→ Energy Principle

- recall have equation of motion for displacement $\underline{\xi}$, in ideal MHD

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = F(\underline{\xi})$$

$$\equiv -\hat{\underline{K}}(\underline{\xi})$$

operator (~'spring constant')

Now, $\hat{K}$ self-adjoint, i.e.

$$\int d^3x \, \vec{\eta} \, \hat{\underline{K}}(\hat{\xi}) = \int d^3x \, \hat{\xi} \, \hat{\underline{K}}(\vec{\eta})$$

for any $\vec{\eta}$, $\vec{\xi}$. Proof incorporates B.C.'s.

(see Kadomtsev 159, 160)

Then, can write, for displacement ξ

kinetic energy

$$T = \frac{1}{2} \int_{V_i} d^3x \, \rho_0 \left(\frac{\partial \xi}{\partial t} \right)^2$$

potential energy

"δW"

$$W = \frac{1}{2} \int_{V_i} d^3x \Big[\gamma p_0 (\nabla \cdot \hat{\xi})^2 + \frac{1}{4\pi} (\nabla \times \underline{\hat{\xi}} \times \underline{B}_0)^2$$

$$+ \underline{\xi} \cdot \nabla p_0 \, \underline{\nabla} \cdot \underline{\xi} - \frac{1}{4\pi} [\hat{\xi} \times (\nabla \times \underline{B}_0)] \cdot [\underline{\nabla} \times \hat{\xi} \times \underline{B}_0] \Big] +$$

⑤ ⑥ $\delta 93.$

$$\frac{1}{8\pi}\int_{V_e} d^3x\,(\underline{\nabla}\times\underline{A})^2 - \frac{1}{2}\int dS_0\left(\frac{\partial p_0}{\partial n} + \frac{1}{8\pi}\frac{\partial}{\partial n}\overline{B_{0i}^2} - \frac{1}{8\pi}\frac{\partial}{\partial n}B_{0e}^2\right)\xi_n^2$$

normal displacement

① → compression > 0 ⟹ always stabilizing.

② $(\underline{\nabla}\times\hat{\xi}\times\underline{B_0})^2 = \left[\underline{B_0}\cdot\underline{\nabla}\,\hat{\xi} - \hat{\xi}\cdot\underline{\nabla}\,\underline{B_0} - \underline{B_0}(\underline{\nabla}\cdot\hat{\xi})\right]^2$

(perturbation magnetic energy $|\delta B|^2$)

if $\underline{\nabla}\cdot\hat{\xi} = 0$, $\left\{\begin{array}{l} B_0 \text{ uniform} \\ \text{or } \hat{\xi}\cdot\underline{\nabla}B_0 = 0 \end{array}\right.$ then

$\delta B^2 = (\underline{B_0}\cdot\underline{\nabla}\,\hat{\xi})^2$ → bending energy, always stabilizing

③ $(\hat{\xi}\cdot\underline{\nabla}p_0)(\underline{\nabla}\cdot\hat{\xi})$ → pressure gradient drive

④ $[\hat{\xi}\times(\underbrace{\underline{\nabla}\times\underline{B_0}}_{\underline{J_0}})]\cdot[\underline{\nabla}\times\hat{\xi}\times\underline{B_0}]$ → kink term (current gradient)

⑤ → gap field energy

⑥ → energy due surface displacement $\frac{\partial}{\partial n} = \hat{n}\cdot\underline{\nabla}$

i.e. $\delta W > 0 \rightarrow$ stable
$\delta W < 0 \rightarrow$ unstable.

Some examples: (simple)

a.) Plasma – B field Boundary (R.-T.)

B-field $\partial B^2/\partial n < 0$ $\rightarrow$ unstable $\dfrac{\partial B_0^2}{\partial n} > 0 \rightarrow$ stable.

i.e. consider no magnetic field in plasma.
$\underline{B_0} = 0$, flat p_0

$$\therefore \delta W = \frac{1}{2}\int_{V_i} d^3x \underbrace{\gamma p_0 (\nabla\cdot\underline{\xi})^2}_{>0} + \frac{1}{8\pi}\int d^3x \underbrace{(\nabla\times\hat{A})^2}_{>0}$$

$$+ \frac{1}{16\pi}\int_{S_0} ds \underbrace{\frac{\partial B_0^2}{\partial n}}_{?} \xi_n^2$$

For ?,

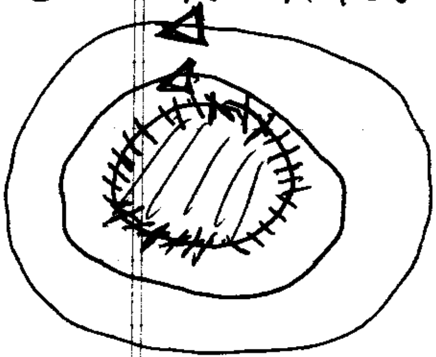
$\dfrac{\partial B_0^2}{\partial n} \rightarrow$ field decreases away from plasma boundary
$\rightarrow \delta W > 0$, so stable.

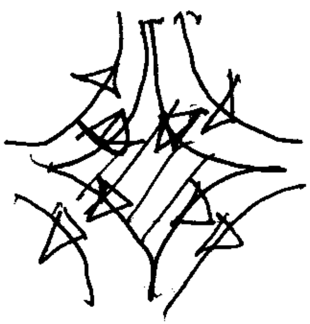
$\partial B_0^2/\partial n < 0 \Rightarrow$ field decreases away from plasma

$\Rightarrow$ instability !?

i.e.

convex lines (from plasma) Concave lines





(i.e. cusp)

(surface current)

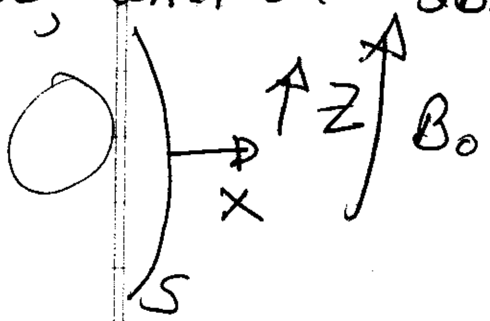
$\frac{\partial B_0^2}{\partial n} < 0$

$\rightarrow$ unstable

$\frac{\partial B_0^2}{\partial n} > 0$

$\rightarrow$ definitely stable

Now, what of $\partial B_0^2/\partial n < 0$?



consider short wavelength perturbation

$\Rightarrow$ surface as planar.

$\therefore \; \xi_n = \xi_x = \xi_0 \exp[ik_y y + ik_z z]$

matter inside S conserved $\Rightarrow \underline{\nabla} \cdot \underline{\xi} = 0$

now $\nabla \times \nabla \times A_0 = 0$

$$\nabla(\nabla\cdot A_0) - \nabla^2 A_0 = 0 \qquad k^2 \underline{A}_0 - \underline{k}(k\cdot \underline{A}_0) = 0$$

and can put $\underline{\nabla}\cdot\underline{A}_0 = 0$ $\quad (\underline{B} = \underline{\nabla}\times\underline{A}_0)$

$\therefore$ $k_x^2 + (k_y^2 + k_z^2) = 0$

$$\Rightarrow k_x = -i(k_y^2+k_z^2)^{1/2} = -iK \qquad (\mathrm{SW})$$

and $\hat{n}\times\underline{\hat{A}} = -\hat{\xi}_n B_{0e}$ $\quad \Rightarrow A_{0z} = 0$

$$A_{0y} = -\hat{\xi}_m B_0$$

$\therefore$ $|\nabla\times\underline{A}|^2 = 2k_z^2 B_0^2 \xi_0^2 e^{-2Kx}$

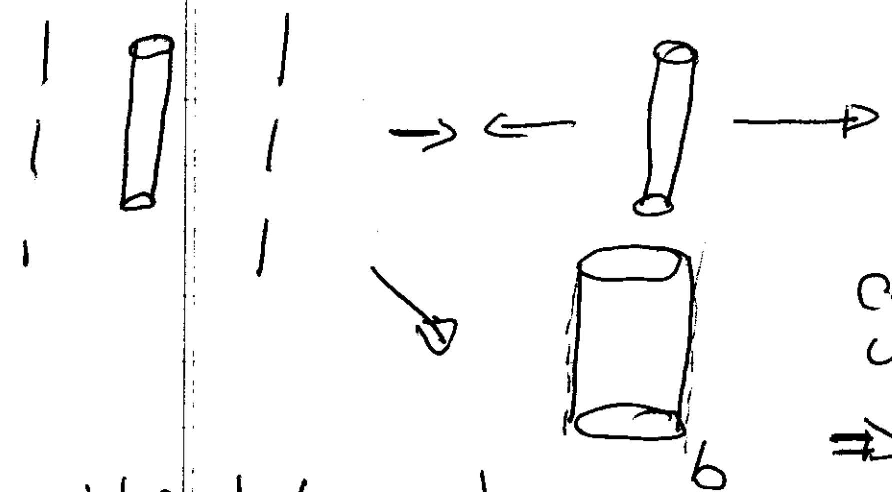
$\therefore$ $\delta W = \int dS\left[\dfrac{k_z^2 B_0^2}{8\pi K} + \dfrac{1}{16\pi}\dfrac{\partial B_0^2}{\partial n}\right]\xi_0^2$

$\Rightarrow$ for $\dfrac{k_z^2}{K} \to 0$ (minimal bending)

$\delta W < 0$ for $\partial B_0^2/\partial n < 0$.

b.) Pinch – { Distributed Current / Localized Modes / Resonant }

Now ⇒ opposite extreme simplifacation from surface current pinch



conducting wall up/ against plasma ⇒ no gap, etc.

internal term only

$$\Rightarrow \delta W = \frac{1}{2}\int_{V_i} d^3x \Big[\gamma p_0 (\underline{\nabla}\cdot\underline{\xi})^2 + \frac{1}{4\pi}(\nabla\times\underline{\xi}\times\underline{B}_0)^2$$

$$+ \underline{\xi}\cdot\underline{\nabla} p_0 \, \underline{\nabla}\cdot\underline{\xi} - \frac{1}{4\pi}(\underline{\xi}\times\underline{\nabla}\times\underline{B}_0)\cdot(\nabla\times\underline{\xi}\times\underline{B}_0)\Big]$$

furthur, assume cylindrical/helical symmetry ⇒

$$\underline{\xi} = \underline{\xi}(r)\, e^{i(m\theta + kz)}$$

∴

$$\frac{\delta W}{\delta \mathcal{E}_\theta} = 0 \Rightarrow \frac{m}{r}\tilde{\mathcal{E}}_\theta + k\tilde{\mathcal{E}}_z = \frac{i}{r}\frac{d}{dr}(r\tilde{\mathcal{E}}_r)$$

$$\frac{\delta W}{\delta \mathcal{E}_z} = 0 \Rightarrow B_z\tilde{\mathcal{E}}_\theta - \tilde{\mathcal{E}}_z B_\theta =$$

$$\frac{-i}{k^2r^2+m^2}\left[(krB_\theta - mB_z)\frac{d\mathcal{E}_r}{dr} - (krB_\theta + mB_z)\frac{\mathcal{E}_r}{r}\right]$$

so, can eliminate $\mathcal{E}_\theta$, $\mathcal{E}_z$ and write (after I. B. P.):

$$\boxed{\delta W = \frac{\pi}{2}\int_0^b dr\left\{f\left(\frac{d\mathcal{E}_r}{dr}\right)^2 + g\,\mathcal{E}_r^2\right\}}$$

1D system!

$$f = \frac{r}{4\pi}\frac{(krB_z + mB_\theta)^2}{k^2r^2+m^2} = \frac{r}{4\pi}\frac{(\underline{k}\cdot\underline{B})^2}{(k^2+m^2/r^2)}$$

$$g = \frac{2k^2r^2}{k^2r^2+m^2}(dP/dr) + \frac{r}{4\pi}(\underline{k}\cdot\underline{B})^2\frac{\left(k^2+\frac{m^2}{r^2}-1/r^2\right)}{(k^2+m^2/r^2)}$$
$$+ \left(2k^2r^3/4\pi(k^2r^2+m^2)^2\right)(k^2B_z^2 - m^2B_\theta^2/r^2)$$

Now, $\frac{\delta W}{\delta \xi_r} = 0 \Rightarrow$

$$\boxed{\begin{aligned} &\frac{d}{dr} f \frac{d\xi_r}{dr} - g\,\xi_r = 0 \\ &\xi_r \big|_b = 0 \\ &\xi_r \big|_0 \text{ finite} \end{aligned}}$$

(E.O.M.) equation of motion for displacement

Now, can further comment:

$\Rightarrow$ full solution is extremum of $L = T - W$

$\therefore \ \omega \neq 0 \Rightarrow g \to \underbrace{g + g_1}_{\text{extra term}} \begin{cases} > 0 \text{ for } \omega^2 < 0 \\ < 0 \text{ for } \omega^2 > 0 \end{cases}$

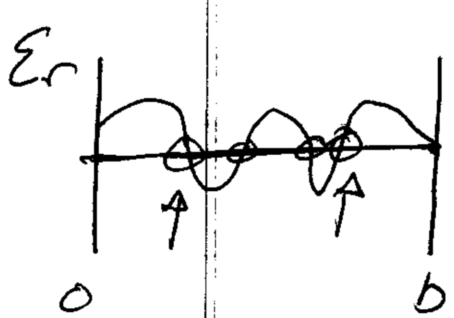
$$\left(\begin{aligned} L &= +\omega^2 |\tilde{\xi}|^2 - W \\ -L &= W - \omega^2 |\tilde{\xi}|^2 \end{aligned} \right)$$

(i.e. $g < 0$)

$\Rightarrow$ assume solution of E.O.M. has more than two zeroes in $(0, b)$.

$\therefore$ by adding $g_1 > 0$ ($\Rightarrow \omega^2 < 0$)

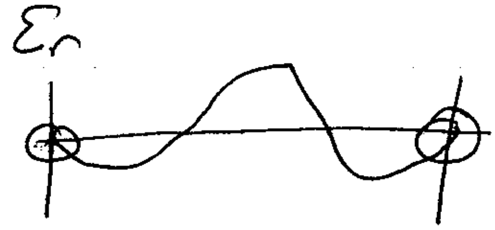
Can shift zeroes:



$\Rightarrow$

adding $\oplus$ to

$g \Rightarrow$



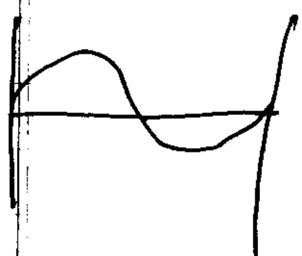
$$\left\{ \begin{array}{l} \frac{d}{dr} f \frac{d\mathcal{E}_r}{dr} - g\,\mathcal{E}_r = 0 \\ \Rightarrow \mathcal{E}_r \text{ wiggles less!} \end{array} \right.$$

(wiggles $\Rightarrow$ $g < 0$)

$\Rightarrow$ modified solution satisfies boundary conditions!

$\therefore$ corresponds to unstable solution. $\omega^2 < 0$.

but if solution E.O.M. has fewer than two zeroes (i.e. 1 zero):



, can only satisfy b.c.'s by $\mathcal{E}_r$ wiggling more.

$\Rightarrow$ must add negative g, to g $\Rightarrow$ $\omega^2 > 0$.

→ Why care about this?

⇒ if more than two zeroes in ξ_r solving E.O.M. ⇒ instability!

⇒ if fewer, stability

∴ establishes connection between oscillations/structure of E.O.M. solution and stability.

Implications ⇒

- consider resonant, large m mode

- now, want $\delta W < 0$ for instability

but: $$g = \frac{r(\underline{k}\cdot\underline{B})^2}{4\pi}\frac{\left(k^2 + \frac{m^2}{r^2} - \frac{1}{r^2}\right)}{\left(k^2 + m^2/r^2\right)} + \ldots$$

and: $m \to \infty$

⇒ $\underline{k}\cdot\underline{B}$ must $\to 0$

i.e. only way to reconcile large m and instability is $\underline{k} \cdot \underline{B} \to 0$

i.e. mode localized at resonant surface, where $q = m/n$. !

Now, define
$$\begin{cases} \iota = B_\theta / r B_z = \dfrac{1}{R q(r)} \\ x = r - r_{m,n} \end{cases} \qquad \frac{m}{n} = q(r_{m,n})$$

so, expanding in x :

$$k r B_z + m B_\theta = m B_z \underbrace{r \iota'}_{\text{shear}} x$$

and

$$f = \left(r^3 B_z^{\cancel{4}\,2} / 4\pi \cancel{B^2} \right) \iota'^2 x^2$$

$$g = \frac{2 B_\theta^2}{B^2} \frac{d p_0}{d r} + \frac{m^2 r B_z^2}{4\pi} \iota'^2 x^2$$

$\therefore$ EOM becomes :

$$\frac{d^2\mathcal{E}_r}{dx^2} + \frac{2}{x}\frac{d\mathcal{E}_r}{dx} + \frac{Q}{x^2}\mathcal{E}_r = K^2\mathcal{E}_r$$

$\Rightarrow$ where: $Q = \dfrac{-8\pi\mu^2}{r(\mu'^2)B_z^2}\left(\dfrac{dP_0}{dr}\right)$

$$K^2 = m^2B^2/r^2B_z$$

$$\mu' = d\mu/dr$$

Now, obviously RHS negligible near $x \to 0$ (i.e. near rational surface)

$$\therefore \quad \frac{d^2\mathcal{E}_r}{dx^2} + \frac{2}{x}\frac{d\mathcal{E}_r}{dx} + \frac{Q}{x^2}\mathcal{E}_r = 0$$

$\mathcal{E}_r \sim x^\nu$, as eqn. homogeneous $\Rightarrow$

$$\Rightarrow \boxed{\nu = -\frac{1}{2} \pm \left(\tfrac{1}{4} - Q\right)^{1/2}}$$

$Q < 1/4 \rightarrow \nu$ real $\rightarrow$ solution has no zeroes.

but if $Q > 1/4$

$$\xi_r = x^{-1/2} \sin\left(\left(Q - 1/4\right)^{1/2} \ln x\right)$$

$\therefore$ infinite # zeroes near $x \to 0$

$\Rightarrow$ unstable!

Now $Q > 1/4$ $\Rightarrow$

$$\boxed{-8\pi r \frac{dp_0}{dr} > \frac{B_z^2}{4}\left(\frac{d\ln \mu}{d\ln r}\right)^2}$$

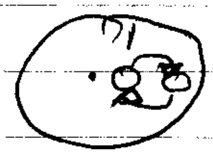
$\Rightarrow$ pressure gradient threshold for instability.

$\boxed{\text{Suydam Criterion}}$

→ More on Physics of Interchange
("Thermodynamic" Picture of Interchange)

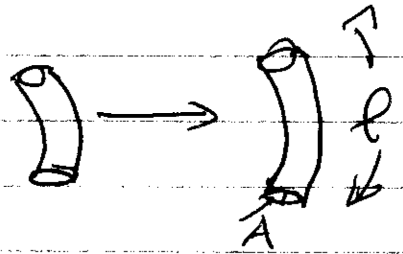
- can consider interchange of plasma as interchange of flux tubes

i.e.



so, problem that of understanding why flux tube ① "lusts" to expand from ① → ②

i.e. why



Approach by calculating volume expansion and extracting force via balance with PV work.

→ tube 'wants' to displace, as will increase its volume $\bar{V}$

i.e. $\bar{V} = \int A \, d\ell$

A ↓ cross-sectional area

(flux)

but $\Phi = AB$ is frozen in

so $A = \Phi / B$

and $V = \Phi \int dl/B \equiv \Phi U$

$$\boxed{U = \int dl/B}$$

effective volume, which lusts to increase

– Now, for expansion force $\leftrightarrow$ effective gravity

$$p\,dV = V F_R\, dR$$

$$F_R = \left(\frac{dV/dR}{V}\right) p \equiv p U'/U$$

effective force for interchange

$$\boxed{F_R = p U'/U}$$

For simple tokamak

$$U = \int \frac{dl}{B} \sim R^2$$

$$dl \sim R\, d\phi, \quad B \sim 1/R$$

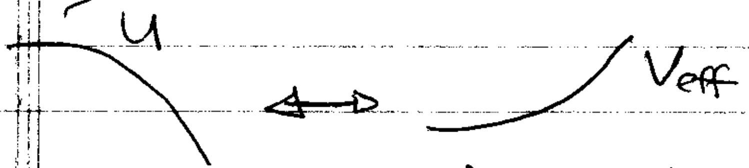
so, $\boxed{\bar{F}_R = 2p/R}$

Note:

- tube expands in direction of increasing $U \Rightarrow -U = V_{eff}$ – effective potential energy for tube

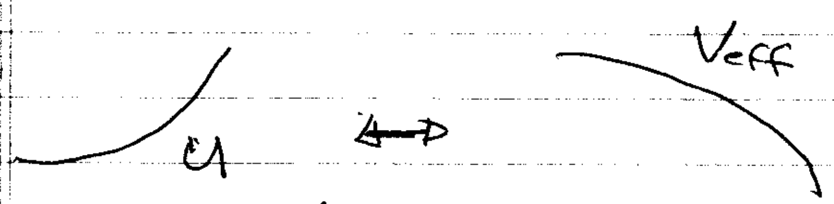
$\therefore$

- U decreases from center to edge



$\Rightarrow$ magnetic "well" $\to$ favorable for stability

- U increases from center to edge



$\Rightarrow$ magnetic "hill" $\to$ unfavorable

→ Suydam Done Simply...

Can write reduced MHD equations:

① $$m_i n_0 \frac{d}{dt} \nabla_\perp^2 \phi = \frac{1}{c} (\underline{B} \cdot \underline{\nabla}) J_\parallel - \frac{K}{r} \frac{\partial p}{\partial \theta}$$

$K \equiv 2B_\theta^3 / r B^2 \sim 1/R_c \equiv$ curvature of field lines

crucial - Suydam is stability limit for ideal interchange.

② and $$\hat{E}_\parallel = 0$$

③ $$\frac{dp}{dt} = 0 \,, \quad \text{as} \quad \underline{\nabla} \cdot \underline{V} = 0$$

so, can immediately write:

$$\omega m_i n_0 \nabla_\perp^2 \hat{\phi} = - \frac{B_0}{4\pi r} (m - nq) \nabla_\perp^2 \tilde{\psi} + \frac{m}{rc} \langle J_\parallel \rangle' \tilde{\psi}$$

$$+ \frac{2m}{q^2 R^2} \tilde{p}$$

$$\omega \tilde{\psi} = - \frac{B_0}{r} (m - nq) \tilde{\phi}$$

$$\omega \tilde{p} = - \frac{m}{r} \tilde{\phi} \langle p \rangle'$$

so, can assemble as:

$$\omega m_i n_0 \nabla_\perp^2 \left(\frac{\omega \tilde{\psi}}{-\frac{B_\theta}{r}(m-n\xi)} \right) = -\frac{B_\theta}{4\pi r}(m-n\xi) \nabla_\perp^2 \tilde{\psi}$$

$$+ \frac{m}{rc} \langle J_\parallel \rangle' \tilde{\psi} + \frac{2m}{\xi^2 R^2} \left(\frac{-m}{\omega r} \hat{\phi} \frac{dp_0}{dr} \right)$$

but $\hat{\phi} = \omega \tilde{\psi} \Big/ -\frac{B_\theta}{r}(m-n\xi)$

$$\Rightarrow \omega^2 \left\{ m_i n_0 \nabla_\perp^2 \left(\tilde{\psi} \Big/ \frac{B_\theta}{r}(m-n\xi) \right) \right\} = -\frac{B_\theta}{4\pi r}(m-n\xi) \nabla_\perp^2 \tilde{\psi}$$

$$+ \frac{m}{cr} \langle J_\parallel \rangle' \tilde{\psi} + \frac{2m}{\xi^2 R^2} \left(\frac{m}{\cancel{\omega r}} \frac{dp_0}{dr} \frac{\cancel{\omega} \tilde{\psi}}{-\frac{B_\theta}{\cancel{r}}(m-n\xi)} \right)$$

Now, if seek determine marginality criterion, take $\omega^2 \to 0$, so:

$$\boxed{\underset{①}{\nabla_\perp^2 \tilde{\psi}} = \underset{②}{\frac{4\pi m}{(m-n\xi) c B_\theta} \langle J_\parallel \rangle' \tilde{\psi}} + \underset{③}{\frac{8\pi m^2}{B_T^2 (m-n\xi)^2 r} \frac{dp_0}{dr} \tilde{\psi}}}$$

→ Above is <u>Newcomb Equation</u> → equation for marginal displacement (i.e. equiv. to $\hookrightarrow \frac{\partial \xi_r}{\partial t} = 0$ ⇒ "perturbed eqbrm" Euler eqn) in ideal MHD <u>i.e.</u>

$$\begin{cases} \frac{1}{c} (\underline{B} \cdot \underline{\nabla}) J_\parallel - \frac{2}{q^2 R^2} \frac{\partial p}{\partial \theta} = 0 \\ \text{with} \quad \underline{B} \cdot \underline{\nabla} p = 0 \qquad \Rightarrow i k_\parallel \tilde{p} = -\frac{\tilde{B}_r}{B_0} \frac{\partial}{\partial r} p_0 \end{cases}$$

$$\Rightarrow \underbrace{\frac{1}{\cancel{c}} i k_\parallel \nabla^2 \tilde{\psi} \frac{\cancel{c}}{4\pi}}_{①} + \underbrace{\frac{1}{c} \tilde{B}_r \frac{\partial \langle J_\parallel \rangle}{\partial r}}_{②} - \underbrace{\frac{2}{q^2 r^2} i m \left(-\frac{\tilde{B}_r}{B_0 i k_\parallel} \frac{dp_0}{dr} \right)}_{③} = 0$$

① current perturbation $\tilde{J}_\parallel$

② displacement of eqbrm. current → <u>drives kinks</u>

③ curvature driven current ("Pfirsch-Schluter") → <u>drives interchanges</u>

→ Obviously, Newcomb equation <u>fails</u> at $x \to 0$, unless $\hat{\psi} \to 0$, on rational surfaces. Need dynamics, inertia, etc. or $\left\{ \begin{array}{l} \text{resistivity} \\ \text{nonlinearity} \ldots \end{array} \right.$

Now:

$$\frac{4\pi m}{(m-nq)cB_0}\langle J_\parallel\rangle' = \frac{4\pi\, m/n\langle J_\parallel\rangle'}{\left(\frac{m}{n}-q\right)cB_0} = \frac{4\pi q\langle J_\parallel\rangle'}{-q' x c B_0} \equiv \frac{\delta}{x}$$

$$\frac{8\pi m^2}{B^2(m-nq)^2 r}\frac{dP_0}{dr} \equiv -\frac{\gamma}{x^2} \quad ; \quad \left\{ \begin{array}{l} \boxed{\gamma>0} \\ \gamma = \dfrac{-8\pi r\, dP_0/dr}{B^2 \hat{s}^2} \\ \hat{s} = r q'/q \end{array} \right.$$

$\hat{s}$ ↓ shear parameter ⇒ rate of pitch rotation.

∴ have:

$$-\nabla_\perp^2 \psi + \frac{\delta}{x}\psi - \frac{\gamma}{x^2}\psi = 0$$

as interested in pressure driven modes (i.e. interchanges), take $\delta = 0$.

$$\therefore \left(\nabla_r^2 - \frac{m^2}{r^2} + \frac{\gamma}{x^2}\right)\psi = 0$$

$$k_r^2 \gg k_\theta^2 \Rightarrow \psi \sim x^\nu$$

$$\nu(\nu-1) + \gamma = 0$$

$$r^2 - r + \gamma = 0$$

$$r = \frac{1}{2} \pm \frac{1}{2}(1-4\gamma)^{1/2}$$

∴ to ~~so~~ avoid instability, need avoid nodes, so

$\boxed{\gamma < 1/4}$ → recovers Suydam criterion.

i.e.

$$\boxed{-\frac{8\pi r}{B^2 \hat{s}^2}\frac{dp_0}{dr} < 1/4}$$ → limit on pressure gradient, due shear

→ Physics of Suydam Criterion

Note can write:

$$r\left(-\frac{dp_0}{dr}\right)\frac{4\pi}{B^2} < \frac{\hat{s}^2}{8}$$

$$\Rightarrow \boxed{\frac{r}{L_p}\beta < \frac{\hat{s}^2}{8} = \left(\frac{rq'}{q}\right)^2 / 8}$$

$$rq'/q = (1/L_s^2)(qR)^2$$

⇒ β-limit, ∇p-limit ⟩ criterion (in terms stability)

— structurally similar to line-tied inter-change criterion,

i.e. schematically

$$\gamma^2 = \gamma_I^2 - k_{\parallel}^2 V_A^2 \quad (m \to \infty)$$

$$= \cancel{\frac{k_\theta^2}{k_\perp^2}}^{\,1} \frac{\kappa}{\rho} \frac{d\rho}{dr} c_s^2 - k_{\parallel}^2 V_A^2$$

$$= \frac{c_s^2}{R_c L_p} - k_{\parallel}^2 V_A^2$$

Now, in sheared system, with resonances

$$k_{\parallel} = \frac{k_\theta x}{L_s} \sim \frac{k_\theta \Delta x}{L_s},$$ If take $(\Delta x) k_\theta \sim 1$
i.e. no other scale ... in ideal MHD

$$\gamma^2 = \frac{c_s^2}{R_c L_p} - \frac{V_A^2}{L_s^2}$$

$\Rightarrow$ stability for $\dfrac{(c_s/V_A)^2}{R_c L_p} < \dfrac{1}{L_s^2}$

if take $1/L_s = \hat{s}/R_q$

$$\Rightarrow \quad \frac{R^2 q^2}{R_c L_p} \beta < \hat{s}^2$$

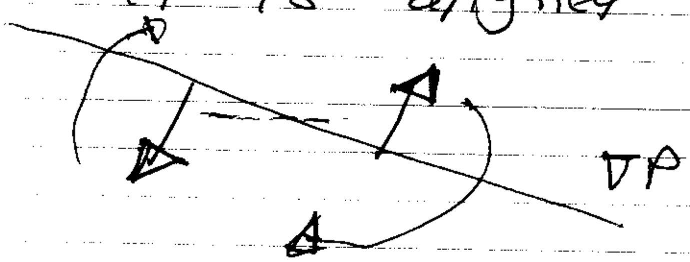
but $\quad 1/R_c = \frac{r}{q^2 R^2}$

$\Rightarrow \boxed{\frac{r}{4\rho}\beta < \hat{s}^2}$ → recovers Suydam, up to #!

i.e.

→ in periodic system with resonances, shear induces "effective line-tying", of sorts

→ physics is penalty in energy to rotate convective cell so that it is aligned with field.



stability is gain of gradient relaxation vs. loss due shear-enforced rotation penalty.

→ "ideal" MHD consistent with $k_\theta \Delta \sim 1$ choice. Apart from boundary (excluded here by mode localization), ideal MHD is scale free.

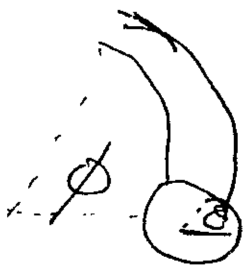
·) Interchange Dynamics in Sheared Magnetic Field

Now in "cylindrical" tokamak : $(\epsilon = a/R \ll 1)$

- $\underline{B}_0 = B_\theta(r)\,\hat{\theta} + B_z\,\hat{z}$ $\qquad |B_\theta| < B_z$

- periodic perturbations $\Rightarrow$

$$\hat{\phi} = \sum_{m,n} \hat{\phi}_{m,n}\, e^{i(m\theta - n\phi)}$$



$\theta \equiv$ poloidal $\measuredangle$
$\phi \equiv$ toroidal $\measuredangle$

Then, note:

$$\underline{B}\cdot\underline{\nabla} = \frac{B_\theta(r)}{r}\frac{\partial}{\partial\theta} + \frac{B_z}{R}\frac{\partial}{\partial\phi}$$

$$\rightarrow i\left(m\frac{B_\theta(r)}{r} - \frac{n}{R}B_z\right)$$

$$= i\,\frac{B_z}{R}\left(\frac{m}{q(r)} - n\right)$$

- $q(r) = rB_z / R B_\theta$ $\equiv$ local pitch of magnetic field ("safety factor")

Thus, $k_{\parallel}^{m,n} = \frac{1}{R}\left(\frac{m}{q(r)} - n\right)$

- tends to be small (i.e. line bending, Landau damping, etc. weak) when

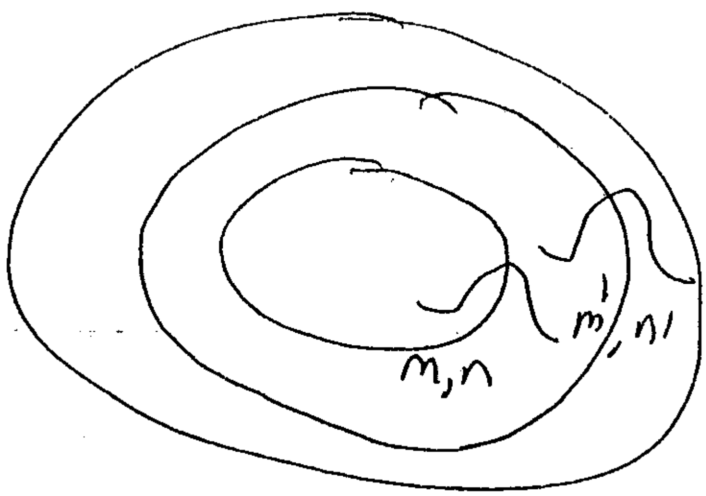
  $q(r) = m/n$ i.e. $\left\{\begin{array}{l}\text{local pitch of field line} \\ = \text{pitch of perturbatio}\end{array}\right.$

- defines $r_{m,n}$ s/t $q(r_{m,n}) = m/n$

<u>i.e.</u> $r_{m,n}$ is radius of $\left\{\begin{array}{l}\text{mode rational surface} \\ \text{resonant surface}\end{array}\right.$

where mode naturally wants to sit, to minimize bending, dissipation etc.

<u>i.e.</u>



fluctuations in tokamaks tie to resonant surf.

natural to write $\vec{\phi}_{m,n} = \vec{\phi}_{m,n}(x)$

where $x = r - r_{m,n}$

Note: $k_{\parallel} = \frac{1}{R}\left(\frac{m}{q(r_{m,n}+x)} - n\right)$

$= \frac{1}{R}\left(-\frac{m q'_{m,n} x}{q^2_{m,n}}\right) + \text{h.o.t}$

$\equiv \frac{k_\theta x}{L_s}$ $\qquad k_\theta = m/r$

$\frac{1}{L_s} = -\frac{r q'}{R q^2} \equiv$ magnetic shear length

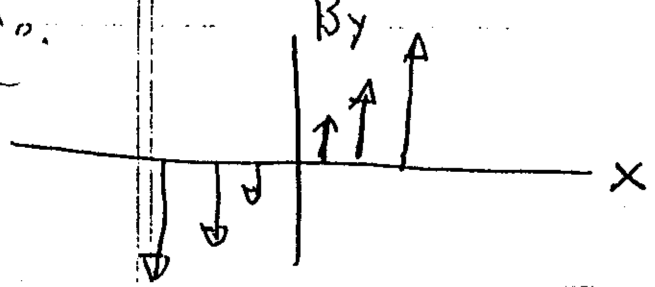
- equivalent to placing a resonant mode in local field

$\underline{B} = B_0\left(\hat{z} + \frac{x}{L_s}\hat{y}\right) \equiv$ sheared slab model.

Now, can furthur observe:

- in sheared system, field lines have radially varying orientation

i.e.



$B_y = B_0 x / L_s$

to interchange two flux tubes, need rotate flux tubes to align (locally) with sheared field (i.e. frozen in)

$\Rightarrow$ expect sheared field will exert significant stabilizing effect in ideal interchange.

i.e. $$\frac{1}{c}\frac{\partial \hat{A}}{\partial t} = \frac{B_z}{4\pi}\nabla_\parallel \hat{\phi}, \qquad J_z = -\nabla_\perp^2 A$$

$$-\frac{\partial^2}{\partial t^2}\nabla_\perp^2\hat{\phi} = \frac{B_0\cdot\nabla}{\rho_0 c}\frac{\partial \hat{J}_z}{\partial t} + \frac{\partial^2\hat{\phi}}{\partial y^2}\frac{g_{eff}}{L_\rho}$$

$$\hat{\phi} = \sum_{m,n} e^{\gamma_{m,n}t}\,\hat{\phi}_{m,n}(x)\,e^{i(m\theta - n\phi)}$$

$$+\gamma^2\left(\frac{\partial^2}{\partial x^2} - k_\theta^2\right)\hat{\phi}_{m,n} = +\frac{\cancel{\gamma} B_z i k_\parallel}{\rho_0 \cancel{c}}\,\frac{\nabla_\perp^2}{\cancel{\gamma}}\left(\frac{\cancel{c} B_z i k_\parallel \hat{\phi}_{m,n}}{4\pi}\right) + k_\theta^2\frac{g_{eff}}{L_\rho}\hat{\phi}_{m,n}$$

$$\gamma^2\left(\frac{\partial^2}{\partial x^2} - k_\theta^2\right)\hat{\phi}_{m,n} = -V_A^2 k_\parallel \nabla_\perp^2\left(k_\parallel\hat{\phi}_{m,n}\right) + k_\theta^2\frac{g_{eff}}{L_\rho}\hat{\phi}_{m,n}$$

$$\gamma_{m,n}^2 = \frac{\left[-k_\theta^2\frac{g_{eff}}{L_\rho}\int|\hat{\phi}_{m,n}|^2dx - V_A^2\int dx\,|\nabla_\perp k_\parallel\hat{\phi}_{m,n}|^2\right]}{\int|\nabla_\perp\hat{\phi}|^2dx}$$

.., For scaling :

$\nabla_\perp \sim 1/a$ (key: No scale for $\vec{\phi}$, other than a!)

$k_\parallel \sim \frac{k_\theta x}{L_s} \sim \frac{k_\theta a}{L_s}$

So, for β_{crit} (transition to instability) :

$$\frac{g_{eff}}{|L_p|} \geq \frac{V_A^2}{L_s^2} \Rightarrow \frac{L_s^2}{|L_p| R_c} \beta_{crit} > 1$$

stability if $\beta \leq \frac{|L_p| R_c}{L_s^2} \sim O(\epsilon^2)$ as $L_s \sim R$.

i.e. shear forces line-tying effect via $\nabla_\parallel \Rightarrow \sim 1/L_s$.

More detailed analysis confirms basic scaling
$\beta \leq \frac{L_p R_c}{L_s^2}$ (Suydam limit).

Now, useful to consider resistive interchange in sheared field

- allows field, fluid to slip (not frozen in!)

- introduces small scale $\Delta x \sim (\eta/\omega)^{1/2} \ll a$

here, basic smallness parameter is

$$1/S = \eta/a^2 / V_A/a$$

$1/S$: Lundquist #

η/a^2: resistive diffusion rate

V_A/a: Alfven rate

For resistive interchange:

$$-\frac{\partial^2}{\partial t^2}\nabla_\perp^2\tilde{\phi} = \frac{B_0\cdot\nabla}{\rho_0 c}\frac{\partial\tilde{J}_z}{\partial t} + \frac{\partial^2\tilde{\phi}}{\partial y^2}\frac{g_{eff}}{\rho_0}$$

$$\frac{\partial\tilde{J}_z}{\partial t} - \eta\nabla_\perp^2\tilde{J}_z = \frac{c}{4\pi}\underline{B_0}\cdot\underline{\nabla}\left(-\nabla_\perp^2\tilde{\phi}\right)$$

Assume $\eta k_\perp^2 > \gamma$ (verify aposteriori!)

$$\Rightarrow \tilde{J}_z = +\frac{c}{4\pi}\frac{B_0\cdot\nabla}{\eta}\tilde{\phi}$$

(not possible in ideal for $k\cdot B \neq 0$)

(electrostatic approximation i.e. $E_\parallel = \cancel{E_\parallel}_{\text{inductive}} + E_{\parallel es} = \eta J_\parallel$)

$\therefore$

$$-\gamma^2\left(\frac{\partial^2}{\partial x^2} - k_\theta^2\right)\tilde{\phi} = \frac{\gamma i B_0 k_\parallel}{\rho_0 c\eta}\frac{cB_0 i k_\parallel}{4\pi}\tilde{\phi} - k_\theta^2\frac{g_{eff}}{\rho_0}\tilde{\phi}$$

$$\rightarrow \left(\frac{\partial^2}{\partial x^2} - k_\theta^2\right)\tilde{\phi} - \frac{k_\parallel^2 V_A^2}{\gamma\eta}\tilde{\phi} - \frac{k_\theta^2 g_{eff}}{L_p \gamma^2}\tilde{\phi} = 0$$

$k_\parallel = k_\theta x/L_s$

$\Rightarrow$ eigenvalue problem for $\gamma_{m,n}$:

$$\left(\frac{\partial^2}{\partial x^2} - k_\theta^2\right)\tilde{\phi}_{m,n} - \frac{k_\theta^2 V_A^2}{L_s^2 \gamma\eta} x^2 \tilde{\phi}_{m,n} - \frac{k_\theta^2 g_{eff}}{\gamma^2 L_p}\tilde{\phi}_{m,n} = 0$$

Now, $\tilde{\phi}_{m,n} = e^{-\alpha_{m,n} x^2/2}$ $\qquad \alpha_{m,n} \sim 1/(\Delta x_{m,n})^2$

$\rightarrow$ "slow" interchange ($k_\theta \Delta x < 1$)

$$\alpha^2 x^2 - \alpha - \frac{k_\theta^2 V_A^2 x^2}{L_s^2 \gamma\eta} - \frac{k_\theta^2 g_{eff}}{\gamma^2 L_p} = 0$$

$\alpha = \left(\frac{k_\theta^2 V_A^2}{L_s^2 \gamma\eta}\right)^{1/2}$ $\rightarrow$ defines basic mode scale (η dependent!, γ dependent)

$\alpha = \frac{k_\theta^2 g_{eff}}{\gamma^2 L_p}$ $\rightarrow$ dispersion relation (need $g_{eff}/L_p < 0$

To determine γ, α explicitly:

$$\left(\frac{k_\theta^2 V_A^2}{L_s^2 \gamma \eta}\right)^{1/2} = \frac{k_\theta^2 g_{eff}}{\gamma^2 L_p}$$

$\Rightarrow$

$$\gamma = \left(\frac{L_s^2}{L_p^2}\left(\eta k_\theta^2\right)\left(\frac{V_A^2}{R_c^2}\right)\beta^2\right)^{1/3}$$

so

$$\begin{cases} \alpha = \left(k_\theta^2 V_A^2 / L_s^2 \left(\frac{L_s^2}{L_p^2}\, \eta k_\theta^2\, \frac{V_A^2}{R_c^2}\, \beta^2\right)^{1/3} \eta\right)^{1/2} \\ \Delta x = 1/\sqrt{\alpha} \end{cases}$$

For validity:

$$\begin{cases} \frac{\eta}{(\Delta x)^2} = \eta\alpha > \gamma \\ k_\theta^2 (\Delta x)^2 = \frac{k_\theta^2}{\alpha} < 1 \end{cases}$$

$\Rightarrow$ for e.s.:

$$\eta^2 \frac{k_\theta^2 V_A^2}{L_s^2 \gamma \eta} > \gamma^2$$

$$\cancel{\eta}\, \frac{\cancel{k_\theta^2}\, \cancel{V_A^2}}{L_s^2} > \gamma^3 = \frac{L_s^2}{L_p^2}\, \cancel{\eta}\, \cancel{k_\theta^2}\, \frac{\cancel{V_A^2}}{R_c^2}\, \beta^2$$

i.e. Need: $\dfrac{\beta L_s^2}{|L_p| R_c} < 1$ for validity of electrostatic approximation.

Note:

i.) $\left.\begin{array}{l} R_c \sim a \\ m/r \sim k_\theta \end{array}\right\} \Rightarrow$

$$\gamma \sim \left(\frac{L_s^2 m^2 \beta^2}{L_p^2}\right)^{1/3} \left(\frac{\eta}{a^2}\,\frac{V_A^2}{a^2}\right)^{1/3}$$

$$\sim \left(1/\tau_R \tau_A^2\right)^{1/3} \Rightarrow \gamma \tau_A \sim S^{-1/3} \beta^{2/3} (L_s/L_p)$$

i.e. growth rate is hybrid of resistive diffusion and Alfven rates

$\Rightarrow$ resistive diffusion allows decoupling of field, fluid, thereby triggering instability.

ii.) for incompressible MHD, have instability for all β (i.e. unlike ideal MHD, no β_{crit} exists!)

iii.) $\Delta x = \left(L_s^2 \gamma \eta / k_\theta^2 V_A^2\right)^{1/4} \ll a$

$$\sim S^{-1/3} \beta^{1/6}$$

i.e. $\Delta x/a \sim S^{-1/3} \Rightarrow$ narrow layer.

(iv) For fast interchange: $k_\theta^2(\Delta x)^2 > 1$

Thus, as before:

$$\alpha^2 x^2 - \alpha - k_\theta^2 - \frac{k_\theta^2 V_A^2}{L_s^2 \gamma \eta} x^2 - \frac{g_{eff} k_\theta^2}{\gamma^2 L_p} = 0$$

$$\Rightarrow \quad \alpha = \left(k_\theta^2 V_A^2 / L_s^2 \gamma \eta\right)^{1/2}$$

$\frac{k_\theta^2}{\alpha} > 1 \quad \Rightarrow$ now obtain dispersion relation:

$$-k_\theta^2 = -g_{eff} k_\theta^2 / \gamma^2 |L_p|$$

$$\gamma^2 = g_{eff}/|L_p| = c_s^2 / R_c |L_p|$$

$$\Delta x \sim S^{-1/2}$$

Note:

(i) Fast regime entered when:

$$\frac{k_\theta^2}{\alpha} > 1 \quad \Rightarrow \quad \frac{k_\theta^2}{\left(k_\theta^2 V_A^2 / L_s^2 \gamma \eta\right)^{1/2}} > 1$$

$$k_\theta^{\cancel{2}4} > \frac{\cancel{k_\theta^2} V_A^2}{L_s^2 \gamma \eta}$$

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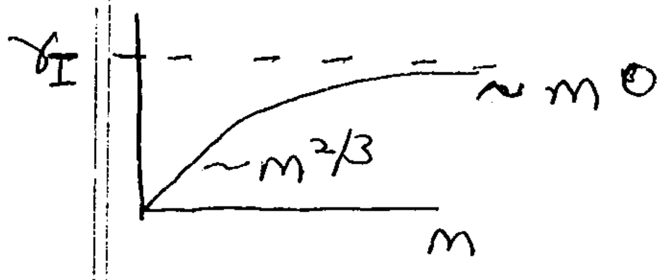
$\eta k_\theta^2 > V_A^2/L_s^2 \, \gamma_I$

i.e. fast interchanges dominate at large m



In practice, large η or high m $\Rightarrow$ fast interchange

ii.) Note essence of fast interchange is:

- high ηk_θ^2
- ideal growth rate.

Physical content is that ηk_θ^2 so large that line-bending destroyed and mode reverts to ideal growth

iii.) Note $\Delta x \sim \delta^{-1/2}$, i.e. mode still localized by η. also $\gamma \Delta^2 \sim \eta$

iv.) In reality, fast interchange eventually cut-off by dissipation (μ, D etc.).

v.) All resistive interchanges localized to $\underline{k} \cdot \underline{B}_0 = 0$ resonant surfaces.