General Formula for Error Propagation

$$q = q(x, y, z)$$

$$q_{best} = q(x_{best}, y_{best}, z_{best})$$

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta y\right)^2 + \left(\frac{\partial q}{\partial z}\delta z\right)^2}$$

for independent random errors δx, δy, and δz

main formula for error propagation
always use this formula

Experiment 1: Measure Density of Earth

- **Calculate average density ρ and determine which elements constitute the major portion of the earth.**
- **Two measurements**
  - **(a) Earth's Radius R_e. (challenging measurement)**
  - **(b) Local acceleration of gravity g. (fairly easy)**
- **Use Newton's constant G=6.67 X 10^{-11} N m^2/kg^2**
- **Aim for 10% or better error on ρ.**

$$F = \frac{GMm}{r^2}$$ **Gravitational force**

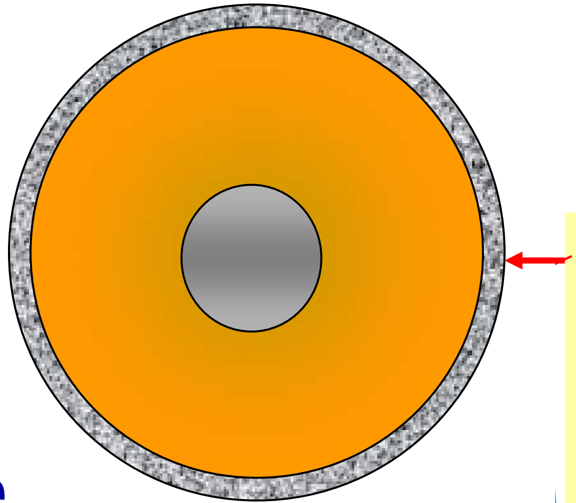
$$g = \frac{F}{m} = \frac{GM}{R_e^{\ 2}} = \frac{G(\frac{4}{3}\pi R_e^{\ 3}\rho)}{R_e^{\ 2}} = \frac{4}{3}\pi G R_e \rho$$

$$\rho = \frac{3}{4\pi}\frac{g}{GR_e}$$

What's the Point

Its an experiment about optimizing measurement technique, error estimation, and error propagation

What Element(s) make up the Earth



- **Assume most of earth's volume is one element.**

Densities

rock	2.7
aluminum	2.7
zinc	7.14
iron	7.20
nickel	8.85
copper	8.89
silver	10.5
lead	11.34
mercury	13.60
gold	19.3

Measure Earth's Radius using Δt Sunset

From right triangle:

$$\cos(\theta) = \frac{R_e}{R_e + h} = \frac{1}{1 + h/R_e} \approx 1 - \frac{h}{R_e}$$

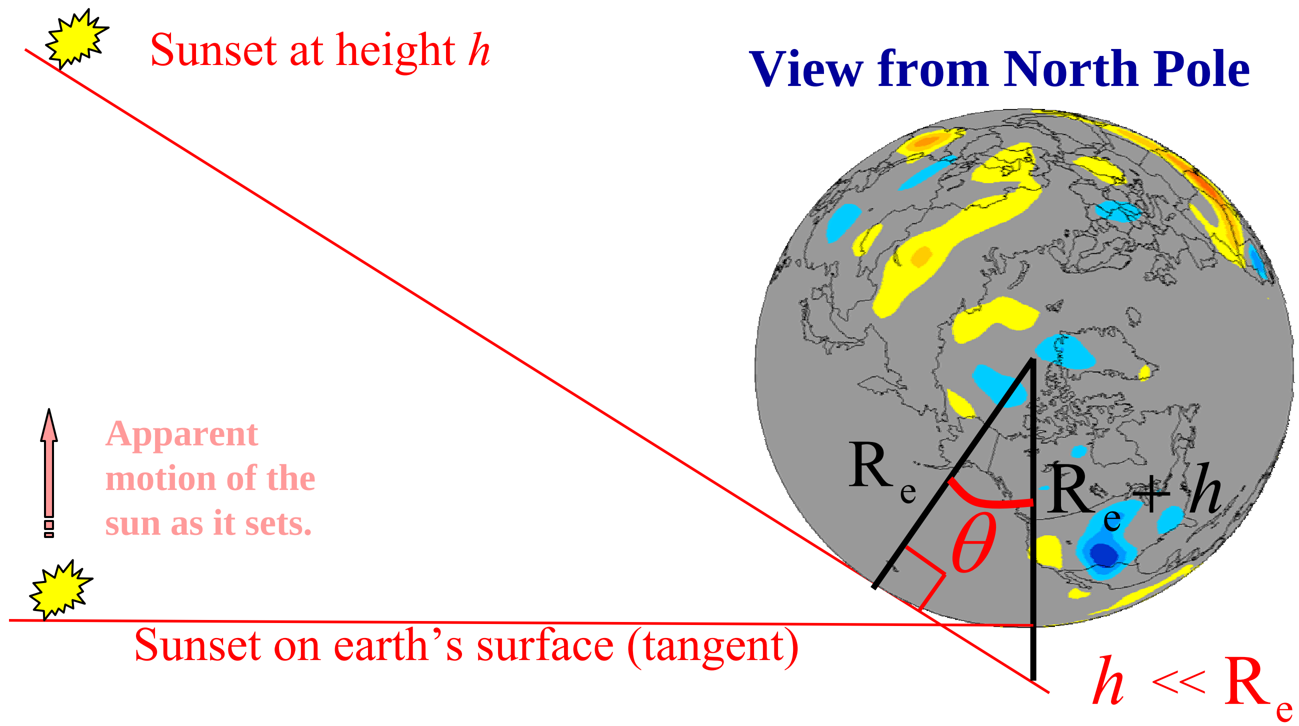
For small θ:

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}$$

Equating:

$$\frac{\theta^2}{2} = \frac{h}{R_e}$$

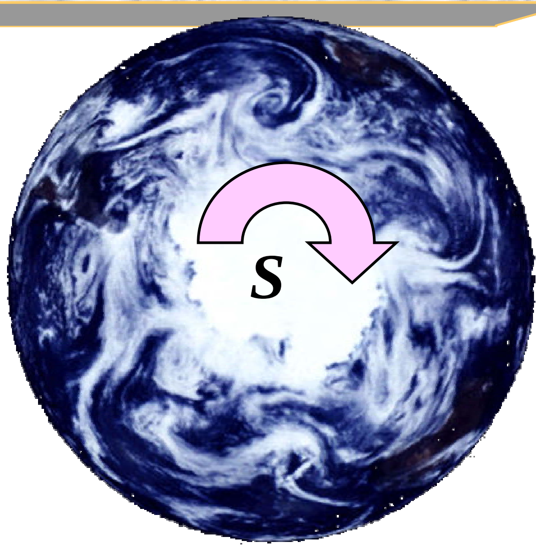
$$\theta = \sqrt{\frac{2h}{R_e}}$$



Assume we are at equator

θ Increases at Earth Rotates

Earth makes (nearly) one rotation per day.
Angular frequency is 2π radians per day.



ω (omega) = earth's angular frequency.

$$\omega = \frac{2\pi \ \frac{\text{radians}}{\text{day}}}{\left(24\,\frac{\text{hours}}{\text{day}}\right)\left(60\,\frac{\text{minutes}}{\text{hour}}\right)\left(60\,\frac{\text{seconds}}{\text{minute}}\right)} = 7.27\times10^{-5}\ \frac{\text{radians}}{\text{second}}$$

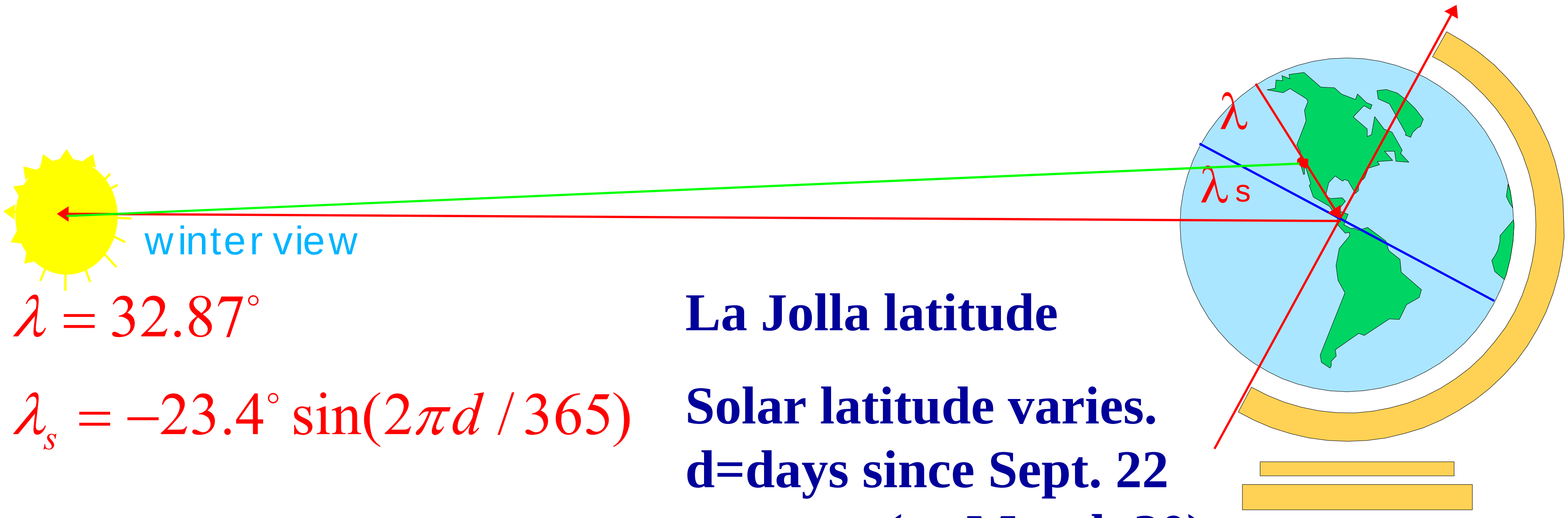
$$\theta = \omega t = \sqrt{\frac{2h}{\mathrm{R_e}}}$$

θ (theta) = angle earth rotates after true sunset.

$$t = \frac{1}{\omega}\sqrt{\frac{2h}{\mathrm{R_e}}}$$

Solving for *t*, we get the time delay of the sunset at height *h* (since the true sunset).

Correct for Latitude and Earth's Axis



$\lambda = 32.87^\circ$ **La Jolla latitude**

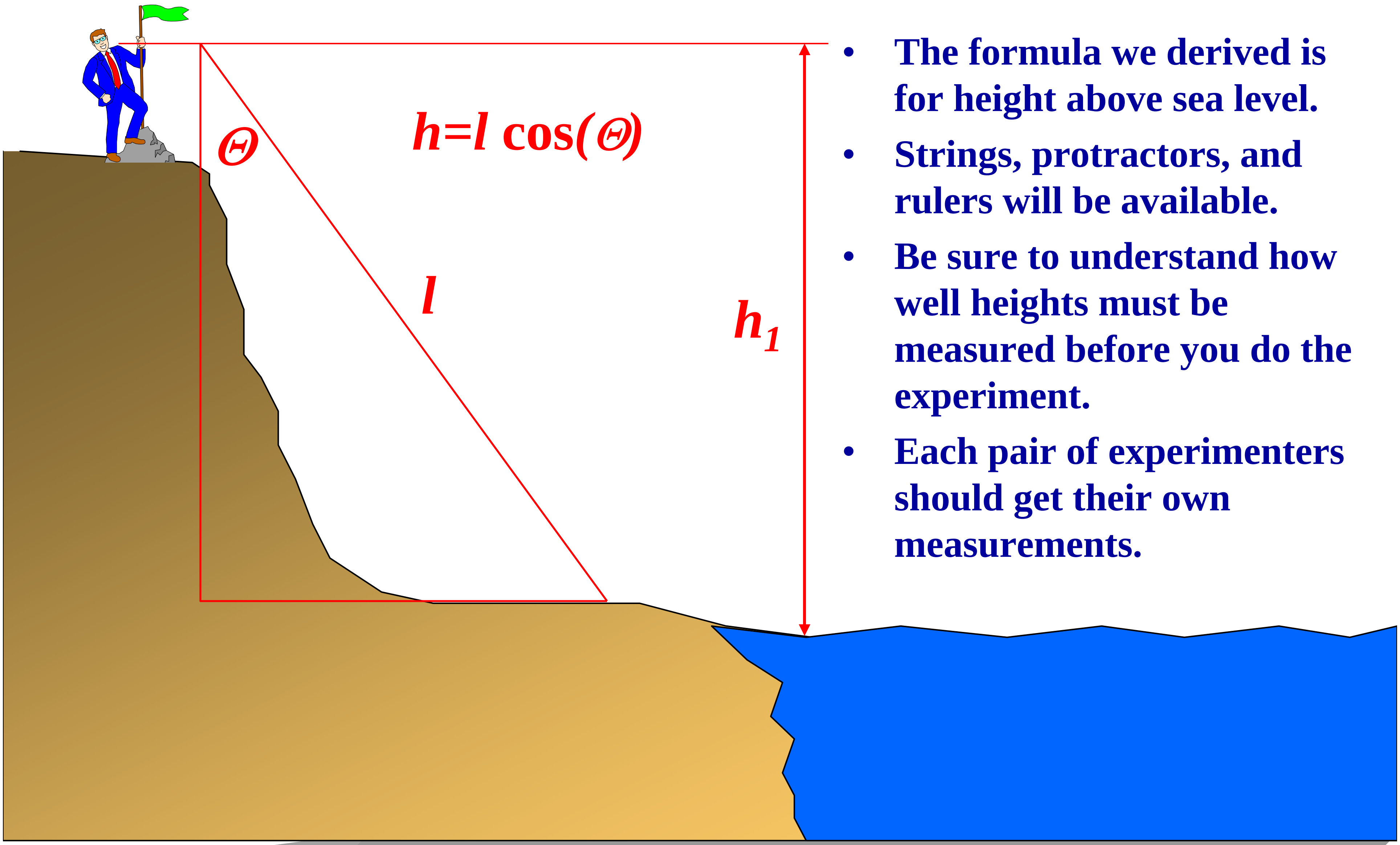
$\lambda_s = -23.4^\circ \sin(2\pi d / 365)$ **Solar latitude varies.**

d=days since Sept. 22
(or March 20).

This formula accounts for our latitude and for the angle of the earth's axis from the plane of its orbit.

$$t = \frac{1}{\omega}\sqrt{\frac{2h}{\mathrm{R_e}[\cos^2(\lambda)\cos^2(\lambda_s) - \sin^2(\lambda)\sin^2(\lambda_s)]}} \equiv \frac{1}{\omega}\sqrt{\frac{2Ch}{\mathrm{R_e}}}$$

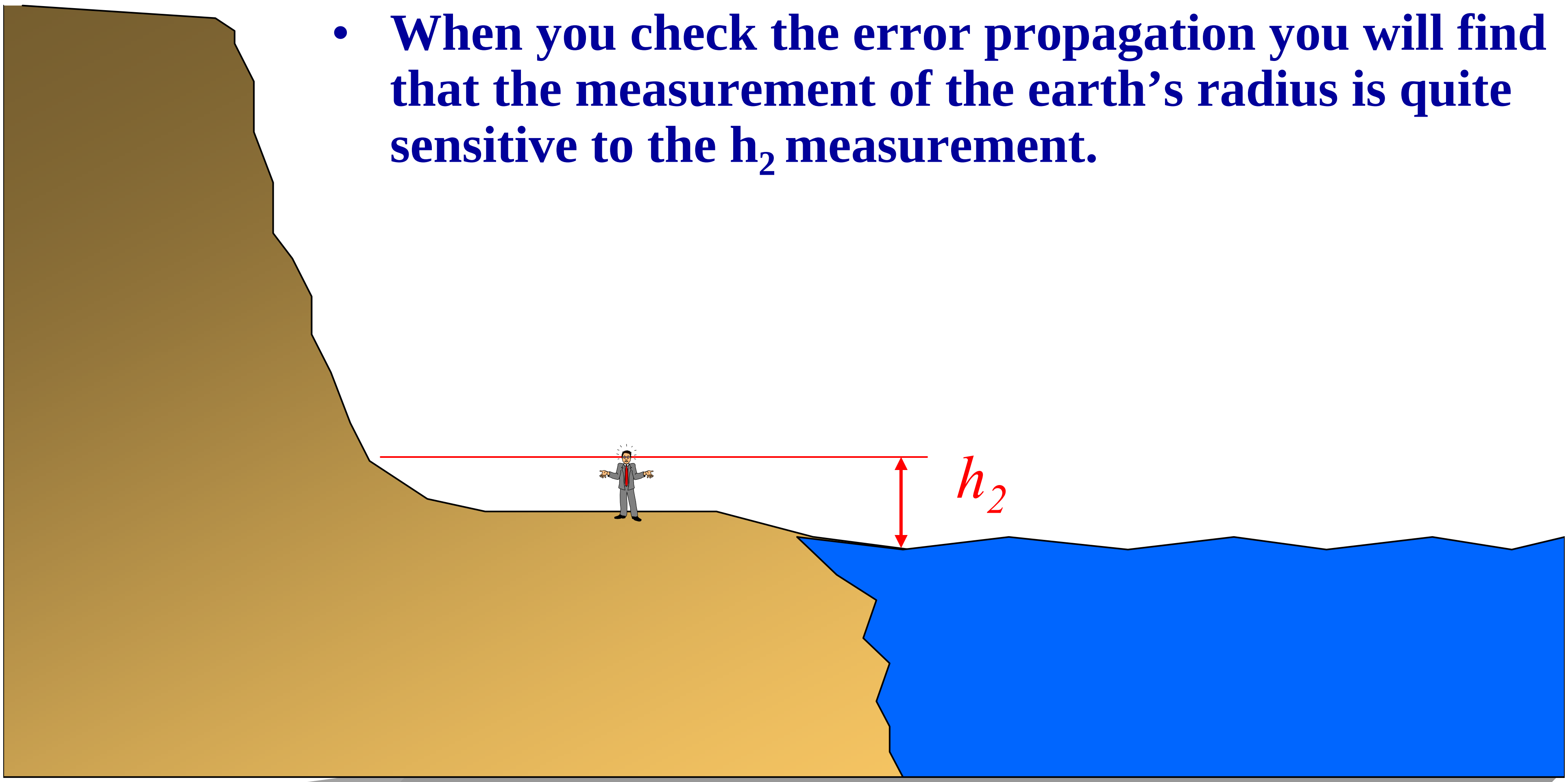
Measuring the Height of the Cliff



- The formula we derived is for height above sea level.
- Strings, protractors, and rulers will be available.
- Be sure to understand how well heights must be measured before you do the experiment.
- Each pair of experimenters should get their own measurements.

Your Height Above Sea Level on Beach

- The experimenter on the beach also views the sunset from above sea level.
- When you check the error propagation you will find that the measurement of the earth's radius is quite sensitive to the h_2 measurement.



"The Equation" for Experiment 1a

$$t = \frac{1}{\omega}\sqrt{\frac{2Ch}{\mathrm{R_e}}}$$

From previous page.

$$\Delta t = t_1 - t_2 = \frac{1}{\omega}\sqrt{\frac{2C}{\mathrm{R_e}}}\left(\sqrt{h_1} - \sqrt{h_2}\right)$$

Time difference between the two sunset observers.

$$C \equiv \frac{1}{\cos^2(\lambda)\cos^2(\lambda_s) - \sin^2(\lambda)\sin^2(\lambda_s)}$$

Season dependant factor slightly greater than 1.

use this formula for your error analysis

$$\mathrm{R_e} = \frac{2C}{\omega^2}\left(\frac{\sqrt{h_1} - \sqrt{h_2}}{\Delta t}\right)^2$$

Propagating Errors for R_e

$$R_e = \frac{2C}{\omega^2}\left(\frac{\sqrt{h_1}-\sqrt{h_2}}{\Delta t}\right)^2$$

basic formula

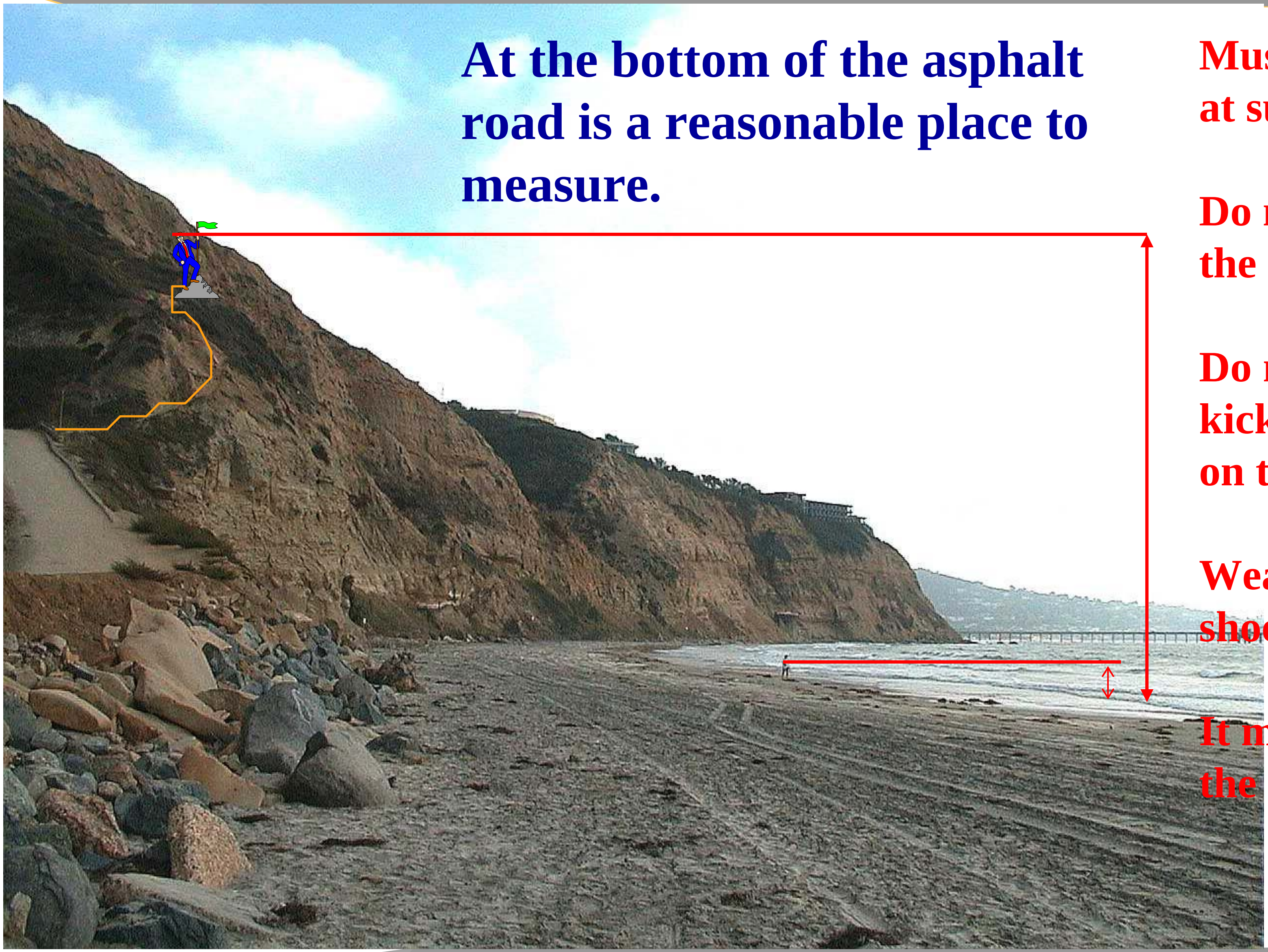
$$\sigma_{R_e} = \frac{\partial R_e}{\partial \Delta t}\sigma_{\Delta t} \oplus \frac{\partial R_e}{\partial h_1}\sigma_{h_1} \oplus \frac{\partial R_e}{\partial h_2}\sigma_{h_2}$$

Propagate errors (use shorthand for addition in quadrature)

$$\sigma_{R_e} = \frac{2R_e}{\Delta t}\sigma_{\Delta t} \oplus \frac{R_e}{\sqrt{h_1}\left(\sqrt{h_1}-\sqrt{h_2}\right)}\sigma_{h_1} \oplus \frac{R_e}{\sqrt{h_2}\left(\sqrt{h_1}-\sqrt{h_2}\right)}\sigma_{h_2}$$

Note that error blows up at h_1=h_2 and at h_2=0.

Cliffs West of Muir Campus



At the bottom of the asphalt road is a reasonable place to measure.

Must return there at sunset.

Do not go too near the cliffs

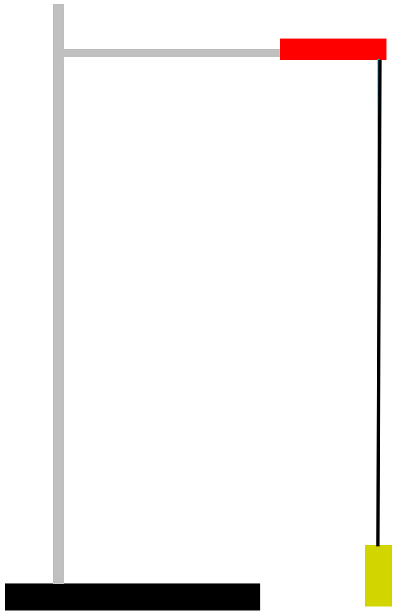
Do not drop or kick objects below on the beach

Wear walking shoes

It may be cold in the evening


Weather plays a role.
Completely clear days are best.
sunset time – a moment when the last point of the Sun disappears

Measuring *g* with a Pendulum



- Period can be measured with electronic timer over one cycle of with a stopwatch over many cycles.
- Frictional forces play a role for light weights.
- Small oscillations are good.
- Heavy weights may cause coupling to other oscillators like unstable stand.
- Short strings may cause moment of inertia to become important.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Period of pendulum

Propagating Errors for Experiment 1

$$\rho = \frac{3}{4\pi}\frac{g}{GR_e}$$

Formula for density.

$$\sigma_\rho = \frac{3}{4\pi}\frac{1}{GR_e}\sigma_g \oplus \frac{-3}{4\pi}\frac{g}{GR_e^2}\sigma_{R_e}$$

Take partial derivatives and add errors in quadrature

$$\frac{\sigma_\rho}{\rho} = \frac{\sigma_g}{g} \oplus \frac{\sigma_{R_e}}{R_e}$$

Statistical analysis

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Period of pendulum

uncertainty

error propagation

statistical analysis

two methods to find uncertainty

The mean

$x_1,\ x_2,\ \ldots\ ,\ x_N$ — N measurements of the quantity x

$x_{best} = \bar{x}$ — the best estimate for $x \rightarrow$ the average or mean

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_N}{N} = \frac{\sum x_i}{N}$$

$$\sum_{i=1}^{N} x_i = \sum_i x_i = \sum x_i = x_1 + x_2 + \ldots + x_N$$

sigma notation

↑ ↑

common abbreviations

The standard deviation

$$d_i = x_i - \bar{x}$$ deviation of x_i from $\bar{x}$

$$\sigma_x = \sqrt{\frac{1}{N}\sum\left(x_i - \bar{x}\right)^2}$$

average uncertainty of the measurements $x_1, \ldots, x_N$

↕

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum\left(x_i - \bar{x}\right)^2}$$

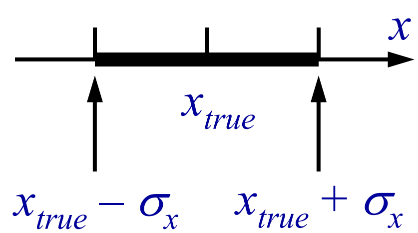
standard deviation of x

RMS (route mean square) deviation

uncertainty in any one measurement of $x \rightarrow$ $\delta x = \sigma_x$

↓

68% of measurements will fall in the range $x_{true} \pm \sigma_x$



The standard deviation of the mean

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

$$\delta x = \sigma_{\bar{x}}$$

uncertainty in $\bar{x}$
is the standard deviation of the mean

based on the N measured values $x_1, \ldots, x_N$ we can state our final answer for the value of x:

$$(\text{value of } x) = x_{best} \pm \delta x$$

$$x_{best} = \bar{x} \qquad \delta x = \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

$$\left(\text{value of } x\right) = \bar{x} \pm \sigma_{\bar{x}}$$

Example

We make measurements of the period of a pendulum 3 times and find the results:
T = 2.0, 2.1, and 2.2 s.

(a) What is the mean period?
(b) What is the RMS error (the standard deviation) in the period?
(c) What is the error in the mean period (the standard deviation of the mean)?
(d) What is the best estimate for the period and the uncertainty in the best estimate.

$$\bar{x} = \frac{1}{N}\sum x_i \qquad \sigma_x = \sqrt{\frac{1}{N-1}\sum (x_i - \bar{x})^2} \qquad \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} \qquad (\text{value of } x) = \bar{x} \pm \sigma_{\bar{x}}$$

$$\bar{T} = \frac{1}{N}\sum T_i = \frac{1}{3}\sum (2 + 2.1 + 2.2) = \underline{2.1\text{ s}}$$

$$\sigma_T = \sqrt{\frac{1}{N-1}\sum (T_i - \bar{T})^2} = \sqrt{\frac{1}{2}\left[(2-2.1)^2 + (2.1-2.1)^2 + (2.2-2.1)^2\right]} = \sqrt{\frac{1}{2}\left[0.1^2 + 0.1^2\right]} = \underline{0.1\text{ s}}$$

$$\sigma_{\bar{T}} = \frac{\sigma_T}{\sqrt{N}} = \frac{0.1}{\sqrt{3}} = 0.057735\text{ s} \rightarrow \underline{0.06\text{ s}}$$

$$T = \bar{T} \pm \sigma_{\bar{T}} = \underline{2.10 \pm 0.06\text{ s}}$$

Systematic errors

$$\delta x = \sqrt{\left(\delta x_{ran}\right)^2 + \left(\delta x_{sys}\right)^2}$$

random component

systematic component