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Quasi-linear Theory

1. Motivation and Overview

The Vlasov plasma

To motivate our understanding of the structure and physics of quasi-linear theory let us consider the simple example of a collisionless plasma in one dimension. The evolution of the particle distribution function $f(x, v, t)$ is given by the Vlasov equation,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \dot{v}\frac{\partial f}{\partial v} = 0 \qquad 1.1$$

which is essentially the Boltzmann equation in one dimension in the absence of collisions. The acceleration dv/dt of an individual particle of charge q and mass m will be due to the electric field of the rest of the particles,

$$\dot{v} = \frac{q}{m}E(x,t) \qquad 1.2$$

where the electric field E must be calculated self-consistently from Gauss' Law,

$$\partial_x E = -4\pi\rho(x,t) = -4\pi q\int dv\, f(x,v,t) \qquad 1.3$$

Here the charge density ρ has been expressed in terms of the distribution function f. Equivalently, we may introduce the plasma permittivity such that

$$\epsilon E = E + 4\pi \mathcal{P} \qquad 1.4$$

where

$$\partial_x \mathcal{P} = -\rho \quad , \quad \partial_x(\epsilon E) = 0 \qquad 1.5$$

Taking the derivative of *(1.4)* with respect to x and moving into Fourier space yields

$$\epsilon_{k\omega} = 0 \qquad 1.6$$

Thus the electric field may be calculated self-consistently from either *(1.3)* or equivalently *(1.6)*.

Let is now consider the mean evolution of f where

$$f = \langle f\rangle + f' \quad , \quad \langle f'\rangle = 0 \qquad 1.7$$

Here, $\langle f\rangle$ is assumed slowly evolving on a timescale γ^{-1} and the brackets $\langle . \rangle$ denote a spatial average. The fluctuation f' in the distribution oscillates rapidly on a timescale ω^{-1} and it will be assumed that this oscillation is very fast compared to the evolution of $\langle f\rangle$: $\omega \gg \gamma$.

Now imagine that in the unperturbed plasma the particles are balanced by immobile ions everywhere: the electric field will be due entirely to the fluctuation in the distribution function:

$$E = E' \quad , \qquad \langle E' \rangle = 0 \tag{1.8}$$

where

$$\partial_x E' = -4\pi q \int dv\, f' \tag{1.9}$$

Taking the average of the Vlasov equation *(1.1)* gives

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{q}{m} \frac{\partial}{\partial v} \langle E' f' \rangle = 0 \tag{1.10}$$

Note that *(1.10)* has the generic form of a continuity equation:

$$\partial_t \langle f \rangle = -\partial_v J_v \tag{1.11}$$

where J_v is the phase-space current. Furthermore, *(1.10)* poses an elementary moment closure problem: $\langle f \rangle$ is determined by $\langle E' f \rangle$ which in turn is determined by $\langle E' E' f \rangle$ and so on. To solve the problem in any statistical sense we must truncate the moment hierarchy.

The Quasi-linear approximation

The simplest closure scheme is to assume that the response of the fluctuation f' to the perturbation E' is purely linear – i.e., the fluctuations are so small that quadratic terms may be neglected. This is the *quasi-linear approximation.* Thus the Vlasov equation becomes

$$\partial_t \left(\langle f \rangle + f' \right) + v \partial_x f' = -\frac{q}{m} E' \partial_v \langle f \rangle \tag{1.12}$$

where the second-order term on the RHS has been discarded. We make further use of the slow evolution of $\langle f \rangle$ when compared to f' by neglecting the time-derivative of $\langle f \rangle$ on the LHS. Fourier-transforming the remainder, we obtain

$$f'_{k\omega} = -i \frac{q}{m} \frac{E'_{k\omega}}{\omega - kv} \partial_v \langle f \rangle \tag{1.13}$$

Thus

$$\partial_t \langle f \rangle = \partial_v \mathcal{D} \partial_v \langle f \rangle \tag{1.14}$$

where

$$\mathcal{D} = i \frac{q^2}{m^2} \sum_{k,\omega} \frac{|E'_{k\omega}|^2}{\omega - kv} \partial_v \langle f \rangle \tag{1.15}$$

and

$$\epsilon_{k\omega} = 1 + \frac{4\pi q^2}{m} E'_{k\omega} \int dv \frac{i}{\omega - kv} \partial_v \langle f \rangle = 0 \tag{1.16}$$

Equation *(1.14)* is the *quasi-linear equation* for a 1D Vlasov plasma.

It may be remarked upon immediately that *(1.14)* has the generic form of a *diffusion* equation for $<f>$ in velocity-space, with $\mathcal{D}$, the diffusion coefficient, defined by *(1.15)* and *(1.16)*

Diffusion is necessarily a non-deterministic process involving some stochastic "kick" in velocity-space. It is intrinsically irreversible; there is no way we can retrace a given particle's trajectory. The Vlasov equation, on the other hand, is collisionless and therefore reversible – all particle motions can be retraced along their trajectories. There is no irreversible increase in the entropy of the plasma. So what is the origin of the irreversibility?

As we shall see in *§2*, overlapping wave-particle resonances can lead to the onset of chaotic trajectories in phase-space and this acts as the fundamental irreversibility for the quasi-linear equation *(1.14)*.

The format for the rest of this chapter will thus be arranged as follows:

- In *§2* we will examine further the issues of irreversibility and resonance and briefly discuss the subject of Hamiltonian chaos.

- In *§3* we will ask the question: "when is quasi-linear theory valid?" and, building on the discussion of *§2* formulate a Ginzberg-style criterion predicting the theory's breakdown.

- In *§4* we discuss the energy budget for the quasi-linear equation and ask: "can we formulate quasi-linear theory from a Fokker-Planck equation?" As we shall see, the answer will require us to carefully differentiate between resonant and non-resonant diffusion.

- Finally in *§5* we address the issue of quasi-linear relaxation in the specific case of the "bump-on-tail" instability.

2. Irreversibility and Resonance

The origin of irreversibility

In *§1* we saw that the quasi-linear equation *(1.14)* had the generic form of a diffusion equation. But what is the origin of the irreversibility underpinning the diffusion?

To answer this question we first note that, generally, quasi-linear theory is concerned with a *broad spectrum* of *unstable waves*. In a finite system of size L the wave-number will be quantized ($k_n = n\pi/L$ where n is an integer) and we have a spectrum of waves with phase velocities $v_{ph,\,n} = \omega(k_n)/k_n$. When a particle with velocity equal to $v_{ph,\,n}$ we expect *wave-particle resonance* with mode *n to* occur: the Doppler-shifted electric fluctuations appear as a DC field to particles whose velocity matches the phase velocity of the field and so can do work on the particles. All other particles see an oscillatory AC field which does no work when averaged over time.

The equation of motion is governed by the Lagrangian

$$\mathcal{L} = \frac{1}{2} m v^2 - q\phi(x,t) \qquad 2.1$$

where the electric potential $\Phi(x, t)$ may be decomposed into

$$\phi(x,t) = \sum_n \phi_n \cos(k_n x - \omega_n t) \qquad 2.2$$

We shall assume (without loss of generality) that the zeroth resonance dominates: in this case, the potential becomes a function of y only, where

$$y = x - \frac{\omega_0}{k_0} t = x - v_{ph,0}\, t \qquad 2.3$$

We now make the assumption that the trajectory of the particle is essentially unperturbed by the electric field, i.e., $x = x_0 + v\,t$. Discussion of this approximation is deferred to *§3*; for the time being we accept it without question, and *(2.3)* becomes

$$y = x_0 + (v - v_{ph,0})t \ , \quad v_y = v - v_{ph,0} \qquad 2.4$$

In this frame, the Hamiltonian of the system is

$$E = \frac{1}{2} m (v_y^2 - v_{ph,0}^2) + q\,\phi_0 \cos k_0 y \qquad 2.5$$

where, in the absence of an explicit dependence upon time, the Hamiltonian is a conserved quantity: the energy E.

The phase-plane portrait of the system represented by *(2.5)* is sketched in *fig.2.1*. The seperatrix $E = 0$ divides the motion into trapped ($E < 0$) and circulating ($E > 0$). The width of the seperatrix is easily shown to be $\Delta V \sim (q\Phi_0/m)^{1/2}$.

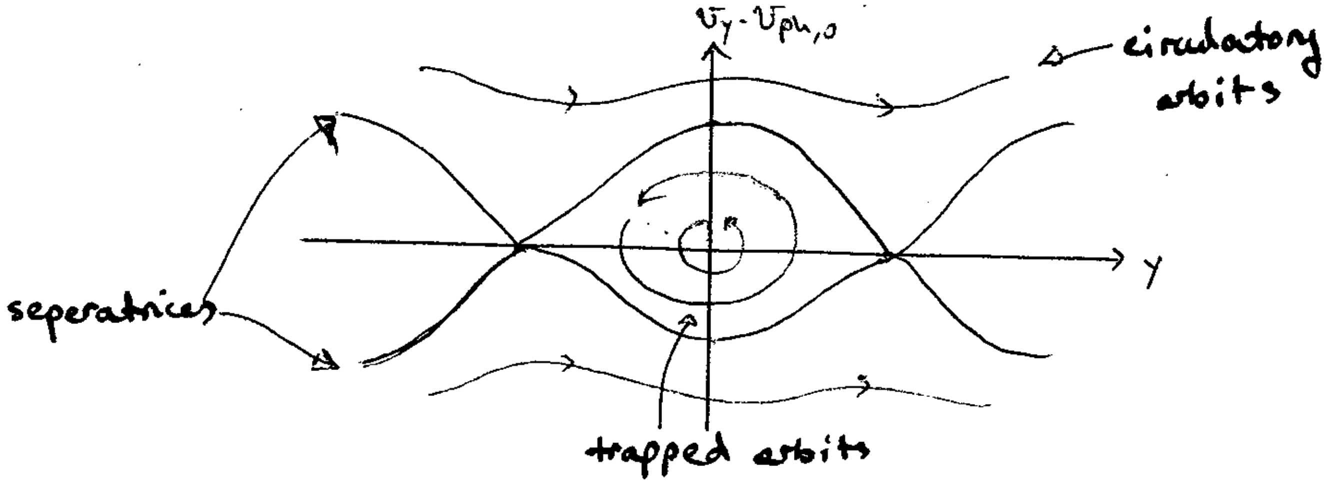

Fig.2.1: Phase-plane portrait of a single resonance.

Systems with a single resonance are *integrable*: for an N degree of freedom system we can obtain N periodic variables and N conjugate constants of the motion: the *action-angle variables* of the system. For an integrable system, any given trajectory is constrained to the surface of an N-torus: *fig.2.2* shows the case of $N = 2$.

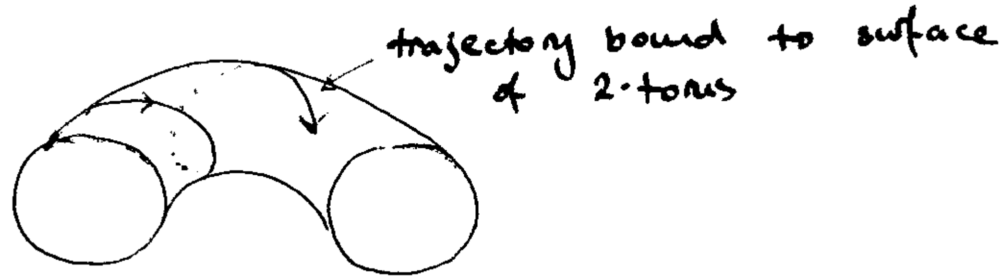

Fig.2.2: Motion on a 2-torus.

If we add a small non-integrable perturbation to the Hamiltonian, i.e., more resonances, the system is no longer integrable. It can be shown (see Ott, Tabor etc.) that for most tori, the addition of another resonance only slightly deforms the torus; however, a few tori are dramatically deformed and, as the non-integrable perturbation is tuned up, these tori deform other tori nearby. Eventually, the trajectories become so deformed that they are essentially locally chaotic and the orbit may be treated as stochastic.

The orbits are still confined within seperatrices however; for global chaos the resonances must be close enough that the seperatrices between them are destroyed and orbits can stochastically wander from resonance to resonance (see *fig.2.3*), leading to irreversible diffusion in v-space. The condition for such *resonance overlap* is the *Chirikov criterion*:

$$\frac{1}{2}\left(\Delta v_m + \Delta v_{m+1}\right) \gtrsim v_{ph,m+1} - v_{ph,m} \qquad 2.6$$

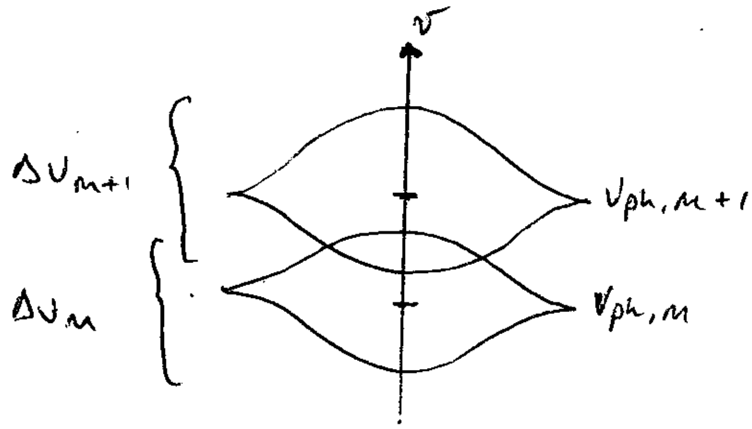

Fig.2.3. Resonance overlap.

Resonance overlap and subsequent orbit stochasticity is the fundamental irreversibility underpinning the quasi-linear equation. Equation *(2.6)* is a necessary condition for quasi-linear theory to be valid; it is, however, not sufficient, as we shall see in the next section.

3. Validity of the quasi-linear approximation

The unperturbed orbits approximation

A key assumption in the discussion of *§2* was the use of unperturbed orbits in calculating the effect of resonances on a particle. Obviously, if the particle orbit is substantially deflected by the changing electric field then the use of such unperturbed orbits is no longer valid and the applicability of quasi-linear theory to the evolution of the distribution function becomes suspect. An appropriate measure of the validity of quasi-linear theory then can be formulated by asking when the use of unperturbed orbits is justified.

We imagine that the particle "feels" the instantaneous superposition of modes of the electric field. There are two time-scales:

- τ_{life} : the lifetime of the instantaneous field pattern
- τ_{bounce} : the 'bounce time' for the particle within the instantaneous field pattern.

Evidently, when $\tau_{life} >> \tau_{bounce}$ the particle will 'bounce' before the electric field pattern changes and, hence, the use of unperturbed orbits is no longer justified. Alternatively, when $\tau_{life} << \tau_{bounce}$ the pattern changes too rapidly for the particle to become trapped and unperturbed orbits are satisfactory approximations.

How do we relate the timescales τ_{life} and τ_{bounce} to physical quantities? The bounce time can be obtained from basic physics:

$$\tau_{bounce}^{-1} \sim k\,(q\phi_0/m)^{1/2} \qquad 3.1$$

The lifetime of the field pattern, however, is related to the spread in phase velocities: it is the time taken to disperse by one wavelength k^{-1} :

$$\tau_{life}^{-1} \approx k\,\Delta(\omega_k/k) \qquad 3.2$$

Thus,

$$\begin{aligned}\tau_{life}^{-1} &= k\left(\frac{d\omega_k}{dk}\frac{\Delta k}{k} - \frac{\omega_k}{k^2}\Delta k\right)\\ &= \left(\frac{d\omega_k}{dk} - \frac{\omega_k}{k}\right)\Delta k\\ &= (v_{gr} - v_{ph})\,\Delta k\end{aligned} \qquad 3.3$$

One immediate consequence of *(3.3)* is that $\tau_{life} \to \infty$ for waves with equal phase and group velocities: such waves are called *non-dispersive*. In fact, quasi-linear theory (and weak turbulence theory in general) performs poorly for non- or weakly dispersive waves.

Let us rederive *(3.3)* in a more systematic fashion. Consider the electric field correlation function defined by

$$C = \langle E(x_1, t_1)\, E(x_2, t_2) \rangle_{x,t} \tag{3.4}$$

For homogeneous and stationary fluctuations C will be a function of the space and time intervals only. If we introduce variables $x_\pm = x_1 \pm x_2$ and $t_\pm = t_1 \pm t_2$ then, for homogeneous and stationary fluctuations, C will be a function of x_- and t_- only, and the space-time average $\langle . \rangle_{x,t}$ can be taken over the variables x_+ and t_+. Thus,

$$C(x_-, t_-) = \left\langle \sum_{k,k'} E_k E_{k'} e^{i[(k+k')x_+ - (\omega_k + \omega_{k'})t_+ + (k-k')x_- - (\omega_k - \omega_{k'})t_-]} \right\rangle_{x_+, t_+} \tag{3.5}$$

Performing the average yields delta-functions centred on $k = -k'$ and $\omega_k = -\omega_{k'}$. Hence

$$C(x_-, t_-) = \sum_k |E_k|^2 e^{ikx_-} e^{-i\omega_k t_-} \tag{3.6}$$

To make further progress we assume the following form for the spectrum of the electric field fluctuations:

$$|E_k|^2 = |E_0|^2 \Big/ \left\{ \frac{(k-k_0)^2}{\Delta k^2} + 1 \right\} \tag{3.7}$$

and, evaluating on unperturbed orbits $x_- = x_{0-} + v\, t_-$ we obtain

$$\langle E^2 \rangle \sim |E_0|^2 e^{ik_0 x_-} e^{-|\Delta k x_{0-}|} e^{i(k_0 v - \omega_{k_0})t_-} \times$$
$$\times\, e^{-|\Delta(kv - \omega_k)t_-|} \tag{3.8}$$

The first two exponents in *(3.8)* contain irrelevant phase information, while the third exponent is oscillatory and will vanish on resonance. The final exponent, however, describes the decay rate of the correlation function due to dispersal. Thus, the decay rate is set by

$$\begin{aligned} \Delta(kv - \omega_k) &= v\,\Delta k - v_{gr}\,\Delta k \\ &= |v - v_{gr}|\,\Delta k \end{aligned} \tag{3.9}$$

Thus, the lifetime of the field pattern, or equivalently the effective *autocorrelation time* of the electric field, for resonant particles ($v = v_{ph}$) is given by

$$\tau_{life}^{-1} = \tau_{ac}^{-1} = |(v_{ph} - v_{gr})\Delta k| \tag{3.10}$$

There are, therefore, four key timescales for the system:

- τ_{ac} : lifetime of electric field pattern for resonant particles
- γ^{-1} : the growth/damping time of the wave
- τ_{bounce} : the trapping time
- τ_{relax} : the average distribution relaxation time.

As we saw in *§2*, for the <*f*> closure to be meaningful we must have

- $\tau_{ac}, \gamma^{-1} << \tau_{relax}$

while the validity of the unperturbed orbit approximation requires

- $\tau_{ac} << \tau_{bounce}$.

Finally, quasi-linear theory is valid when you have overlapping resonances *and* the following time ordering:

- $\tau_{ac} < \gamma^{-1} < \tau_{relax}$

4. The energy budget

Cooking the books

So far in this game we have divided the players into two camps: *particles* (governed by f) versus *fields* (governed by E). An equivalent and arguably more intuitive approach is to look at the interaction between *resonant particles* and the *wave*, where the latter consists of both the field and the non-resonant particles.

Consider, for example, a plasma oscillation. The permittivity is

$$\epsilon(\omega) = 1 - \omega_p^2/\omega^2 \tag{4.1}$$

Thus, the wave energy $\mathcal{W}$, defined by

$$\mathcal{W} = \frac{d}{d\omega}(\omega\epsilon)\bigg|_{\omega_k} \frac{|E|^2}{8\pi} \tag{4.2}$$

is found to be

$$\mathcal{W} = 2 \cdot \frac{|E|^2}{8\pi} \tag{4.3}$$

The prefactor of *2* in *(4.3)* indicates that both the field and the "sloshing" of the non-resonant particles contribute to the total wave energy.

The kinetic energy density $\mathcal{K}^{total}$ of the particles

$$\mathcal{K}^{total} = \int dv \, \frac{mv^2}{2} \langle f \rangle \tag{4.4}$$

may be calculated from the spatially-averaged Vlasov equation

$$\partial_t \langle f \rangle = -\partial_v \frac{q}{m} \langle E'f' \rangle \tag{4.5}$$

yielding

$$\partial_t \mathcal{K}^{total} = \int dv \, qv \langle E'f' \rangle \tag{4.6}$$

where $\langle E'f' \rangle$ was assumed to vanish at infinity in the calculation of the RHS of *(4.6)*.

If, as in *§1*, we now insert the linear response of f' to the electric field fluctuations E', we obtain

$$\partial_t \mathcal{K}^{total} = -i \int dv \, \frac{vq^2}{m} \sum_k |E_k|^2 \left\{ \frac{\mathcal{P}}{\omega - kv} - i\pi\delta(\omega - kv) \right\} \frac{\partial\langle f \rangle}{\partial v} \tag{4.7}$$

where $\mathscr{P}$ is the principle part of the contour integral over ω. Thus the kinetic energy density is divided into two parts: the *non-resonant particle* kinetic energy density $\mathcal{K}^{non\text{-}res}$ associated with the principle part of the integral, and the *resonant particle* kinetic energy density $\mathcal{K}^{res}$ associated with the delta function:

$$\partial_t \mathcal{K}^{res} = -\int dv \, {}^{\pi q^2}\!/_M \sum_k \frac{\omega}{k|k|} \delta\left(\frac{\omega}{k} - v\right) \partial_v \langle f \rangle \, |E_k|^2$$

$$= -{}^{\pi q^2}\!/_M \sum_k \frac{\omega}{k|k|} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\omega/k} |E_k|^2 \qquad 4.8$$

To calculate the wave energy density, $\mathcal{W}$, we recall that

$$\epsilon = 1 + {}^{\omega_p^2}\!/_k \int dv \frac{1}{\omega - kv} \partial_v \langle f \rangle \qquad 4.9$$

which will have both a real and an imaginary part, $\epsilon = \epsilon^R(\omega_k + i\,\gamma_k) + i\,\epsilon^{Im}$, where γ_k is the growth/damping rate of the electric field fluctuations. Since $\gamma_k << \omega_k$ we can expand ϵ^R about ω_k to obtain, on rearranging

$$\gamma_k = -\epsilon^{Im} \left[\frac{\partial \epsilon^R}{\partial \omega}\right]^{-1} \qquad 4.10$$

Now,

$$\mathcal{W} = \sum_k \frac{\partial}{\partial \omega} (\omega \epsilon) \frac{|E_k|^2}{8\pi} = \sum_k \omega_k \left.\frac{\partial \epsilon^R}{\partial \omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi} \qquad 4.11$$

so that

$$\partial_t \mathcal{W} = \sum_k 2\gamma_k \omega_k \left.\frac{\partial \epsilon^R}{\partial \omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi} \qquad 4.12$$

Substituting *(4.10)* into the expression gives

$$\frac{\partial \mathcal{W}}{\partial t} = -\sum_k \epsilon^{Im}_{k,\omega_k} \cdot \omega_k \cdot \frac{|E_k|^2}{4\pi} \qquad 4.13$$

From *(4.9)* we have

$$\epsilon^{Im} = \frac{\omega_p^2}{k} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\omega/k} \frac{(-i\pi)}{|k|} \qquad 4.14$$

and so

$$\partial_t \mathcal{W} = \frac{\pi q^2}{M} \sum_k \frac{\omega}{k|k|} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\omega/k} |E_k|^2 \qquad 4.15$$

Comparing this with *(4.8)*, we see that

$$\partial_t \mathcal{K}^{res} + \partial_t \mathcal{W} = 0 \qquad 4.16$$

Note that *(4.16)* is essentially the Poynting theorem for plasma waves, where, for a homogeneous system the divergence of the wave energy density flux is zero, and the <***E·J***> coupling is mediated by the resonant particles, who observe E as a DC field.

Before we proceed further it is worth considering the fate of $\mathcal{K}^{res}$ when the wave saturates. In this case, $\partial_t \mathcal{W} = 0$ and so $\partial_t \mathcal{K}^{res} = 0$. In this case a plateau forms in the distribution function as the system relaxes. We shall discuss this situation in more detail in *§5*.

Now, recall that $\mathcal{W} = \mathcal{K}^{non\text{-}res} + \mathcal{E}^{field}$, where $\mathcal{E}^{field}$ is the field energy density. If we then regroup the terms in *(4.16)* we see that

$$\partial_t \mathcal{E}^{field} + \partial_t \mathcal{K}^{total} = 0 \qquad 4.17$$

where $\mathcal{K}^{total} = \mathcal{K}^{res} + \mathcal{K}^{non\text{-}res}$ is the total particle kinetic energy density. Thus, fields and particles also conserve energy. A direct proof of *(4.17)* is left to the Appendix at the end of this chapter.

Fokker-Planck and all that

The upside of all this is that the quasi-linear diffusion coefficient $\mathcal{D}$ of *§1* has more going on than meets the eye. Consider, for a weakly, non-stationary state

$$\mathcal{D} = \sum_k q^2/M^2 |E_k|^2 \cdot |\gamma_k| \Big/ \left[(\omega - kv)^2 + |\gamma_k|^2\right]$$

$$\cong \sum_k \frac{q^2}{M^2} |E_k|^2 \left\{ \pi\delta(\omega - kv) + \frac{\gamma_k}{\omega_k^2} \right\} \qquad 4.18$$

where γ_k is taken to be positive to satisfy causality (the diffusion coefficient cannot be negative). The first term in the curly brackets in *(4.18)* will be associated with *resonant diffusion* and the second with *non-resonant diffusion*. Resonant diffusion, as the name implies, is rooted in resonance overlap and, hence, particle stochasticity. Thus it is true, irreversible diffusion, and can be obtained from a Fokker-Planck formulation (left as an exercise to the reader). The non-resonant diffusion coefficient, on the other hand, given by

$$\mathcal{D}^{non\text{-}res} = \sum_k \frac{q^2}{M^2} |E_k|^2 \frac{\gamma_k}{\omega_k^2} = \frac{1}{2} \partial_t \sum_k |v_k|^2 \qquad 4.19$$

where we used the equation of motion in the form

$$|v_k|^2 = \frac{q^2}{M^2} |E_k|^2 / \omega_k^2 \qquad 4.20$$

and replaced a factor of $2\gamma_k$ by a time derivative in *(4.19)*. Thus, $\mathcal{D}^{non\text{-}res}$ is proportional to the time derivative of the "sloshing" energy of the particles in the wave. The motion is therefore reversible, and the non-resonant diffusion coefficient *cannot* be obtained from a Fokker-Planck formulation.

In the stationary state $\gamma_k = 0$ and $\mathcal{D}^{non\text{-}res}$ vanishes. This result is related to the fact that in any Hamiltonian system there is a partial cancellation between the drag and drift terms in the Fokker-Planck equation:

$$\partial_t \langle f \rangle = -\partial_v \left[\frac{\langle \Delta v \rangle}{\Delta t} \langle f \rangle - \partial_v \frac{\langle \Delta v^2 \rangle}{2\Delta t} \langle f \rangle \right] \qquad 4.21$$

For a Hamiltonian system it can be shown that

$$\frac{\langle \Delta v \rangle}{\Delta t} = \partial_v \frac{\langle \Delta v^2 \rangle}{2\Delta t} \qquad 4.22$$

and thus

$$\partial_t \langle f \rangle = \partial_v \frac{\langle \Delta v^2 \rangle}{2\Delta t} \frac{\partial \langle f \rangle}{\partial v} \qquad 4.23$$

Thus there is no drag term in the Fokker-Planck equation of a Hamiltonian system.

The point of all this is that we can consider non-resonant diffusion due to the particle "sloshing" energy as either

- part of the *wave* energy density, or
- part of the *total particle* kinetic energy density.

Physically, we now have a picture of the plasma as a gas of real particles (the resonant particles) interacting with quasi-particles (the wave). The evolution of the real particles, as we have seen, is governed by a kinetic equation (the Vlasov equation) and we will show later on that the quasi-particles are governed by a wave-kinetic equation. This is an intuitively appealing picture and will pervade throughout the rest of this text.

5. An application of quasi-linear theory

The "bump-on-tail" instability

The prototypical linear instability in plasma theory is the well-known "bump-on-tail" instability. A beam of particles of density n_b and velocity v_b is directed through plasma at rest. When the beam is switched on, the distribution function of the particles looks something like *fig.5.1*, with most of the particles clustered around the centre of the distribution, and the eponymous "bump" on the tail of the distribution representing the beam particles superimposed upon the bulk.

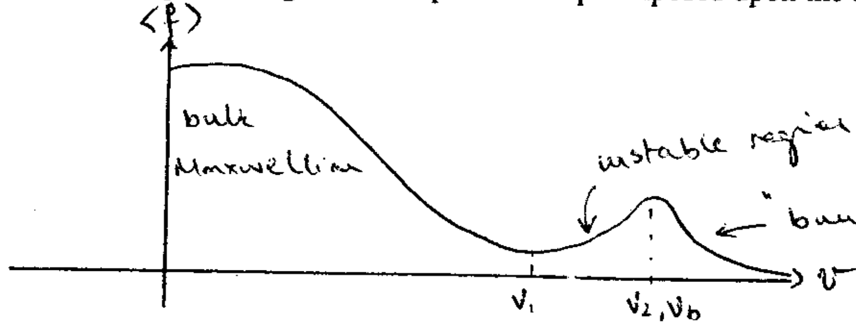

Fig.5.1: The bump-on-tail instability

In the region between v_1 and v_2 the slope of the distribution function is positive and so we expect *inverse Landau damping* to carry energy from the waves to the resonant particles – hence the system is unstable. In the time-asymptotic limit we expect the linear instability to saturate; as we shall see, this comes about by a *flattening* of the distribution function around v_b.

The self-consistent field equation for this system is

$$\epsilon_{k,\omega_k} = 0 \tag{5.1}$$

where $\omega(k)$ *and* γ_k are calculated from $\langle f \rangle$ which, as before, is governed by the diffusion equation

$$\partial_t \langle f \rangle = \partial_v \mathcal{D} \, \partial_v \langle f \rangle \tag{5.2}$$

where, as per *§4*, we divide the diffusion coefficient $\mathcal{D}$ into resonant and non-resonant parts

$$\mathcal{D} = \mathcal{D}^{res} + \mathcal{D}^{non\text{-}res} = \frac{q^2}{M^2}\sum_k |E_k|^2 \left[\pi\,\delta(\omega_k - kv) + \frac{\gamma_k}{\omega_k^2}\right] \tag{5.3}$$

Observe that *resonant* diffusion describes the dynamics of the particles in the *tail* of the distribution function whereas the bulk Maxwellian is governed by *non-resonant* diffusion.

Consider first the resonant particles:

$$\partial_t \int dv \, \frac{\langle f \rangle^2}{2} = -\int dv \, \mathcal{D}^{res} \left(\frac{\partial \langle f \rangle}{\partial v}\right)^2 \tag{5.4}$$

Stationarity implies that the integrand of the RHS of this equation is zero: this can be satisfied by either a *plateau* in the distribution function (i.e., $\partial\langle f\rangle/\partial v=0$), or $\mathcal{D}^{res}=0$, indicating that the wave grows and then damps. In the latter case, we expect that, as $t \rightarrow \infty$, damping requires that $\partial\langle f\rangle/\partial v < 0$. We shall now show that this situation leads to a contradiction and so we expect a plateau to form.

The resonant particle diffusion coefficient is, from *(5.3)*,

$$\mathcal{D}^{res} \simeq 16\pi^2 \frac{q^2}{m^2} \int dk\, \mathcal{E}_k^{field}\, \delta(\omega - kv) \tag{5.5}$$

where $\mathcal{E}^{field}$ is the field energy density. Thus

$$\mathcal{D}^{res} = \frac{16\pi^2 q^2}{m^2} \frac{1}{v} \mathcal{E}^{field}(\omega_p/v) \tag{5.6}$$

and so

$$\partial_t \mathcal{D}^{res} = \frac{16\pi^2 q^2}{m^2 v} 2\gamma_{\omega_p/v}\, \mathcal{E}^{field}(\omega_p/v) \tag{5.7}$$

Now, from *(4.10)* and *(4.14)* we have

$$\gamma_{\omega_p/v} = \pi v^2 \omega_p \frac{\partial\langle f\rangle}{\partial v} \tag{5.8}$$

so

$$\begin{aligned} \partial_t \mathcal{D}^{res} &= \frac{16\pi^2 q^2}{m^2 v} \cdot 2\pi v^2 \omega_p \frac{\partial\langle f\rangle}{\partial v}\, \mathcal{E}^{field}(\omega_p/v) \\ &= \pi\omega_p v^2 \frac{\partial\langle f\rangle}{\partial v} \mathcal{D}^{res}. \end{aligned} \tag{5.9}$$

using *(5.7)*. Hence,

$$\mathcal{D}^{res}(v,t) = \mathcal{D}^{res}(v,0)\exp\left[\pi\omega_p v^2 \int_0^t dt' \frac{\partial\langle f\rangle}{\partial v}\right] \tag{5.10}$$

where,

$$\mathcal{D}^{res}(v,0) = \frac{16\pi^2 q^2}{m^2 v} \mathcal{E}^{field}_{\omega_p/v}(t=0) \tag{5.11}$$

and the evolution equation for $\langle f\rangle$ for the resonant particles becomes,

$$\partial_t \langle f\rangle = \partial_t \partial_v \left\{ \mathcal{D}^{res}/\pi\omega_p v^2 \right\} \tag{5.12}$$

Thus the generic solution to the problem is *(5.10)* and

$$\langle f(v,t)\rangle = \langle f(v,0)\rangle + \partial_v \left\{ \frac{\mathcal{D}^{res}(v,t) - \mathcal{D}^{res}(v,0)}{\pi\omega_p v^2} \right\} \tag{5.13}$$

If we now assume that $\mathcal{D}^{res}=0$, we find

$$\langle f(v,t)\rangle \approx \langle f(v,0)\rangle - \partial_v \left[\frac{\mathcal{D}^{res}(v,0)}{\pi \omega_p v^2} \right] \qquad 5.14$$

However, for $n << n_b$

$$\frac{16\pi^2 q^2}{M^2 v} \frac{\mathcal{E}^{field}(0)}{\pi \omega_p v^2} = \frac{2E^{fluctuation}(0)}{n M v_b^2/2} << 1 \qquad 5.15$$

and so $<f(v, t)> \approx <f(v, 0)>$ to a good approximation. For resonant velocities we saw that linear instability requires $\partial<f>/\partial v > 0$. Thus, if $\mathcal{D}^{res}=0$, the slope of the unstable region must be *positive* for all time.

On the other hand, we saw that if we assume $\mathcal{D}^{res}=0$ then the waves must eventually damp and the slope of the unstable region must become *negative* in the time-asymptotic limit: the assumption of $\mathcal{D}^{res}=0$ in the stationary case leads to a contradiction and hence we have established that $\partial<f>/\partial v \to 0$ as $t \to \infty$ and a plateau forms.

Saturation of the instability

We can now immediately determine the saturation level of the instability from the conservation of energy – in its resonant particle/wave formulation – by a Maxwell-like construction. The initial and final configurations of the distribution function are shown in *fig.5.2a,b*. Physically, we expect the bump to be "slowed down". The non-resonant particles in the bulk must adjust, however, to conserve total momentum. Thus, we expect the bulk to spread outwards, leading to a shift in the "effective temperature" of the bulk.

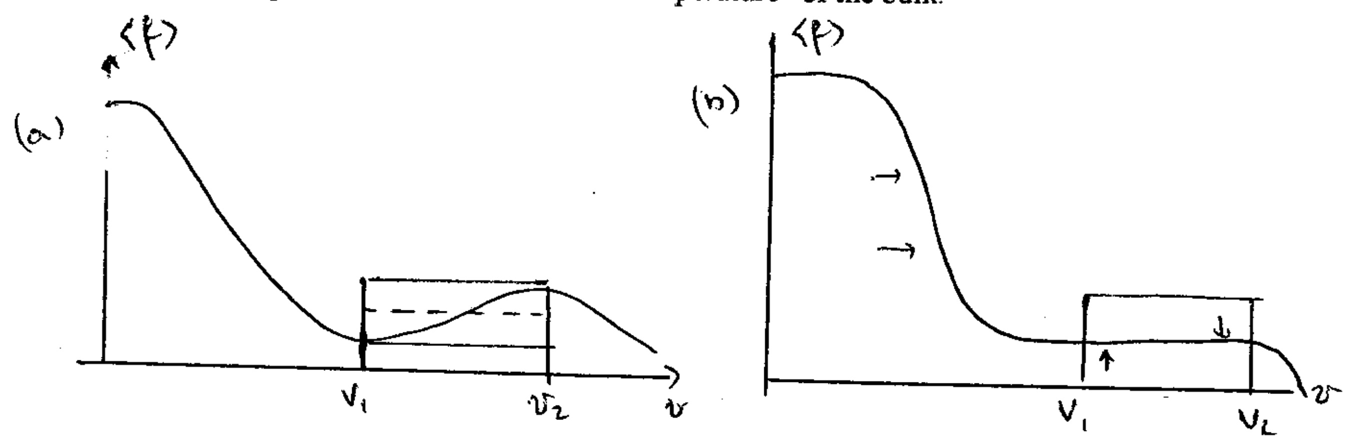

Fig.5.2: (a) the initial distribution. (b) The saturated distribution. Note that the bulk has been shifted outwards

To estimate the effect of the saturated instability on the bulk, we consider the non-resonant particles:

$$\partial_t \langle f\rangle = \partial_v \frac{q^2}{M^2} \sum_k |E_k|^2 \frac{\gamma_k}{(\omega - kv)^2} \frac{\partial}{\partial v}\langle f\rangle \qquad 5.16$$

using our expression for $\mathcal{D}^{non\text{-}res}$ we found in *§4*. In terms of the field energy then, *(5.16)* becomes

$$\partial_t \langle f \rangle \cong \frac{8\pi}{M^2} q^2 \int dk\, \mathcal{E}_k^{field} \frac{\gamma_k}{\omega_p^2} \frac{\partial^2 \langle f \rangle}{\partial v^2} \tag{5.17}$$

Replacing a factor of $2\gamma_k$ with a time derivative and using $\omega_p^2 = 4\pi n q^2/m$ we have

$$\partial_t \langle f \rangle = \frac{1}{nm} \partial_t \int dk\, \mathcal{E}_k^{field} \frac{\partial^2 \langle f \rangle}{\partial v^2} \tag{5.18}$$

Now, let us define an "effective temperature" $T_{eff}(t)$ by

$$T_{eff}(t) = \frac{2}{n} \int dk\, \mathcal{E}_k^{field}(t) \tag{5.19}$$

We may use this new parameter to rewrite *(5.18)* in the form

$$\frac{\partial \langle f \rangle}{\partial T_{eff}} = \frac{1}{2m} \frac{\partial^2 \langle f \rangle}{\partial v^2} \tag{5.20}$$

which can be shown to have the solution

$$\langle f \rangle = \left\{ \frac{m}{2\pi [T_0 + T_{eff}(t) - T_{eff}(0)]} \right\}^{1/2} \exp\left[-\frac{mv^2}{2} \Big/ [T_0 + T_{eff}(t) - T_{eff}(0)] \right] \tag{5.21}$$

At saturation, then, the *non-resonant* particles are heated to a higher "effective temperature" due to a net increase in the field energy. It should be noted, however, that this heating is one-sided: only particles on same side of the distribution function as the bump are heated.

Appendix

Direct proof of $\partial_t (\mathcal{K}^{total} + \mathcal{E}^{field}) = 0$

From the quasi-linear equation *(1.14)* we have

$$\partial_t \mathcal{K}^{total} = -\sum_k \int dv \, \frac{\omega_p^2}{k} kv \frac{|E_k|^2}{4\pi} \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v} \qquad \text{A1}$$

where, as usual

$$\epsilon_{k\omega} = 1 + \frac{\omega_p^2}{k} \int \frac{dv}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v} \qquad \text{A2}$$

Thus,

$$\begin{aligned} \partial_t \mathcal{K}^{total} &= -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \, \frac{\omega_p^2}{k} (kv - \omega + \omega) \cdot \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v} \\ &= -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \, \frac{\omega_p^2}{k} \frac{\omega}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v} \end{aligned} \qquad \text{A3}$$

Finally, using $\epsilon_{k w} = 0$,

$$\begin{aligned} \partial_t \mathcal{K}^{total} &= i \sum_k \frac{|E_k|^2}{4\pi} \omega_k \\ &= -i \sum_k \frac{|E_k|^2}{8\pi} 2\gamma_k \\ &= -\partial_t \mathcal{E}^{field}. \end{aligned} \qquad \text{A4}$$

as required.