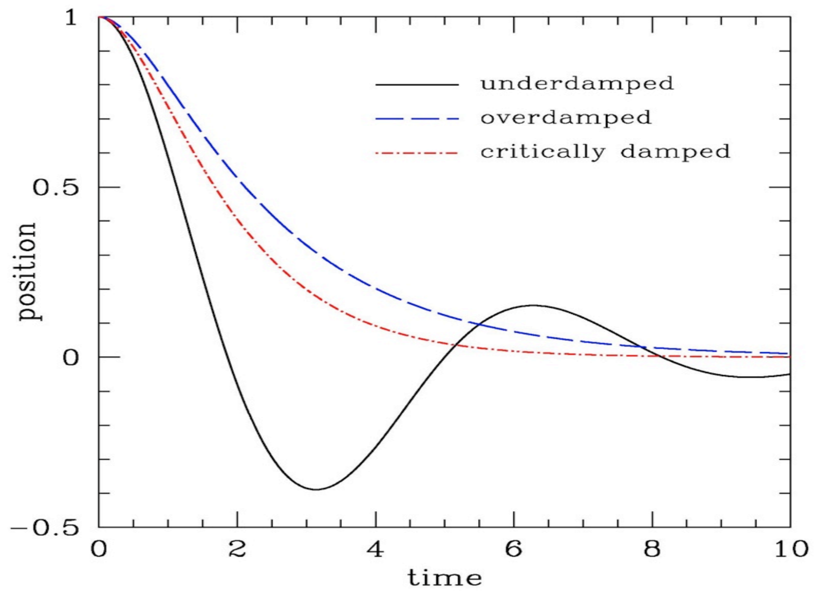
adjust constants in $x_{\rm h}(t)$ to satisfy boundary conditions

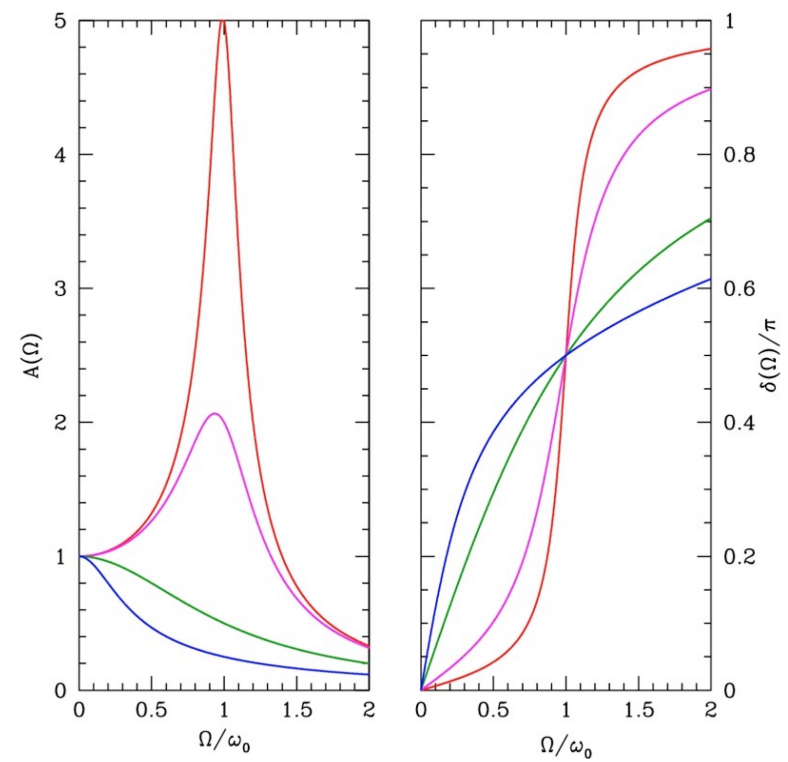
$A'(\Omega) = 0 \;\Rightarrow\; \Omega = 0,\; \Omega^2 = \omega_0^2 - 2\beta^2 \equiv \Omega_{\rm R}^2$

if $\omega_0^2 > 2\beta^2$, $\Omega = 0$ local minimum and $\Omega = \Omega_{\rm R}$ global maximum

if $\omega_0^2 < 2\beta^2$, $\Omega = 0$ global maximum

quality factor: $Q = \dfrac{\Omega_{\rm R}}{2\beta}$





$\beta/\omega_0 =$ 0.1 / 0.25 / 1.0 / 2.0