0.1 DISCUSSION #6 PROBLEM

0.1.1 Problem and Constraints

Consider the point mass m inside the hoop of radius R, depicted in Fig. 1. We choose as generalized coordinates the Cartesian coordinates (X, Y) of the center of the hoop, the Cartesian coordinates (x, y) for the point mass, the angle ϕ through which the hoop turns, and the angle θ which the point mass makes with respect to the vertical. These six coordinates are not all independent. Indeed, there are only two independent coordinates for this system, which can be taken to be θ and ϕ. Thus, there are *four* constraints:

$$X - R\phi \equiv G_1 = 0 \tag{1}$$

$$Y - R \equiv G_2 = 0 \tag{2}$$

$$x - X - R\sin\theta \equiv G_3 = 0 \tag{3}$$

$$y - Y + R\cos\theta \equiv G_4 = 0\ . \tag{4}$$

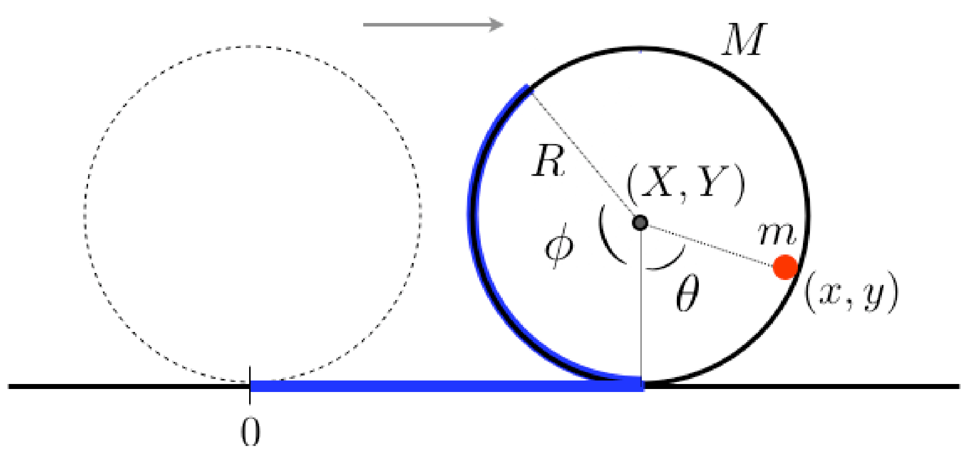

Figure 1: A point mass m inside a hoop of mass M, radius R, and moment of inertia I.

The kinetic and potential energies are easily expressed in terms of the Cartesian coordinates, aside from the energy of rotation of the hoop about its CM, which is expressed in terms of $\dot{\phi}$:

$$T = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 \tag{5}$$

$$U = MgY + mgy\ . \tag{6}$$

The moment of inertia of the hoop about its CM is $I = MR^2$, but we could imagine a situation in which I were different. For example, we could instead place the point mass inside a very short cylinder with two solid end caps, in which case $I = \frac{1}{2}MR^2$. The

Lagrangian is then

$$L = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 - MgY - mgy\ . \tag{7}$$

Note that L as written is completely independent of θ and $\dot{\theta}$!

0.1.2 Continuous Symmetry

Note that there is an continuous symmetry to L which is satisfied by all the constraints, under

$$\tilde{X}(\zeta) = X + \zeta \qquad\qquad \tilde{Y}(\zeta) = Y \tag{8}$$

$$\tilde{x}(\zeta) = x + \zeta \qquad\qquad \tilde{y}(\zeta) = y \tag{9}$$

$$\tilde{\phi}(\zeta) = \phi + \frac{\zeta}{R} \qquad\qquad \tilde{\theta}(\zeta) = \theta\ . \tag{10}$$

Thus, according to Noether's theorem, there is a conserved quantity

$$\begin{aligned}\Lambda &= \frac{\partial L}{\partial \dot{X}} + \frac{\partial L}{\partial \dot{x}} + \frac{1}{R}\frac{\partial L}{\partial \dot{\phi}} \\ &= M\dot{X} + m\dot{x} + \frac{I}{R}\,\dot{\phi}\ . \end{aligned} \tag{11}$$

This means $\dot{\Lambda} = 0$. This reflects the overall conservation of momentum in the x-direction.

0.1.3 Energy Conservation

Since neither L nor any of the constraints are explicitly time-dependent, the Hamiltonian is conserved. And since T is homogeneous of degree two in the generalized velocities, we have $H = E = T + U$:

$$E = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 + MgY + mgy\ . \tag{12}$$

0.1.4 Equations of Motion

We have $n = 6$ generalized coordinates and $k = 4$ constraints. Thus, there are four undetermined multipliers $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ used to impose the constraints. This makes for ten unknowns:

$$X, Y, x, y, \phi, \theta, \lambda_1, \lambda_2, \lambda_3, \lambda_4\ . \tag{13}$$

Accordingly, we have ten equations: six equations of motion plus the four equations of constraint. The equations of motion are obtained from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial L}{\partial q_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial q_\sigma} \ . \tag{14}$$

Taking each generalized coordinate in turn, the equations of motion are thus

$$M\ddot{X} = \lambda_1 - \lambda_3 \tag{15}$$

$$M\ddot{Y} = -Mg + \lambda_2 - \lambda_4 \tag{16}$$

$$m\ddot{x} = \lambda_3 \tag{17}$$

$$m\ddot{y} = -mg + \lambda_4 \tag{18}$$

$$I\,\ddot{\phi} = -R\,\lambda_1 \tag{19}$$

$$0 = -R\cos\theta\,\lambda_3 - R\sin\theta\,\lambda_4 \ . \tag{20}$$

Along with the four constraint equations, these determine the motion of the system. Note that the last of the equations of motion, for the generalized coordinate $q_\sigma = \theta$, says that $Q_\theta = 0$, which means that the force of constraint on the point mass is radial. Were the point mass replaced by a rolling object, there would be an angular component to this constraint in order that there be no slippage.

0.1.5 Implementation of Constraints

We now use the constraint equations to eliminate X, Y, x, and y in terms of θ and ϕ:

$$X = R\phi \quad , \quad Y = R \quad , \quad x = R\phi + R\sin\theta \quad , \quad y = R(1-\cos\theta) \ . \tag{21}$$

We also need the derivatives:

$$\dot{x} = R\,\dot{\phi} + R\cos\theta\,\dot{\theta} \quad , \quad \ddot{x} = R\,\ddot{\phi} + R\cos\theta\,\ddot{\theta} - R\sin\theta\,\dot{\theta}^2 \ , \tag{22}$$

and

$$\dot{y} = R\sin\theta\,\dot{\theta} \quad , \quad \ddot{x} = R\sin\theta\,\ddot{\theta} + R\cos\theta\,\dot{\theta}^2 \ , \tag{23}$$

as well as

$$\dot{X} = R\,\dot{\phi} \quad , \quad \ddot{X} = R\,\ddot{\phi} \quad , \quad \dot{Y} = 0 \quad , \quad \ddot{Y} = 0 \ . \tag{24}$$

We now may write the conserved charge as

$$\Lambda = \frac{1}{R}(I + MR^2 + mR^2)\,\dot{\phi} + mR\cos\theta\,\dot{\theta} \ . \tag{25}$$

This, in turn, allows us to eliminate $\dot{\phi}$ in terms of $\dot{\theta}$ and the constant Λ:

$$\dot{\phi} = \frac{\gamma}{1+\gamma}\left(\frac{\Lambda}{mR} - \dot{\theta}\cos\theta\right) , \tag{26}$$

where

$$\gamma = \frac{mR^2}{I + MR^2} . \tag{27}$$

The energy is then

$$\begin{aligned} E &= \tfrac{1}{2}(I + MR^2)\,\dot{\phi}^2 + \tfrac{1}{2}m\big(R^2\,\dot{\phi}^2 + R^2\,\dot{\theta}^2 + 2R^2\cos\theta\,\dot{\phi}\,\dot{\theta}\big) + MgR + mgR(1-\cos\theta) \\ &= \tfrac{1}{2}mR^2\left\{\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\dot{\theta}^2 + \frac{2g}{R}\,(1-\cos\theta) + \frac{\gamma}{1+\gamma}\left(\frac{\Lambda}{mR}\right)^2 + \frac{2Mg}{mR}\right\} . \end{aligned} \tag{28}$$

The last two terms inside the big bracket are constant, so we can write this as

$$\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\dot{\theta}^2 + \frac{2g}{R}\,(1-\cos\theta) = \frac{4gk}{R} . \tag{29}$$

Here, k is a dimensionless measure of the energy of the system, after subtracting the aforementioned constants. If $k > 1$, then $\dot{\theta}^2 > 0$ for all θ, which would result in 'loop-the-loop' motion of the point mass inside the hoop – provided, that is, the normal force of the hoop doesn't vanish and the point mass doesn't detach from the hoop's surface.

0.1.6 Equation of Motion

The equation of motion for θ obtained by eliminating all other variables from the original set of ten equations is the same as $\dot{E} = 0$, and may be written

$$\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\ddot{\theta} + \left(\frac{\gamma\sin\theta\cos\theta}{1+\gamma}\right)\dot{\theta}^2 = -\frac{g}{R} . \tag{30}$$

We can use this to write $\ddot{\theta}$ in terms of $\dot{\theta}^2$, and, after invoking eqn. 29, in terms of θ itself. We find

$$\dot{\theta}^2 = \frac{4g}{R}\cdot\left(\frac{1+\gamma}{1+\gamma\sin^2\!\theta}\right)\big(k - \sin^2\tfrac{1}{2}\theta\big) \tag{31}$$

$$\ddot{\theta} = -\frac{g}{R}\cdot\frac{(1+\gamma)\sin\theta}{\big(1+\gamma\sin^2\!\theta\big)^2}\left[4\gamma\,\big(k - \sin^2\tfrac{1}{2}\theta\big)\cos\theta + 1 + \gamma\sin^2\!\theta\right] . \tag{32}$$

0.1.7 Forces of Constraint

We can solve for the λ_j, and thus obtain the forces of constraint $Q_\sigma = \sum_j \lambda_j \frac{\partial G_j}{\partial q_\sigma}$.

$$\lambda_3 = m\ddot{x} = mR\,\ddot{\phi} + mR\cos\theta\,\ddot{\theta} - mR\sin\theta\,\dot{\theta}^2$$

$$= \frac{mR}{1+\gamma}\Big[\ddot{\theta}\,\cos\theta - \dot{\theta}^2\sin\theta\Big] \tag{33}$$

$$\lambda_4 = m\ddot{y} + mg = mg + mR\sin\theta\,\ddot{\theta} + mR\cos\theta\,\dot{\theta}^2$$

$$= mR\Big[\ddot{\theta}\,\sin\theta + \dot{\theta}^2\sin\theta + \frac{g}{R}\Big] \tag{34}$$

$$\lambda_1 = -\frac{I}{R}\,\ddot{\phi} = \frac{(1+\gamma)I}{mR^2}\,\lambda_3 \tag{35}$$

$$\lambda_2 = (M+m)g + m\ddot{y} = \lambda_4 + Mg\ . \tag{36}$$

One can check that $\lambda_3\cos\theta + \lambda_4\sin\theta = 0$.

The condition that the normal force of the hoop on the point mass vanish is $\lambda_3 = 0$, which entails $\lambda_4 = 0$. This gives

$$-(1+\gamma\sin^2\!\theta)\,\cos\theta = 4(1+\gamma)\left(k - \sin^2\tfrac{1}{2}\theta\right)\ . \tag{37}$$

Note that this requires $\cos\theta < 0$, *i.e.* the point of detachment lies above the horizontal diameter of the hoop. Clearly if k is sufficiently large, the equality cannot be satisfied, and the point mass executes a periodic 'loop-the-loop' motion. In particular, setting $\theta = \pi$, we find that

$$k_{\rm c} = 1 + \frac{1}{4(1+\gamma)}\ . \tag{38}$$

If $k > k_{\rm c}$, then there is periodic 'loop-the-loop' motion. If $k < k_{\rm c}$, then the point mass may detach at a critical angle θ^*, but only if the motion allows for $\cos\theta < 0$. From the energy conservation equation, we have that the maximum value of θ achieved occurs when $\dot{\theta} = 0$, which means

$$\cos\theta_{\rm max} = 1 - 2k\ . \tag{39}$$

If $\frac{1}{2} < k < k_{\rm c}$, then, we have the possibility of detachment. This means the energy must be large enough but not too large.