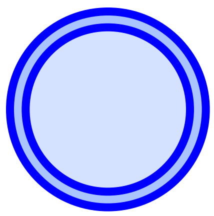
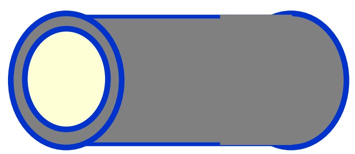
# Experiment 2

- Devise a simple, fast, and non-destructive method to measure the variation in thickness of the shell of a large number of racquet balls to determine if the variation in thickness is much less than 10%.

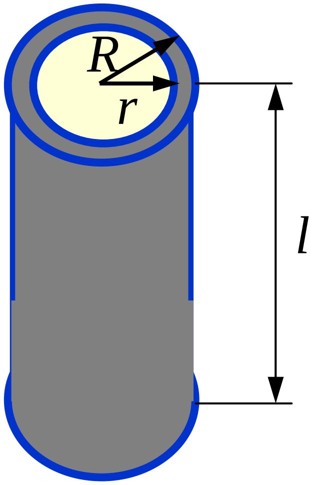


- Devise a method to measure the density of the outer cylinder without damaging the rod so that rods outside 5% tolerance will not be used in a machine.



# The Rods


- Measure the radii $R$ and $r$ and the total mass $M_{total} = M + m$
- Measure $I$ by torsion pendulum
- Determine the density $\rho$ of the outer cylinder




$M$ is the mass of the outer cylinder
$m$ is the mass of the inner cylinder

$$I = \frac{1}{2} m r^2 + \frac{1}{2} M \left( R^2 + r^2 \right)$$

$$M = \frac{1}{R^2} \left( 2I - M_{total} r^2 \right)$$

$$\rho = \frac{M}{\pi \left( R^2 - r^2 \right) l}$$

find $\sigma_\rho \leftarrow$ propagate errors in your proposal

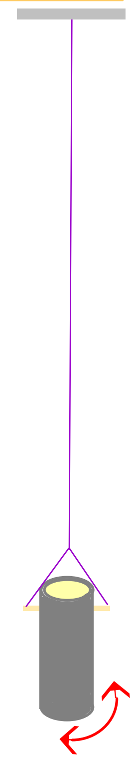
# Measuring *I* by the Torsion Pendulum

$$N = -\kappa\theta = I\ddot{\theta}$$ torque equation gives diff. eq. in $\theta$.

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$ the period

1. calibrate the restoring torque constant $k$ of the wire by measuring period $T_{solid}$ of a solid cylinder for which moment of inertia $I_{solid}$ can be computed

$$I_{solid} = \frac{1}{2} m_{solid} r_{solid}^{2}$$



2. measure period of the torsion pendulum with the rod using the calibrated wire

$$I = \frac{\kappa T^2}{4\pi^2}$$

$$I = \left(\frac{T}{T_{solid}}\right)^2 I_{solid}$$

minimize the wobble of the pendulum since this couples it to other modes and changes the period

# Rejection of Data

3.8, 3.5, 3.9, 3.9, 3.4, 1.8 ← suspect

$$\bar{x} = \frac{1}{N}\sum x_i \quad \sigma_x = \sqrt{\frac{1}{N-1}\sum (x_i - \bar{x})^2}$$

$\bar{x} = 3.4$ s

$\sigma = 0.8$ s

$\bar{x} - x_{sus} = 3.4 - 1.8 = 1.6 = 2\sigma$

Table A

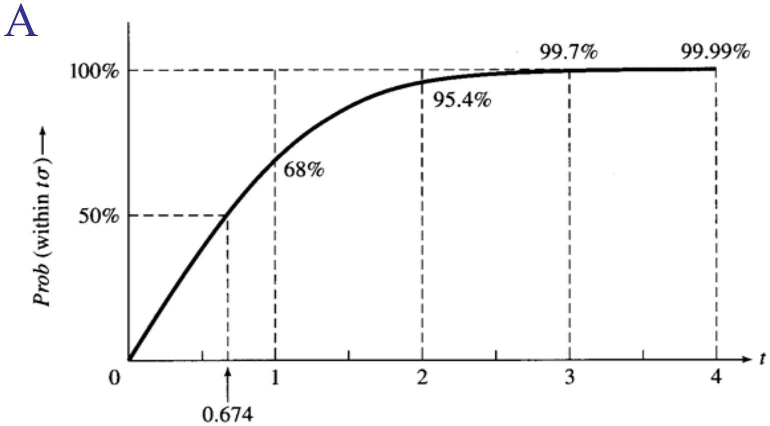
$\text{Prob(outside } 2\sigma) = 1 - \text{Prob(within } 2\sigma) =$

$= 1 - 0.95 = 0.05$

$n = \text{(expected number as deviant as 1.8)} =$

$= N \times \text{Prob(outside } 2\sigma) =$

$= 6 \times 0.05 = 0.3$




*erf(t)* – error function

if $n < 0.5$ the measurement is "improbable" and can be rejected according to Chauvenet's criterion

$x_1, \ldots, x_N$

$$t_{sus} = \frac{|x_{sus} - \bar{x}|}{\sigma}$$

$n = N \times \text{Prob(outside } t_{sus}\sigma)$

if $n < \frac{1}{2}$, then $x_{sus}$ can be rejected

← Chauvenet's criterion

## Example

A student makes 5 measurements of the period of a pendulum and gets
$T$ = 2.8, 2.5, 2.7, 2.7, 2.3 s.
Should any of these measurements be dropped?

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Calculate the average

$$\bar{T} = \frac{2.8 + 2.5 + 2.7 + 2.7 + 2.3}{5} = 2.6 \text{ s} \longleftarrow \quad \bar{x} = \frac{1}{N}\sum x_i$$

Calculate the standard deviation

$$\sigma = \sqrt{\frac{1}{4}\left(0.2^2 + 0.1^2 + 0.1^2 + 0.1^2 + 0.3^2\right)} = 0.2 \text{ s} \longleftarrow \quad \sigma_x = \sqrt{\frac{1}{N-1}\sum\left(x_i - \bar{x}\right)^2}$$

The measurement furthest from the mean is 2.3 s
giving $t_{sus}$ = 0.3/0.2 = 1.5 $\longleftarrow \quad t_{sus} = \frac{\left|x_{sus} - \bar{x}\right|}{\sigma}$

Look up the probability to be further off, $P$ = 13.36 % ⟵ Table A

Multiply by the number of trials to get the expected
number of events that far off, $n = 5 \times 0.1336 \approx 0.67$

$0.67 \geq 0.5 \rightarrow$ Do not drop this measurement (or any other)

# Weighted Averages

$$A:\quad x = x_A \pm \sigma_A$$
$$B:\quad x = x_B \pm \sigma_B$$

combining separate measurements: what is the best estimate for $x$ ?

assume that measurements are governed by Gauss distribution with true value $X$

$$\text{Prob}_X(x_A) \propto \frac{1}{\sigma_A} e^{-(x_A - X)^2/2\sigma_A^2}$$

$$\text{Prob}_X(x_B) \propto \frac{1}{\sigma_B} e^{-(x_B - X)^2/2\sigma_B^2}$$

probability that A finds $x_A$

$$\text{Prob}_X(x_A x_B) = \text{Prob}_X(x_A) \cdot \text{Prob}_X(x_B)$$

probability that A finds $x_A$ and B finds $x_B$

$$\propto \frac{1}{\sigma_A \sigma_B} e^{-\chi^2/2}$$

find maximum of probability

**principle of maximum likelihood**

the best estimate for $X$ is that value for which $Prob_X(x_A, x_B)$ is maximum

$$\chi^2 = \left(\frac{x_A - X}{\sigma_A}\right)^2 + \left(\frac{x_B - X}{\sigma_B}\right)^2$$

chi squared – "sum of squares"

find minimum of $\chi^2$

**method of least squares**

$$\frac{d\chi^2}{dX} = 0 \quad \Rightarrow \quad -2\frac{x_A - X}{\sigma_A{}^2} - 2\frac{x_B - X}{\sigma_B{}^2} = 0$$

$$(\text{best estimate for } X) = \left(\frac{x_A}{\sigma_A^2} + \frac{x_B}{\sigma_B^2}\right) \Big/ \left(\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2}\right)$$

$$= \frac{w_A x_A + w_B x_B}{w_A + w_B} = x_{wav}$$

weighted average

$$w_A = \frac{1}{\sigma_A^2} \qquad w_B = \frac{1}{\sigma_B^2}$$

weights

# Weighted Averages

$x_1, x_2, \ldots, x_N$ - measurements of a single quantity $x$ with uncertainties $\sigma_1, \sigma_2, \ldots, \sigma_N$

$$x_1 \pm \sigma_1,\ x_2 \pm \sigma_2,\ \ldots,\ x_N \pm \sigma_N$$

$$x_{wav} = \frac{\sum w_i x_i}{\sum w_i}$$ ← weighted average

$$w_i = \frac{1}{\sigma_i^2}$$ ← weights

$$\sigma_{wav} = \frac{1}{\sqrt{\sum w_i}}$$ ← uncertainty in $x_{wav}$

can be calculated
using error propagation

# Example of Weighted Average

$R_1 = 11 \pm 1 \quad (\Omega)$
$R_2 = 12 \pm 1$
$R_3 = 10 \pm 3$

three measurements of a resistance
what is the best estimate for *R* ?

$\sigma_1 = 1 \qquad w_1 = 1$
$\sigma_2 = 1 \qquad w_2 = 1$
$\sigma_3 = 3 \qquad w_3 = \frac{1}{9}$

$\longleftarrow \; w_i = \frac{1}{\sigma_i^2}$

$$R_{WAV} = \frac{\sum w_i R_i}{\sum w_i} = \frac{(1 \times 11) + (1 \times 12) + (\frac{1}{9} \times 10)}{1 + 1 + \frac{1}{9}} = 11.42\ \Omega$$

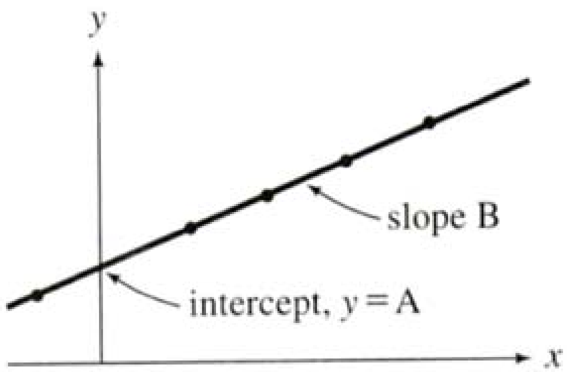
$$\sigma_{WAV} = \frac{1}{\sqrt{\sum w_i}} = \frac{1}{\sqrt{1 + 1 + \frac{1}{9}}} = 0.69$$

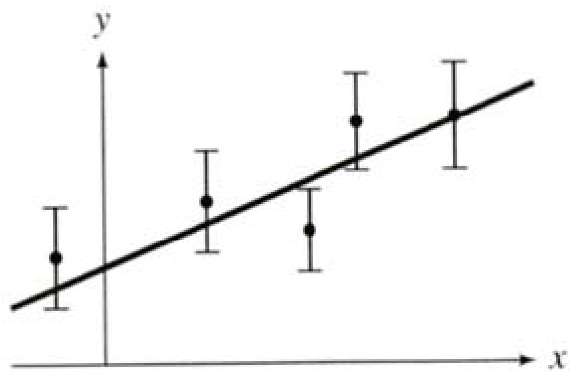
$R = 11.4 \pm 0.7\ \Omega$

# Least-Squares Fitting

consider two variables $x$ and $y$ that are connected by a linear relation

$$y = A + Bx$$




graphical method of finding the best straight line to fit a series of experimental points

$$\begin{matrix} x_1, x_2, \ldots, x_N \\ y_1, y_2, \ldots, y_N \end{matrix} \longrightarrow \text{find } A \text{ and } B$$

analytical method of finding the best straight line to fit a series of experimental points is called **linear regression** or **the least-squares fit for a line**

# Calculation of the Constants *A* and *B*

$$(\text{true value for } y_i) = A + Bx_i$$

$$\text{Prob}_{A,B}(y_i) \propto \frac{1}{\sigma_y} e^{-(y_i - A - Bx_i)^2/2\sigma_y^2}$$ ← probability of obtaining the observed value of $y_1$

$$\text{Prob}_{A,B}(y_1,\ldots,y_N) = \text{Prob}_{A,B}(y_1)\cdots\text{Prob}_{A,B}(y_N)$$ ← probability of obtaining the set $y_1, \ldots, y_N$

$$\propto \frac{1}{\sigma_y^N} e^{-\chi^2/2}$$ ← find maximum of probability

$$\chi^2 = \sum_{i=1}^{N} \frac{(y_i - A - Bx_i)^2}{\sigma_y^2}$$ ← chi squared – “sum of squares”
find minimum of $\chi^2$
least squares fitting

$$\frac{\partial \chi^2}{\partial A} = \frac{-2}{\sigma_y^2} \sum_{i=1}^{N} (y_i - A - Bx_i) = 0$$

$$\frac{\partial \chi^2}{\partial B} = \frac{-2}{\sigma_y^2} \sum_{i=1}^{N} x_i (y_i - A - Bx_i) = 0$$

$$AN + B\sum x_i = \sum y_i$$

$$A\sum x_i + B\sum x_i^2 = \sum x_i y_i$$

→

$$A = \frac{\sum x^2 \sum y - \sum x \sum xy}{\Delta}$$

$$B = \frac{N\sum xy - \sum x \sum y}{\Delta}$$

$$\Delta = N\sum x^2 - \left(\sum x\right)^2$$

# Uncertainties in *y*, *A*, and *B*

$$\sigma_y = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}\left(y_i - A - Bx_i\right)^2}$$

uncertainty in the measurement of *y*

$$\sigma_A = \sigma_y\sqrt{\frac{\sum x^2}{\Delta}}$$

$$\sigma_B = \sigma_y\sqrt{\frac{N}{\Delta}}$$

uncertainties in the constants *A* and *B*

given by error propagation in terms of uncertainties in $y_1, \ldots, y_N$

# Example of Calculation of the Constants *A* and *B*

$T = A + BP$ (T ← y, P ← x)

if volume of an ideal gas is kept constant,
its temperature is a linear function of its pressure

| i | $P_i$ | $T_i$ |
|---|---|---|
| 1 | 65 | -20 |
| 2 | 75 | 17 |
| 3 | 85 | 42 |
| 4 | 95 | 94 |
| 5 | 105 | 127 |

absolute zero of temperature $A = ?$

$$A = \frac{\sum x^2 \sum y - \sum x \sum xy}{\Delta}$$

$$B = \frac{N\sum xy - \sum x \sum y}{\Delta}$$

$$\Delta = N\sum x^2 - \left(\sum x\right)^2$$

$\Sigma P = 425$

$\Sigma P^2 = 37{,}125$

$\Sigma T = 260$

$\Sigma PT = 25{,}810$

$\Delta = N\Sigma P^2 - (\Sigma P)^2 = 5{,}000$

$$A = \frac{\Sigma P^2 \Sigma T - \Sigma P \Sigma PT}{\Delta} = -263.35$$

$$B = \frac{N\Sigma PT - \Sigma P \Sigma T}{\Delta} = 3.71$$

$$\sigma_y = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}\left(y_i - A - Bx_i\right)^2}$$

$$\sigma_A = \sigma_y\sqrt{\frac{\sum x^2}{\Delta}}$$

$$\sigma_T = \sqrt{\frac{1}{N-2}\Sigma(T_i - A - BP_i)^2} = 6.7$$

$$\sigma_A = \sigma_T\sqrt{\frac{\Sigma P^2}{\Delta}} = 18$$

$A = -263.35 \pm 18\,^\circ C$

$A = -260 \pm 20\,^\circ C$

absolute zero of temperature = – 273.15$^0$ C