Gasdynamic Shocks

Read: Kulsrud Chapt 6
refs: Landau & Lifshitz, Fluids

Now, proceed from:

- Kinematic waves/shocks, with $V = V(\rho)$ specified $\Rightarrow$ single equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\underbrace{\left(\rho V(\rho)\right)}_{Q \to \text{flux}} - \nu \partial_x^2 \rho = 0$$

or

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} - \nu \partial_x^2 \rho = 0$$

$$c(\rho) = dQ/d\rho = V(\rho) + \rho \underbrace{V'(\rho)}_{>0 \text{ or } <0}$$

to:

- gas dynamic shocks.

Simple Waves etc.

① Consider 1D gasdynamics:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V) = 0$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

gas adiabatic $\Rightarrow$ s homogeneous ab-initio
$\therefore$ s = const for all times, till shock forms

Now, ~~in~~ : $V = V_x$

$V_y = V_z = 0$

so

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho V = \frac{\partial \rho}{\partial t} + \frac{d(\rho V)}{d\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{\partial V}{\partial t} + \left(V + \frac{1}{\rho}\frac{dP}{dV}\right)\frac{\partial V}{\partial x} = 0$$

→ to write in form $\frac{d}{dt}\begin{pmatrix}\rho \\ V\end{pmatrix} = 0$ ↔ exploit characteristics

Now

$$\frac{\partial \rho/\partial t}{\partial \rho/\partial x} = -\left(\frac{\partial x}{\partial t}\right)_\rho$$

$$\Rightarrow \quad \left(\frac{\partial x}{\partial t}\right)_\rho = \frac{d}{d\rho}(\rho V) = V + \rho\frac{dV}{d\rho}$$

[from characteristic → ρ equation]

and similarly,

$$\left(\frac{\partial x}{\partial t}\right)_V = V + \frac{1}{\rho}\frac{dP}{dV}$$

Now, since $V = V(\rho)$ [i.e. ρ determines V]

we have:

$$\left(\frac{\partial x}{\partial t}\right)_\rho = \left(\frac{\partial x}{\partial t}\right)_V$$

$\Rightarrow$

$$\rho \frac{dV}{d\rho} + V = V + \frac{1}{\rho}\frac{dP}{dV}$$

$$\rho \frac{dV}{d\rho} = \frac{c_s^2}{\rho}\frac{d\rho}{dV} \qquad\qquad dP = c_s^2 \, d\rho$$

$$\therefore \quad (dV/d\rho)^2 = c_s^2/\rho^2$$

and $\quad dV/d\rho = \pm c_s/\rho$

$$\Rightarrow \quad \boxed{V = \pm \int \frac{c}{\rho}\, d\rho = \pm \int \frac{dP}{\rho c_s}}$$

→ relation between fluid element speed V and density or pressure

→ in accord with expectation, V increases with ρ i.e. for ideal gas

$$P\rho^{-\gamma} = \text{const} \qquad \gamma = 5/3$$

$$dP = c_s^2\, d\rho = \underset{\text{const.}}{\alpha}\gamma\, \rho^{\gamma-1}\, d\rho$$

$$c_s^2 = \alpha\gamma\,\rho^{\gamma-1} \qquad c_s = \sqrt{\alpha\gamma}\,\rho^{(\gamma-1)/2}$$

$$V = \pm\int \frac{c_s^2}{\rho c_s}\,d\rho = \pm\int c_s\,\frac{d\rho}{\rho}$$

$$= \pm\sqrt{\alpha\gamma}\int \frac{\rho^{1/3}}{\rho}\,d\rho$$

$$= \pm 3\sqrt{\alpha\gamma}\,\rho^{1/3}$$

so

$$\left(\frac{\partial x}{\partial t}\right)_V = v + \frac{1}{\rho}\frac{dP}{dV}$$

$$V = \pm\int \frac{dP}{\rho c_s} \Rightarrow dV = \frac{dP}{\rho c_s}$$

$$dP/dV = \rho c_s$$

$\Rightarrow$

$$\left(\frac{\partial x}{\partial t}\right)_V = v \pm c_s(V) \qquad \text{since } \rho = \rho(V)$$

$$\therefore \quad x = t\,\left[V \pm c_s(V)\right] + f(V)$$

thus, have "simple wave" solution:

$$\boxed{x = t\left[v \pm c_s(v)\right] + f(v)}$$

obviously, for linearized limit,

$$x = t\left[\cancel{v} \pm c_s\right] + \cancel{f(v)} + x_0$$

$$x = x_0 \pm c_s t \qquad \text{sign} \Rightarrow \text{direction}$$

Why "simple"? $\Rightarrow$ 1D Similarity Flow
$\rightarrow$ characteristic velocity
$\rightarrow$ no characteristic length

so ... assume all quantities depend only on $\xi = x/t$

$$\Rightarrow \quad \frac{\partial}{\partial x} = \frac{1}{t}\frac{d}{d\xi} \qquad \text{i.e. } \begin{pmatrix}\rho \\ v\end{pmatrix} = F(\xi)$$

$$\xi = (x/t)$$

$$\frac{\partial}{\partial t} = -\frac{\xi}{t}\frac{d}{d\xi}$$

so

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial v}{\partial x} + v\frac{\partial \rho}{\partial x} = 0$$

n.b. $\frac{x}{t}$ is "velocity" formed by x, t in absence scales.

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = \frac{-1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow \quad -\frac{\xi}{t}\rho' + \frac{\rho}{t}V' + \frac{V}{t}\rho' = 0 \qquad ' = \frac{d}{d\xi}$$

$$-\frac{\xi}{t}V' + \frac{V}{t}V' = -\frac{c_s^2}{t}\frac{\rho'}{\rho} \qquad \text{const entropy}$$

$$\Rightarrow \quad (V-\xi)\rho' + \rho V' = 0$$

$$(V-\xi)V' = -c_s^2\frac{\rho'}{\rho}$$

$\therefore$ as $\omega(k)$, treat ξ as an <u>eigenvalue</u>, tbd:

$$(V-\xi)\rho' + \rho V' = 0$$

$$-\frac{c_s^2}{\rho}\rho' + (V-\xi)V' = 0$$

$$(V-\xi)^2 = c_s^2$$

$$\Rightarrow$$

$$\xi = V \pm c_s = \frac{x}{t}$$

$\Rightarrow \quad x = (v \pm c_s)\, t$ ✓

From "eigenvector", relate v, ρ etc.

i.e. $\quad v - \xi = -c_s$

and $\quad (v-\xi)\rho' + \rho v' = 0$

$-c_s \rho' + \rho v'$

$\Rightarrow \quad \rho\, dv = c_s\, d\rho$

$\therefore \quad \boxed{dv/d\rho = c_s/\rho}$

$\Rightarrow \quad \boxed{v = \int c_s\, d\rho/\rho = \int dP/c_s\rho}$

- equivalent to previous result with $f(v) = 0$ (Rederived version of "simple wave")
- can also write as

$\boxed{v = \int \sqrt{-dp\, dV}}$

$d(1/\rho) = dV = -\frac{1}{\rho^2}\, d\rho$

$dp = c_s^2\, d\rho$

$v = \int \left(c_s^2\, d\rho \frac{d\rho}{\rho^2} \right)^{1/2}$

$= \int \frac{d\rho}{\rho} c_s$ ✓

→ Physics of Simple Wave (in Gasdynamics)

$$X = t\left[V \pm C_s(V)\right] + f(V)$$

- similarity flow is simple wave with $f(V) = 0$

- can write general solution for simple wave, for adiabatic process

i.e. $P\rho^{-\gamma} = \text{const}$

$$T\rho^{-(\gamma-1)} = \text{const}$$

$$\Rightarrow \rho T^{1/\gamma-1} = \text{const.}$$

but $C_s^2 \sim T$ $\quad (C_s \sim \sqrt{T}) \Rightarrow$

$$\boxed{\rho = \rho_0 \left(c/c_0\right)^{2/\gamma-1}}$$

but $\quad V = \pm\int c \frac{d\rho}{\rho}$

$$d\rho = \rho_0 \frac{2}{\gamma-1} \frac{dc}{c_0} \left(\frac{c}{c_0}\right)^{\left(\frac{2}{\gamma-1}-1\right)}$$

$\Rightarrow$

$$= \rho_0 \frac{2}{\gamma-1} \frac{dc}{c_0} \left(\frac{c}{c_0}\right)^{\left(\frac{2}{\gamma-1}-1\right)}$$

$$V = \pm \int \frac{c \, \frac{2}{\gamma-1} \, \rho_0 \, \frac{dc}{c_0} \left(\frac{c}{c_0}\right)^{\left(2/\gamma-1 \, -1\right)}}{\rho_0 \, (c/c_0)^{2/\gamma-1}}$$

$$= \pm \frac{2}{\gamma-1} \int dc \quad = \pm \frac{2}{\gamma-1} (c - c_0)$$

$$\Rightarrow \quad V = \pm \frac{2}{\gamma-1} (c - c_0)$$

$$\therefore \quad \boxed{\begin{aligned} c &= c_0 \pm \tfrac{1}{2}(\gamma-1)\, V \\ \rho &= \rho_0 \left(1 \pm \tfrac{1}{2}(\gamma-1)\, V/c_0\right)^{2/\gamma-1} \\ P &= P_0 \left(1 \pm \tfrac{1}{2}(\gamma-1)\, V/c_0\right)^{2\gamma/\gamma-1} \end{aligned}}$$

then can reduce simple wave expression:

$$x = t\left[V \pm c_0(V)\right] + f(V)$$

$$\Rightarrow \quad x = t\left[V \pm \left(c_0 \pm \tfrac{1}{2}(\gamma-1)V\right)\right] + f(V)$$

+ sign for compression (steepening):

$$\boxed{x = t\left[\pm c_0 + \tfrac{1}{2}(\gamma+1)v\right] + f(v)}$$

→ point on wave profile moves at speed
$u = v \pm c_s$

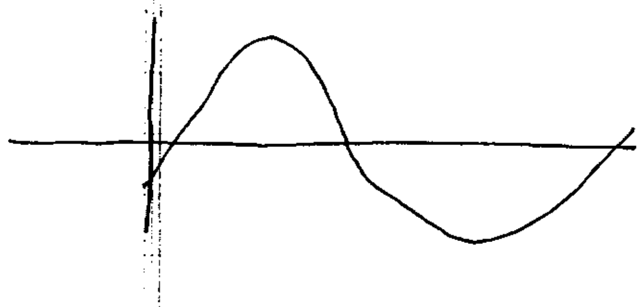
→ have shown $du/d\rho > 0$
i.e. speed increases with density

so

→ <u>if</u> wave in $+x$, then $d\rho/dx < 0$ anywhere in I.C. ⇒

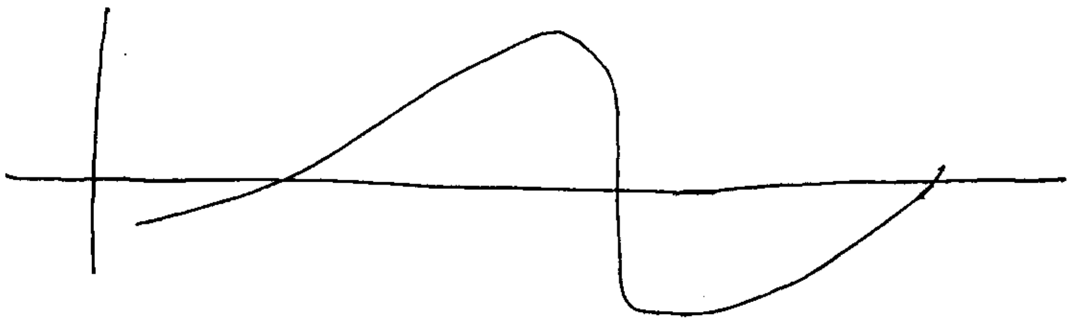
⇒ discontinuity will form ⇒ shock!

<u>i.e.</u> via over-taking and breaking mechanism



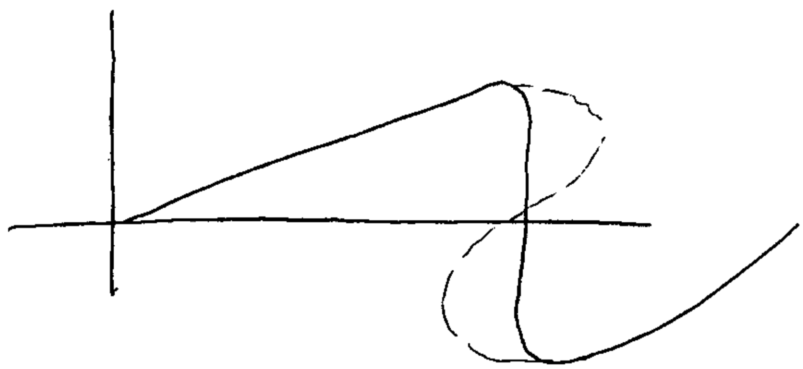
<u>wave</u>

→



steepening

→



shock

(dissipation resolves singularity)

→ when will breaking occur?

⇒ multi-valued position, i.e.

$$\left(\partial x/\partial V\right)_t = 0, \quad \left(\partial^2 x/\partial V^2\right)_t = 0$$

inflection

$$x = t\left[\pm c_0 + \frac{1}{2}(\gamma+1)V\right] + f(V)$$

$$\left.\frac{\partial x}{\partial V}\right|_t = \frac{1}{2}(\gamma+1)t + f'(V) = 0$$

$$t_{\substack{break \\ shock}} = -2f'(V)/(\gamma+1)$$

(need $f(V)' < 0$ i.c.)

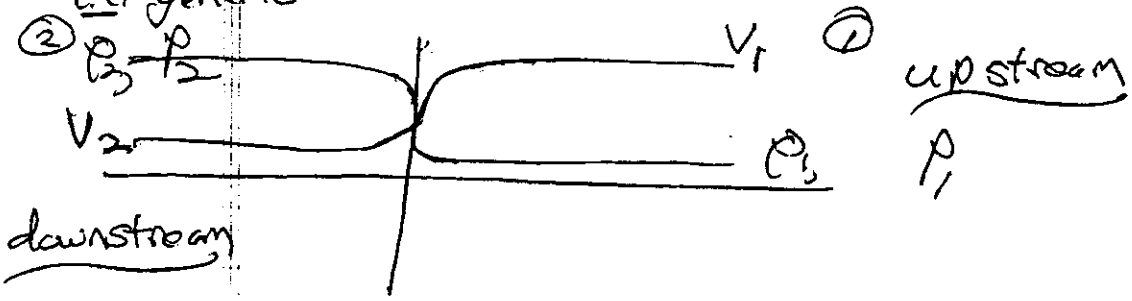
and $f''(v) = 0$

(2) Shocks - Flows with Discontinuity

- once waves steepen and break/shock
  $\Rightarrow$ have flow with $\underbrace{\text{discontinuities}}$
  distinguishing feature of shocks.

- "shock" - { localized region of rapid change / discontinuity

i.e. generic



$x = 0$

$\hookrightarrow$ location of shock in its rest frame

layer thickness $\approx l_{mfp}$

Now, have conservation equations for ideal fluid:

i.e. recall
$$\underbrace{u}_{\text{shock speed}} = \frac{\overbrace{Q(\rho_1) - Q(\rho_2)}^{\text{flux discon.}}}{(\rho_1 - \rho_2)}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0 \quad \rightarrow \text{continuity}$$

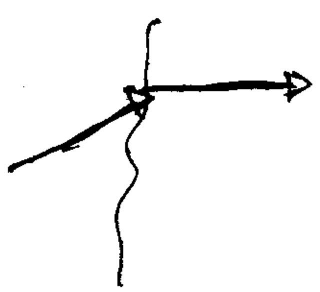
$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2 + \rho \epsilon\right) + \nabla \cdot \left(\rho \underline{V}\left(\tfrac{1}{2}V^2 + \underbrace{\frac{\gamma P}{(\gamma-1)\rho}}_{\text{specific enthalpy}}\right)\right) = 0$$

$\rightarrow$ energy

$$\frac{\gamma P}{\gamma - 1} = \underbrace{\frac{P}{\gamma-1}}_{\substack{\text{energy}\\\text{density}}} + \underbrace{P}_{\substack{\overrightarrow{PV}\text{ work}\\\text{on surroundings}}}$$

$$\frac{\partial}{\partial t}\rho V_i = -\frac{\partial}{\partial x_k}\Pi_{ik}$$

$$\Pi_{ik} = P\delta_{ik} + \rho V_i V_k$$



x=0

General

so $V^{(2)}_{y,z} = V^{(1)}_{y,z}$

tangential components continuous

so

→ Integrating conservation equations

→ work in shock frame ⇒ $U = 0$

continuity ⇒ $$\rho V_x \Big|_{②} = \rho V_x \Big|_{①}$$

energy conservation ⇒

$$\rho V_n \left(\frac{1}{2} V^2 + \frac{\gamma P}{\gamma - 1} \right)_{②} = \rho V_n \left(\frac{V^2}{2} + \frac{\gamma P}{\gamma - 1} \right)_{①}$$

but $\rho V_n \Big|_{②} = \rho V_n \Big|_{①}$

$$V_{y,z\,②}^2 = V_{y,z\,①}^2$$

⇒

$$\left(\frac{V_x^2}{2} + \frac{\gamma P}{\gamma - 1} \right)_{②} = \left(\frac{V_x^2}{2} + \frac{\gamma P}{\gamma - 1} \right)_{①}$$

and momentum conservation ⇒

$$\left(p + \rho V_x^2\right)\Big|_{②} = \left(p + \rho V_x^2\right)\Big|_{①}$$

(again, normal component varies, only).

$\Rightarrow$ have 3 Rankine-Hugoniot jump/continuity conditions:

$$[\;\;] = (\;)_{②} - (\;)_{①}$$

$$\boxed{\begin{aligned} &[\rho V_x] = 0 \\ &\left[\frac{V_x^2}{2} + \frac{\gamma p}{\gamma - 1}\right] = 0 \\ &\left[\rho V_x^2 + p\right] = 0 \end{aligned}}$$

in shock Frame

if in fixed coordinate,

$V_x \rightarrow V_n - U$

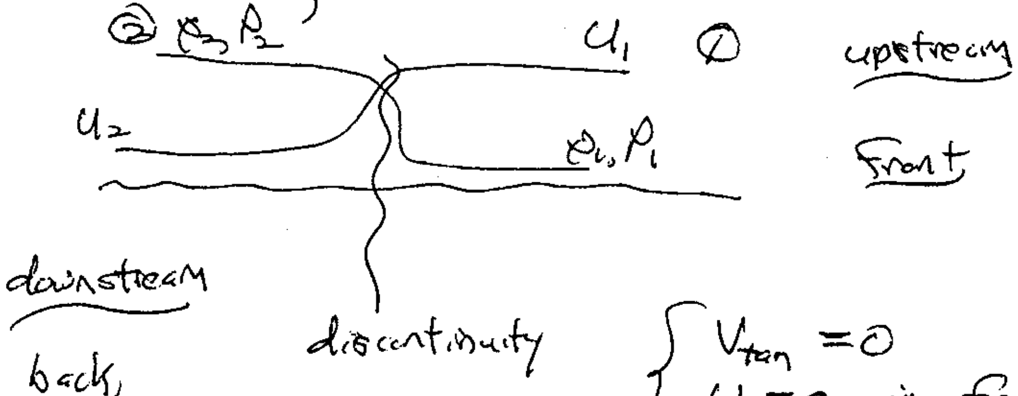
V_n: normal V in fixed coords

U ↳ shock velocity

⑤ Shock Structure

→ can derive very general relation between thermodynamic quantities on 2 sides of shock discontinuity

→ as before, consider case:



$\begin{cases} V_{tan} = 0 \\ u = 0, \text{ in frame} \end{cases}$

then Rankine – Hugoniot conditions ⇒

$$\rho_1 V_1 = \rho_2 V_2 = j$$

$j \equiv$ mass flux density at surface of discontinuity

$$P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2$$

$$w_1 + \frac{V_1^2}{2} = w_2 + \frac{V_2^2}{2}$$

where $w = \dfrac{\gamma P}{(\gamma-1)\rho}$ ≡ enthalpy density

Now, $\bar{V}_1 = 1/\rho_1$, $\bar{V}_2 = 1/\rho_2$ } specific volumes

$$\Rightarrow \quad \boxed{\begin{aligned} v_1 &= j\,\bar{V}_1 \\ v_2 &= j\,\bar{V}_2 \end{aligned}}$$

→ relates flow speeds to flux and volumes

so

$$p_1 + j^2\bar{V}_1^{\,2} = p_2 + j^2\bar{V}_2^{\,2}$$

and $$\boxed{j^2 = (p_2 - p_1)/(\bar{V}_1 - \bar{V}_2)}$$

→ relates flux and speed to pressure difference and volumes on both sides

→ as $j^2 > 0$ must either have:

$$\begin{pmatrix} p_2 > p_1 \\ V_1 > V_2 \end{pmatrix} \quad \underline{\underline{\text{or}}} \quad \begin{pmatrix} p_2 < p_1 \\ V_1 < V_2 \end{pmatrix}$$

$(\rho_2 > \rho_1)$ $\qquad$ $(\rho_2 < \rho_1)$

will see that only $p_2 > p_1$, $\bar{V}_1 > \bar{V}_2$ is physical.

Why? → why only $P_2 > P_1$, $\rho_2 > \rho_1$ physical?

Answer: → $\boxed{\text{Shock must increase entropy}}$

i.e. $S_2 > S_1$

→ as
- entropy can only increase during gases motion
- microscopic diffusion (ν, κ, etc.) in shock effect entropy increase on scale of shock thickness
- amount of entropy increase set by macro jump conditions

→ and, $P_2 > P_1$
$V_1 > V_2$ $(\rho_2 > \rho_1)$

clearly correspond to $\left\{ \begin{array}{l} \text{compression} \\ \text{heating} \end{array} \right.$

⇒ entropy increase.

(n.b. for rigorous proof see: Landau Section 87)

∴ $S_2 > S_1$ ⇒ $P_2 > P_1$, $\rho_2 > \rho_1$ is only physical solution

n.b. need: $V_1/C_{s_1} > 1$, $V_2/C_{s_2} < 1$, too

Now, can go further, i.e.

$$V_1 - V_2 = j\,(\mathcal{V}_1 - \mathcal{V}_2)$$

$$j^2 = (P_2 - P_1) / \mathcal{V}_1 - \mathcal{V}_2$$

$$\Rightarrow \boxed{\underbrace{V_1 - V_2}_{\substack{\Downarrow \\ \text{velocity} \\ \text{difference}}} = \sqrt{(P_2 - P_1)(\mathcal{V}_1 - \mathcal{V}_2)}}$$

$P_2 > P_1 \Rightarrow$
$V_2 < V_1$
$\Leftrightarrow$ + sqr. root

and similarly,

$$w_1 + \frac{V_1^2}{2} = w_2 + \frac{V_2^2}{2}$$

$$\Rightarrow \quad w_1 + \frac{j^2}{2}\mathcal{V}_1^2 = w_2 + \frac{j^2}{2}\mathcal{V}_2^2$$

and since $j^2 = (P_2 - P_1)/(\mathcal{V}_1 - \mathcal{V}_2)$

$$\Rightarrow \boxed{w_1 - w_2 + \frac{1}{2}(\mathcal{V}_1 + \mathcal{V}_2)(P_2 - P_1) = 0}$$

Now, let $w = \underset{\text{internal energy}}{\epsilon} + pV$

$$\frac{p/\rho}{(\gamma-1)} + \frac{p}{\rho} = \frac{\gamma}{\gamma-1}\, p/\rho$$

$\Rightarrow$

$$\boxed{\epsilon_1 - \epsilon_2 + \frac{1}{2}(V_1 - V_2)(p_1 + p_2) = 0}$$

What have we gained from this?

given inflow state thermo variables $\quad p_1, V_1$

then

$$w_1 - w_2 + \frac{1}{2}(V_1 + V_2)(p_2 - p_1) = 0$$

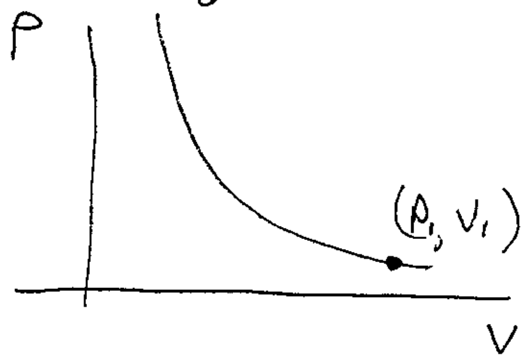
and / or

$$\epsilon_1 - \epsilon_2 + \frac{1}{2}(V_1 - V_2)(p_2 - p_1) = 0$$

$\Rightarrow$ outflow/downstream state relation between p_2, V_2.

curve of possible P_2, V_2 points

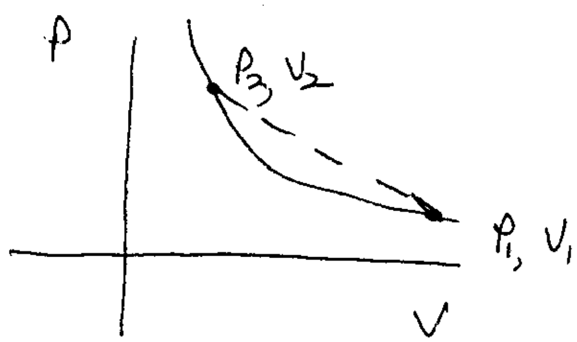
$\Rightarrow$



$\rightsquigarrow$ $\left\{ \begin{array}{l} \text{shock adiabatic} \\ \text{Hugoniot adiabatic} \end{array} \right.$

- Hugoniot adiabatic is curve on which downstream thermodynamic states (P_2, V_2 fall), for given P_1, V_1

- can graphically determine $\left\{ \begin{array}{l} \text{flux } j \\ \text{velocity} \end{array} \right.$



$$\frac{P_2 - P_1}{V_2 - V_1} = \text{slope} = -j^2$$

$\therefore$ $\left. \begin{array}{l} \text{flux } j \\ \text{velocity } v \end{array} \right\}$ determined at each point of shock adiabatic

[Now, can <u>use</u> shock adiabatic relations/equations to <u>characterize</u> <u>discontinuity</u>.

→ Characterizing the Discontinuity ⟺ $\left\{ \begin{array}{l} \underline{\text{Jump Ratios}} \\ \text{for} \\ \text{Polytropic Gas} \\ \left[P\rho^{-\gamma} = \text{const} \right] \end{array} \right.$

Now, have shown:

$$w_1 - w_2 + \frac{1}{2}(V_1 + V_2)(P_2 - P_1) = 0$$

$$w = \frac{\gamma P \bar{V}}{\gamma - 1} = \frac{\gamma P/\rho}{\gamma - 1} = \frac{c_s^2}{\gamma - 1}$$

$\Rightarrow$

$$\frac{\gamma}{\gamma-1}\left[P_1 V_1 - P_2 V_2\right] + \frac{1}{2}(\bar{V}_1 + \bar{V}_2)(P_2 - P_1) = 0$$

$$\frac{\gamma}{\gamma-1}\left[P_1 - P_2 \frac{\bar{V}_2}{\bar{V}_1}\right] + \frac{1}{2}\left(1 + \frac{\bar{V}_2}{\bar{V}_1}\right)(P_2 - P_1) = 0$$

$$\frac{\gamma}{\gamma-1} P_1 + \frac{1}{2}(P_2 - P_1) = \frac{\gamma}{\gamma-1} P_2 \frac{\bar{V}_2}{\bar{V}_1} - \frac{\bar{V}_2}{\bar{V}_1}\frac{(P_2 - P_1)}{2}$$

$$\frac{\gamma}{\gamma-1} P_1 + \frac{1}{2}(P_2 - P_1) = \frac{\bar{V}_2}{\bar{V}_1}\left[\frac{\gamma}{\gamma-1} P_2 - \frac{P_2 + P_1}{2}\right]$$

$\Rightarrow$

$$\frac{\bar{V}_2}{\bar{V}_1} = \frac{2\gamma P_1 + (\gamma-1)(P_2 - P_1)}{2\gamma P_2 - (\gamma-1)(P_2 - P_1)}$$

so finally,

$$\boxed{\frac{V_2}{V_1} = \frac{(\gamma+1)\,P_1 + (\gamma-1)P_2}{(\gamma-1)\,P_1 + (\gamma+1)\,P_2}}$$

Volume/density ratio

⇒ $\boxed{\text{compression ratio}}$

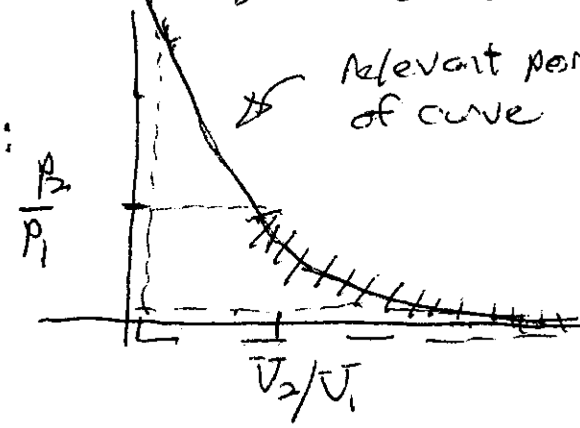
$$\Rightarrow \frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = \frac{\gamma+1 + (\gamma-1)(P_2/P_1)}{\gamma-1 + (\gamma+1)(P_2/P_1)}$$

[note: $P_2 \gg P_1$

$\Rightarrow \frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}$

$= 4$]

Graphically:



hyperbola

$P_2 > P_1$

$\rho_2 > \rho_1$

x-asymptote: $P_2/P_1 \to \infty \Rightarrow V_2/V_1 = (\gamma-1)/(\gamma+1)$

y-asymptote: $V_2/V_1 \to \infty \Rightarrow P_2 = -(\gamma-1)/(\gamma+1)$

then can also extract temperature, velocity, flux etc.

i.e. $P = \rho T \Rightarrow T = PV$

$$\boxed{\frac{T_2}{T_1} = \frac{P_2}{P_1}\left(\frac{V_2}{V_1}\right) = \frac{P_2}{P_1}\left[\frac{(\gamma+1)\,P_1 + (\gamma-1)\,P_2}{(\gamma-1)\,P_1 + (\gamma+1)\,P_2}\right]}$$

⇝ Temperature ratio: note no limit for $P_2 \gg P_1$

For flux j $\qquad j^2 = (p_2 - p_1)/(\mathcal{V}_1 - \mathcal{V}_2)$

(Hugoniot)

and using $\dfrac{\mathcal{V}_2}{\mathcal{V}_1} = \dfrac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2}$

$$\Rightarrow \boxed{j^2 = \left[(\gamma-1)p_1 + (\gamma+1)p_2\right]/2\mathcal{V}_1}$$

and,

$$v_1^2 = j^2 \mathcal{V}_1^2 \quad , \quad v_2^2 = j^2 \mathcal{V}_2^2$$

so using expressions for:
- j^2
- $\mathcal{V}_2/\mathcal{V}_1$

$\Rightarrow$

$$\boxed{\begin{aligned} v_1^2 &= \tfrac{1}{2}\mathcal{V}_1\left\{(\gamma-1)p_1 + (\gamma+1)p_2\right\} \\ &= \tfrac{1}{2}(c_{s1}^2/\gamma)\left[\gamma - 1 + (\gamma+1)p_2/p_1\right] \end{aligned}}$$

$$\boxed{\begin{aligned} v_2^2 &= \tfrac{1}{2}\mathcal{V}_1\left\{(\gamma+1)p_1 + (\gamma-1)p_2\right\}^2/\left\{(\gamma-1)p_1 + (\gamma+1)p_2\right\} \\ &= \tfrac{1}{2}\frac{c_2^2}{\gamma}\left[\gamma - 1 + (\gamma+1)p_1/p_2\right] \end{aligned}}$$

Now, often convenient to describe density ratio, etc. in terms of Mach number of upstream flow

$$M_1 = V_1 / Cs_1$$

now $\quad V_1^2/Cs_1^2 = M_1^2 = \frac{1}{2\gamma}\left[(\gamma-1)+(\gamma+1)P_2/P_1\right]$

so can show

$$\boxed{\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{V_1}{V_2} = (\gamma+1)\, M_1^2 \Big/ \left[(\gamma-1)M_1^2+2\right] \\ \frac{P_2}{P_1} &= \left(2\gamma M_1^2 \big/ \gamma+1\right) - \frac{(\gamma-1)}{(\gamma+1)} \\ \frac{T_2}{T_1} &= \left\{2\gamma M_1^2-(\gamma-1)\right\}\left\{(\gamma-1)M_1^2+2\right\} \Big/ (\gamma+1)^2 M_1^2 \end{aligned}}$$

and sometimes useful to use:

$$M_2^2 = \left\{2+(\gamma-1)M_1^2\right\} \Big/ \left\{2\gamma M_1^2-(\gamma-1)\right\}$$

Interesting to note case of strong shock waves
$\Rightarrow$

$$M_1 \gg 1$$

$\therefore$

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} \rightarrow \sim 4, \text{ maximum}$$

$$\frac{P_2}{P_1} = 2\gamma M_1^2/\gamma+1 \rightarrow \sim M_1^2$$

$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_1^2 \rightarrow \sim M_1^2$$

also check: $M_1 = 1$

$\rho_2/\rho_1 = 1$ ✓, $P_2/P_1 = 1$ ✓, $T_2/T_1 = 1$ ✓

So

1) $\rightarrow$ simple waves $\Rightarrow$ shock formation

2) $\rightarrow$ jump conditions, shock adiabatic $\Rightarrow$ shock {physics, structure}

3) $\rightarrow$ shock jump conditions $\Rightarrow$ shock properties.

To address shock thickness:

→ better/easier to return to kinematic waves

→ recall, $\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial X} = \nu \frac{\partial^2 \rho}{\partial X^2}$

$$\rho = \rho(X) \qquad X = x - Ut$$

$$\Rightarrow \quad \left(-U + c(\rho)\right)\frac{\partial \rho}{\partial X} = \nu \frac{\partial^2 \rho}{\partial X^2}$$

$$\Rightarrow \quad Q(\rho) - U\rho + A = \nu \frac{\partial \rho}{\partial X}$$

so $$dX = \int \nu\, d\rho \Big/ \left(Q(\rho) - U\rho + A\right)$$

$$\therefore \quad \frac{dX}{\nu} = \int d\rho \Big/ \left[Q(\rho) - U\rho + A\right]$$

ν just is scaling parameter of the layer!

Now, for simplicity, consider:

$$Q(\rho) = \alpha\rho^2 + \beta\rho + \gamma$$

$$\alpha > 0$$

Now, write: $\quad Q(\rho) - U\rho + A = -\alpha(\rho-\rho_1)(\rho_2-\rho)$

so $\quad U = \beta + \alpha(\rho_1+\rho_2)$
$\qquad A = \alpha\rho_1\rho_2$

$\Rightarrow$ have: $\quad \dfrac{X}{\nu} = -\displaystyle\int \frac{d\rho}{\alpha(\rho-\rho_1)(\rho_2-\rho)} = \frac{1}{\alpha(\rho_2-\rho_1)} \ln \frac{\rho_2-\rho}{\rho-\rho_1}$

clearly, $\boxed{\underbrace{\Delta X}_{\text{thickness}} \simeq \nu/\alpha(\rho_2-\rho_1)}$

in gas-dynamic case expect

$$\Delta X \sim \mu/\Delta V$$

and as $\quad \Delta V \gtrsim V_{Th} \quad \Rightarrow \quad \Delta X \sim l_{mfp}!$

Lesson: In shocks, viscosity, thermal diffusivity, etc. simply scale discontinuity thickness $\sim l_{mfp}$.

Size + Location of discontinuity set by macroscopics.

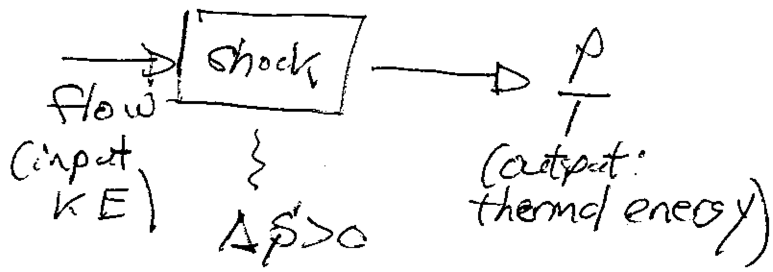
MHD Shocks

a.) Review of Gasdynamic shocks

Read:
- B+S, Chapt. 5
- L+L, ECM Sect. 69-73

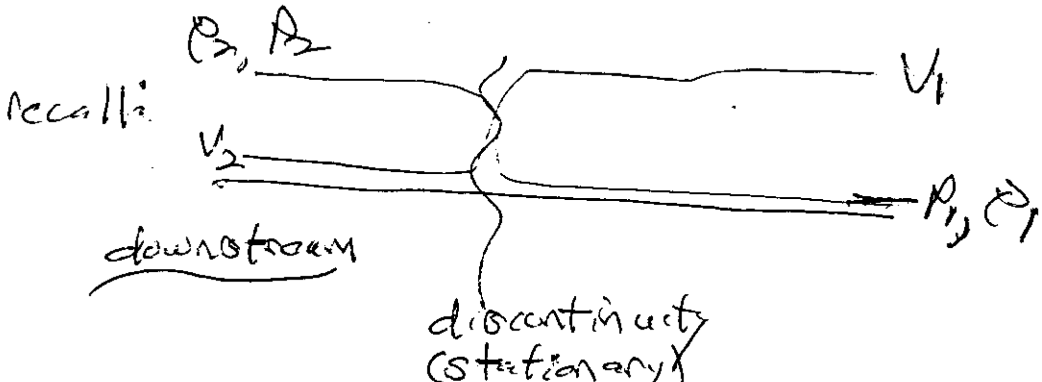
shock?
- discontinuity in flow (not unique sort)
- divides $V > c_s$, $V < c_s$ regions
- produced by input, i.c.'s   i.e. {explosion, wave steepening

(*) - heaters/converters



{old house heated by stream K.E.

recall:



typical shock parameters:

i.e. $r = r(M_1)$, $R = R(M_1)$ > "the answer"

$r = \rho_2/\rho_1 \rightarrow$ compression ratio

$R = P_2/P_1 \rightarrow$ strength

etc.

- exact NL solutions! → tractability from piecewise continuity!

rules of operation :

$$\text{R-H conditions}\quad \begin{cases} s_2 > s_1 \\ [\rho V_n] = 0 \\ \left[w + \frac{V_n^2}{2}\right] = 0 \\ \left[p + \frac{\rho V_n^2}{2}\right] = 0 \end{cases} \qquad w = \gamma p / \rho(\gamma-1)$$

Performance:

$M = V_1/C_{s_1} \gg 1$

(upstream Mach)

$$\begin{cases} r = r(M) \rightarrow \frac{\gamma+1}{\gamma-1} \\ R = R(M) \rightarrow 2\gamma M^2/(\gamma+1) \\ T_2/T_1 \rightarrow \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M^2 \end{cases}$$

indep M (constr. by flux, mom-flux balance)

n.b. for large M #:

$$w_1 + \frac{V_1^2}{2} = w_2 + \frac{V_2^2}{2} \qquad \text{(energy flux balance)}$$

$$\Rightarrow \underset{\text{(all KE)}}{\frac{V_1^2}{2}} \sim \underset{\text{(all ThE)}}{w_2} \qquad \Rightarrow M^2 C_{s_1}^2 \sim C_{s_2}^2$$

$$T_2/T_1 \sim M^2$$

aside: Is shock only form of discontinuity ?

need Π continuous accross discontinuty

$$\Pi_i = p n_i + \rho V_i V_k n_k$$

i.e. $\underline{\underline{\Pi}} = p\,\underline{\underline{I}} + \rho\,\underline{V}\,\underline{V}$

$\underline{\underline{\Pi}} \cdot \underbrace{\hat{n}}_{\text{unit normal to shock surface}} \equiv$ Momentum flux thru surface

$\Pi_x = p + \rho V_x^2$ $\qquad \Pi_y = \rho V_x V_y$

so $[\rho V_x V_y] = 0 \Rightarrow$ either:

$[\rho V_x] \neq 0 \Rightarrow [V_y] = 0$

(tangential flow continuous at shock)

or $[\rho V_x] = 0 \Rightarrow [V_y] \neq 0$

tangential discontinuity

T

i.e. $\Rightarrow$ KH, eddy, vortex etc.

n.b. shock $\Leftrightarrow$ acoustic wave

t-discon $\Leftrightarrow$ vortex mode

MHD – $\underline{B}$ enters!

(n.b. Bomb $\rightarrow$ γ's ionize ahead of blast $\rightarrow$ shock in plasma)

G-D shock $\Rightarrow$ "evolved sound wave"

$$\underset{(KE)}{\text{flow}} \longrightarrow \boxed{S} \begin{array}{l} \longrightarrow p, T \; (Th\,E) \\ \longrightarrow \underline{B} \; (Magn\,E) \end{array}$$

(new channel!)

$\Rightarrow$ so MHD shock...

· MHD shock ⇒ evolved $\begin{cases} \text{fast} \\ \text{intermediate} - ? \\ \text{slow} \end{cases}$

as before, will elucidate via extremes :

fast → $\underline{V}_1 \perp \underline{B}$ → perpendicular shock
(from magnetosonic wave)

slow → $\underline{V}_1 \parallel \underline{B}$ → parallel shock
(from // acoustic wave)

then consider → oblique shock

and intermediate wave — $\begin{cases} \text{transverse} \\ \text{EM} \end{cases}$
→ rotational discontinuity

First : Jump Conditions for MHD shock ($\hat{n} = \hat{x}$)

(i) $\underline{\nabla} \cdot \underline{B} = 0 \Rightarrow [B_x] = 0$

(ii) $\underline{\nabla} \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t} \Rightarrow [E_y] = 0$ $\qquad \underline{E} = -\underline{v} \times \underline{B}/c$

$[E_z] = 0$

$\therefore$
$$[V_z B_x - V_x B_z] = 0$$
$$[v_x B_y - v_y B_x] = 0$$

$\left(\frac{d}{dx} E_{y,z} = 0\right)$

$\gamma P / (\gamma - 1)\rho$

(iii) $[\rho v_x] = 0$

(iv) $$\frac{\partial}{\partial t}\left(\underbrace{\epsilon}_{P/(\gamma-1)} + \frac{1}{2}\rho v^2 + \frac{B^2}{8\pi}\right) + \nabla\cdot\left(\rho \underline{v}\left[\frac{v^2}{2} + w\right] + \frac{c}{4\pi}(\underline{E}\times\underline{B})\right) = 0$$

(from energetics)

so

$$\left[\rho v_x\left[\frac{v^2}{2} + w\right] + \frac{c}{4\pi}(\underbrace{\underline{E}}_{-\frac{\underline{v}\times\underline{B}}{c}}\times\underline{B})_x\right] = 0$$

↳ Poynting flux

(v) $$\frac{\partial}{\partial t}\rho v_i = -\underline{\nabla}\cdot T_{ik}$$

↳ stress tensor (full)

Reyn. tension total pressure.

$$T_{ik} = \left(\rho \underline{v}\,\underline{v} - \frac{\underline{B}\,\underline{B}}{4\pi} + \underline{\underline{I}}\left(P + \frac{B^2}{8\pi}\right)\right)_{i,k}$$

$(\Pi_i = T_{i,k} n_k)$

so (x, x) component:

$$\left[\rho v_x^2 - \frac{B_x B_x}{4\pi} + P + \frac{B^2}{8\pi}\right] = 0$$

$\therefore$

$$\left[\rho v_x^2 + \underbrace{\frac{B_y^2 + B_z^2}{8\pi}}_{B_t^2/8\pi} + P\right] = 0$$

but now off-diagonal terms non-trivial, i.e.

$$\Pi_y = \rho V_x V_y - \frac{B_x B_y}{4\pi}$$

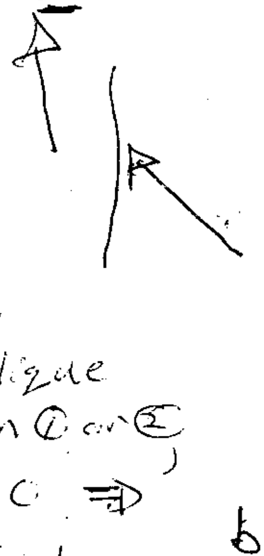
so $[\Pi_y] = 0$ $\qquad$ $\left[\rho V_x V_y - \frac{B_x B_y}{4\pi}\right] = 0$

$[\Pi_z] = 0$ $\qquad$ $\left[\rho V_x V_z - \frac{B_x B_z}{4\pi}\right] = 0$

Note: – no longer have $[\rho V_x V_y] = 0$

– point is that magnetic stresses deliver impulse to fluid element as it crosses the shock.....

i.e. now $[V_t] \neq 0$, due $\underline{J} \times \underline{B}$



$$\Delta(\rho V_y) \sim F_{J\times B}\, \tau_c$$

τ_c → time fluid element to cross shock

$$\sim B_x J_z \tau_c$$

$$\sim B_x J_z \frac{\delta}{V_x}$$

$\tau_{crossing} \sim \delta / V_x$

δ → thickness, V_x → flow

point: if oblique field in ① or ②, $B_{y_{1,2}} \neq 0 \Rightarrow$ tangential impulse imparted.

but $J_z \sim B_y / \delta$

$$\Delta(\rho V_y) \sim \frac{B_x B_y}{V_x} \Rightarrow \left[\rho V_x V_y - \frac{B_x B_y}{4\pi}\right] = 0$$

so, writing out full conditions:

$$\left\{\begin{array}{l}
[\rho V_n] = 0 \qquad \left(\hat{n} = \text{normal to plane of interface}\right) \\
\left[\rho \underline{V} V_n + \left(P + \frac{B^2}{8\pi}\right)\hat{n} - \frac{(\underline{B}\cdot\hat{n})}{4\pi}\underline{B}\right] = 0 \\
\qquad \hookrightarrow \text{magn. tension} \\
\left[\rho V_n\left(\frac{V^2}{2} + w\right) + \hat{n}\cdot\frac{c}{4\pi}(\underline{E}\times\underline{B})\right] = 0 \\
\underline{\text{or}} \\
\left[\rho V_n\left(\frac{V^2}{2} + w + \frac{B^2}{4\pi}\right) - \frac{(\underline{B}\cdot\hat{n})(\underline{V}\cdot\underline{B})}{4\pi}\right] = 0 \\
[B_n] = 0 \\
[\hat{n}\times\underline{E}] = [\hat{n}\times\underline{V}\times\underline{B}] = 0.
\end{array}\right.$$

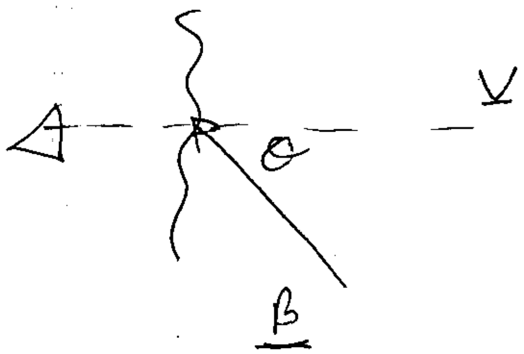
$\Rightarrow$ full set of MHD jump conditions.

Now consider:

a) parallel field, flow

b) perpendicular field, flow.

in general, have:



$\theta = 0^\circ \Rightarrow$ parallel shock

$\underline{V} \parallel \underline{B}$

$\theta = \pi/2 \Rightarrow \perp$ shock

ⓐ $\theta = 0^\circ$ $\quad \underline{B} = B_x \hat{x}$ (only) $\quad$ (both sides)

(parallel shock)

$$[B_x] = 0, \quad [\rho V_x] = 0$$

$$\left[\rho V_x^2 + P + \cancel{\frac{B_y^2 + B_z^2}{8\pi}}\right] = 0$$

$$[\hat{n} \times \underline{V} \times \underline{B}] = 0$$

$$\left[\rho V_x\left(\frac{V^2}{2} + w + \cancel{\frac{B^2}{4\pi}}\right) - \cancel{\frac{B_x V_x B_x}{4\pi}}\right] = 0$$

then can simplify, as $\underline{B}$ field drops out, i.e.

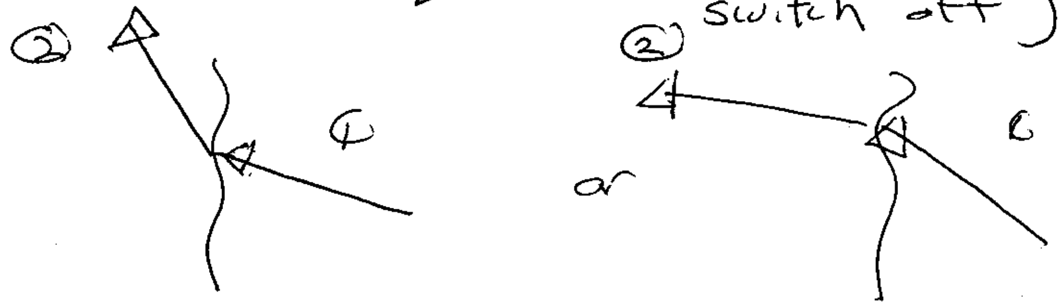
$$[\rho V_x] = 0$$

$$[\rho V_x^2 + P] = 0$$

$$\left[\frac{V_x^2}{2} + w\right] = 0$$

n.b.: B drops out as imposed $\underline{B} = B\hat{x}$

n.b.: If oblique $\Rightarrow$ switch on / switch off $\}$ phenomena

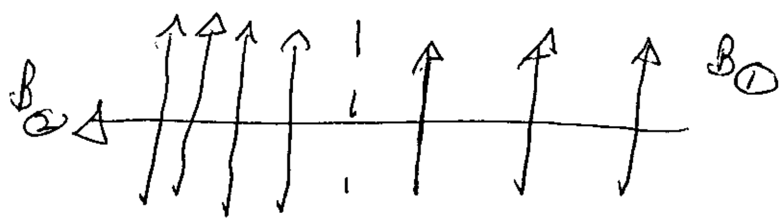


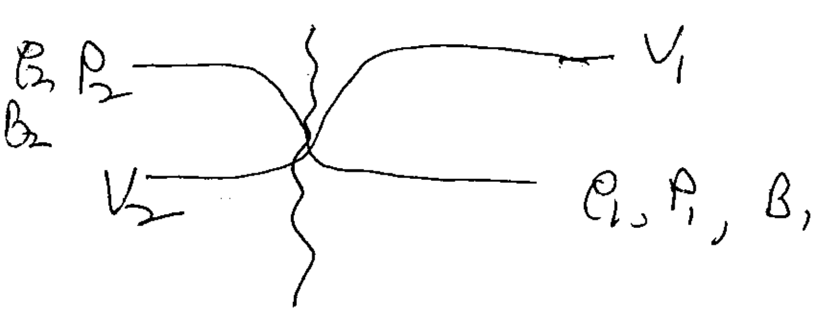
so

- and, as before, $RH \Rightarrow$ same results
- akin to parallel propagation limit of slow wave as un-affected by B_0, in wave theory.

  i.e. $\omega^2 = k_x^2 C_s^2$

b) Perpendicular Shock $\qquad \theta = \pi/2$





Some general comments:

- issue: what defines "shock" here?
(i.e. β large?)

$V_1 > C_s$, $\rightarrow$ $V > V_{MS}$, $\quad V_{MS} = (C_s^2 + V_A^2)^{1/2}$

- how much field compression possible?

(i.e. field amplification in SNR?)

point: $\rho_2/\rho_1 \leq \frac{\gamma+1}{\gamma-1}$!

so, since $B/\rho = \text{const.}$ $\Rightarrow$ $B_1/\rho_1 = B_2/\rho_2$

$\therefore$ $\frac{B_2}{B_1} < \frac{\gamma+1}{\gamma-1}$

- how much restriction on heating does $\underline{B}$ place?

$\Rightarrow$ not much, i.e. for $V/V_{MS} \equiv M_{eff} \gg 1$

should get $\sim$ same as before

Point: $\Delta(B^2)$ constrained by compression ratio
$\underline{\text{via}}$ freezing in.

Proceeding:

jump conditions $\Rightarrow$

$[\rho V_n] = 0 \qquad [\cancel{B_n}] = 0$

$$\left[\rho V_n^2 + P + \frac{B^2}{8\pi}\right] = 0 \qquad (\underline{B}\cdot\hat{n}) = 0$$

$$\left[\rho V_n\left(\frac{V^2}{2} + w + \frac{B^2}{4\pi}\right) - (\underline{B}\cdot\hat{n})(\underline{B}\cdot\underline{V})\right] = 0$$

$$[\hat{n} \times \underline{V} \times \underline{B}] = 0$$

$\Rightarrow$ simplifies to:

$$\left[\frac{V_x^2}{2} + w + \frac{B^2}{4\pi}\right] = 0 \qquad [\hat{n}\times\underline{V}\times\underline{B}] = 0$$

$$\left[\rho V_x^2 + P + \frac{B^2}{8\pi}\right] = 0$$

$$[\rho V_x] = 0$$

(two non-trivial conditions)

convenient to work with: $M = V_1/C_s$
inflow Mach #

$r = \rho_2/\rho_1 = B_2/B_1 \rightarrow$ compression ratio
$R = P_2/P_1 \longrightarrow$ strength parameter

$M \rightarrow$ control parameter
$r, R \rightarrow$ output

now, for stress balance jump condition:

$$\underset{①}{\rho_1 V_1^2} + \underset{②}{P_1} + \underset{③}{\frac{B_1^2}{8\pi}} = \underset{④}{\rho_2 V_2^2} + \underset{⑤}{P_2} + \underset{⑥}{\frac{B_2^2}{8\pi}}$$

① $\rho_1 V_1^2 = \rho_1 C_{s_1}^2 M^2$

② $P_1 = \frac{\rho_1}{\gamma} C_{s_1}^2$

③ $\frac{B_1^2}{8\pi} = \frac{\rho_1 C_{s_1}^2}{\gamma \beta}$ $\qquad \beta \equiv P_{Th}/P_M$

$\Rightarrow$

$$\rho_1 C_{s_1}^2 \left[M^2 + \frac{1}{\gamma} + \frac{1}{\gamma\beta} \right] = \rho V_2^2 + P_2 + \frac{B_2^2}{8\pi}$$

④ $= \frac{\rho_2 V_2^2}{\rho_1 C_{s_1}^2} = \frac{\rho_2^2 V_2^2}{\rho_2 \rho_1 C_{s_1}^2} = \frac{\rho_1^2 V_1^2}{\rho_2 \rho_1 C_{s_1}^2} = \frac{M^2}{r}$

$$⑤ = \frac{\rho_2 c_{s_2}^2}{\gamma \rho_1 c_{s_1}^2} = \frac{R}{\gamma}$$

$$⑥ = B_2^2 / 8\pi\rho_1 c_{s_1}^2 = \frac{B_1^2 (\rho_2/\rho_1)^2}{8\pi \rho_1 c_{s_1}^2}$$ freezing in!

$$= \frac{B_1^2 r^2}{8\pi\rho_1 c_{s_1}^2}$$

$$= r^2/\gamma\beta$$

so

$$M^2 + \frac{1}{\gamma} + \frac{1}{\gamma\beta} = \frac{M^2}{r} + \frac{R}{\gamma} + \frac{r^2}{\gamma\beta}$$

$$\Rightarrow \boxed{\gamma M^2 (1 - 1/r) = (R-1) + \frac{1}{\beta}(r^2-1)}$$

stress balance

and similarly, energy jump condition $\Rightarrow$

$$\boxed{\gamma M^2 \left(1 - \frac{1}{r^2}\right) = \frac{2\gamma}{\gamma-1}\left(\frac{R}{r} - 1\right) + \frac{4(r-1)}{\beta}}$$

Proceeding:

→ eliminate R, using stress balance

$$R = 1 + \gamma M^2 (1 - 1/r) - \frac{1}{\beta}(r^2 - 1)$$

→ plug into energy balance

→ exclude trivial root $r = 1$, $R = 1$ (no shock)

i.e. cancels factor $(r-1)$, $(R-1)$

∴ have for r:

$$\boxed{2(2-\gamma)r^2 + [2\gamma(1+\beta) + \beta\gamma(\gamma-1)M^2]r - \beta\gamma(\gamma+1)M^2 = 0}$$

now → 2 roots r_1, r_2

$$r_1 r_2 = -\beta\gamma(\gamma+1)M^2 / 2(2-\gamma)$$

(i.e. $(r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + r_1 r_2$)

<u>as</u> $\gamma < 2$ $\quad r_1 r_2 < 0$

⇒ 1 value: $r_1 < 0$
1 value: $r_2 > 0$

then condition $\beta_2 > 1 \Rightarrow$

$$\boxed{\gamma M^2 > \gamma + 2/\beta}$$

$$\Rightarrow \gamma \frac{V_1^2}{C_{s_1}^2} > \gamma + \frac{2\,(B_1^2/8\pi)}{p}$$

$$\Rightarrow \boxed{V_1^2 > C_s^2 + V_A^2}$$

✓ ⇒ for ⊥ shock, inflow must beat magnetosonic speed!

For shock strength:

$$\boxed{R = 1 + \gamma M^2 (1 - 1/r) - (r^2 - 1)/\beta}$$

Note:

→ for $\beta \ll 1$ (strong field)

need $M^2 \gg 1/\beta \Rightarrow V_1^2 \gg V_A^2$
for significant heating to occur.

i.e. ⊥ magnetic field always absorbs some inflow kinetic energy.

$\Rightarrow$ in strong field, should use $M = V/V_{MS} \sim V/V_A$
(Alfvenic Mach #)

$\rightarrow$ if $M_A \gg 1$,

$$R = 1 + \gamma M^2 \left(\frac{r-1}{r}\right) - (r^2-1)/\beta$$

$$\approx \gamma M^2 \left(\frac{r-1}{r}\right)$$

and can recall from before:

$$R = \frac{P_2}{P_1} \rightarrow \frac{\gamma+1}{\gamma-1} \frac{T_2}{T_1}$$

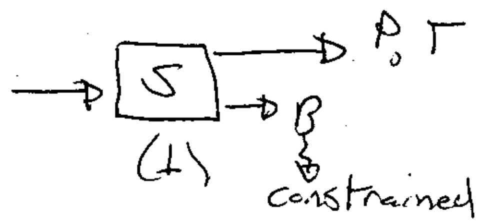
$$\therefore \quad \frac{T_2}{T_1} \cong \frac{\gamma-1}{\gamma+1} \gamma M^2 \left(\frac{\frac{\gamma+1}{\gamma-1} - 1}{\frac{\gamma+1}{\gamma-1}} \right)$$

$$= \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M^2$$

agrees with unmagnetized shock.

$\rightarrow$ Key Message: – Freezing in ties B-compression to density compression

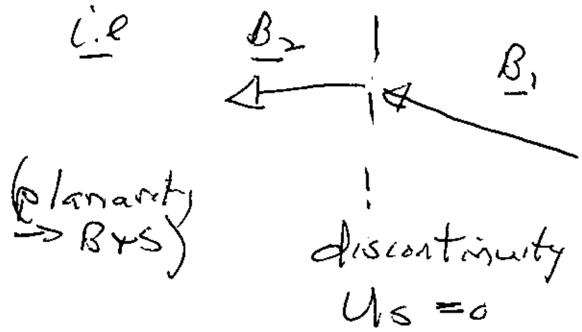
$\therefore$ – limits amount of inflow KE deposited in magnetic field i.e.



c) Oblique Shocks

"Oblique" → $\underline{V}$ inclined relative to $\underline{B}$ (not $\hat{n}$!)

i.e.



i.e. $\underline{V}_{1,2} = V_{1,2\,x}\,\hat{x}$

but $\underline{B}_{1,2} = (B_x, B_{y\,1,2})$

$([B_x] = 0)$

→ point/novelty: ① jumps in B_y $\quad B_{y_1} \to B_{y_2}$

② correspondance with oblique fast, slow wave (→ fast, slow shock ?!)

fast (⊥ → magn.-sonic)
slow (∥ → acoustic)

[n.b. intermediate wave → rotational discontinuity]

→ proceed: take $V_n = V_x$

and grind along with B_x, $B_{y\,1,2}$

→ or use a trick ?

trick: use of Teller / De Hoffmann frame

introduce a V_{y1} ! $\left\{ \begin{array}{l} \text{gain } V_{y1}\text{, but} \\ \text{eliminate } \underline{E} \times \underline{B} \end{array} \right.$

— here, convenient to introduce (as can boundr) frame of motion in V_{y_1}

i.e., choose frame with V_{y_1} s/t

$$\underline{E}_1 = -\frac{\underline{V}_1 \times \underline{B}}{c} = 0$$

(surface of discontinuity still at $x=0$)

$\Rightarrow$

$$V_{y_1} = \frac{V_{x_1} B_{y_1}}{B_{x_1}} = \frac{V_{x_1} B_{y_1}}{B_x}$$

as $[B_x] = 0$

— $\underline{V}_1$ s/t $\underline{E}_1 = 0 \Rightarrow \underline{V}_{y_1} = V_{y_1}\hat{y}$

$\Rightarrow$ (F. De Hoffmann, E. Teller, 1950)

— trade-off ? — gain $V_{y_{1,2}}$ and

frame change only a trade-off.

concommitant $[\rho V_x V_y]$ (would have anyway, due $\underline{J} \times \underline{B}$ impulse!)

— lose $\frac{c}{4\pi}(\underline{E} \times \underline{B})$ in energy jump,

since $\underline{E}_1 = 0$ and

$[\underline{V} \times \underline{B}] = 0$

— so, $V_{y_1} \equiv V_{x_1} B_y / B_x$

$[E_t] = 0$, as $[\hat{n} \times \underline{v} \times \underline{B}] = 0$

$[(\underline{v} \times \underline{B})_t] = 0$

$\Rightarrow \quad \underline{E}_{t} = 0$

$$\Rightarrow \quad V_{y_2} = \frac{V_{x_2} B_{y_2}}{B_x}$$

so
$$\frac{V_{y_2}}{V_{y_1}} = \frac{V_{x_2} B_{y_2}}{\cancel{B_x} \frac{V_{x_1} \cancel{B_{y_1}}}{\cancel{B_x}}}$$

$$= \frac{V_{x_2}}{V_{x_1}} \frac{B_{y_2}}{B_{y_1}} = \frac{\rho_1}{\rho_2} \frac{B_{y_2}}{B_{y_1}} \qquad [\rho v_n] = 0$$

$$\boxed{\frac{V_{y_2}}{V_y} = \frac{1}{r} \frac{B_{y,2}}{B_{y,1}}}$$

$$[E_t] = 0 \Rightarrow B_x V_{y_1} - V_{x_1} B_{y_1} = B_x V_{y_2} - V_{x_2} B_{y_2}$$

$$\Rightarrow \boxed{V_{y_1} - \frac{V_{x_1} B_{y_1}}{B_x} = V_{y_2} - \frac{V_{x_2} B_{y_2}}{B_x}}$$

(redundant with above)

(hold)

now, as oblique, also have:

$$\left[\rho V_x V_y - \frac{B_x B_y}{4\pi}\right] = 0$$

$$\Rightarrow \quad \rho_2 V_{x_2} V_{y_2} - \rho_1 V_{x_1} V_{y_1} = \frac{B_{x_2} B_{y_2}}{4\pi} - \frac{B_{x_1} B_{y_1}}{4\pi}$$

$$\overbrace{\frac{\rho_2 V_{x_2}}{\rho_1 V_{x_1}}}^{1} \frac{V_{y_2}}{V_{y_1}} - 1 = \left(\frac{1}{\rho V_{x_1} V_{y_1}}\right)\left(\frac{B_{x_2} B_{y_2}}{4\pi} - \frac{B_{x_1} B_{y_1}}{4\pi}\right) \quad / \rho_1 V_{x_1} V_{y_1}$$

$$\frac{V_{y_2}}{V_{y_1}} - 1 = \left(\frac{B_x B_{y_1}}{4\pi \rho_1 V_{x_1} V_{y_1}}\right)\left(\frac{B_{y_2}}{B_{y_1}} - 1\right)$$

$V_{y_1} = V_{x_1} B_{y_1} / B_{x_1} \quad \Rightarrow$

$$\boxed{\frac{V_{y_2}}{V_{y_1}} - 1 = \frac{B_{x_1}^2}{4\pi \rho_1 V_{x_1}^2}\left(\frac{B_{y_2}}{B_{y_1}} - 1\right)}$$

so

$$\boxed{\begin{aligned} \frac{V_{y_2}}{V_{y_1}} - 1 &= \frac{B_{x_1}^2}{4\pi\rho_1 V_{x_1}^2}\left(\frac{B_{y_2}}{B_{y_1}} - 1\right) \\ \frac{V_{y_2}}{V_{y_1}} &= \frac{V_{x_2}}{V_{x_1}}\,\frac{B_{y_2}}{B_{y_1}} = \frac{1}{r}\,\frac{B_{y_2}}{B_{y_1}} \end{aligned}}$$

$\rightarrow$ tang Π

$\rightarrow$ frame

$$V_A^2 \equiv B_{x_1}^2 / 4\pi\rho_1$$

$\Rightarrow$

$$\frac{V_{y_2}}{V_{y_1}} = 1 + \frac{V_A^2}{V_{x_1}^2}\left(\frac{B_{y_2}}{B_{y_1}} - 1\right)$$

$$= 1 + \frac{V_A^2}{V_{x_1}^2}\left(r\,\frac{V_{y_2}}{V_{y_1}} - 1\right)$$

$\Leftrightarrow$

$$\boxed{V_{y_2}/V_{y_1} = \frac{V_{x_1}^2 - V_A^2}{V_{x_1}^2 - rV_A^2} = \frac{1}{r}\,\frac{B_{y_2}}{B_{y_1}}}$$

Now, finally use $\left[w + \frac{V^2}{2}\right] = 0$ $\quad (\underline{E} = 0)$

$V^2 = V_x^2 + V_y^2$ (both components!)

note: r undetermined here $\Rightarrow$ normal stress

so plugging in $\left(w = \gamma P/\rho(\gamma-1)\right)$

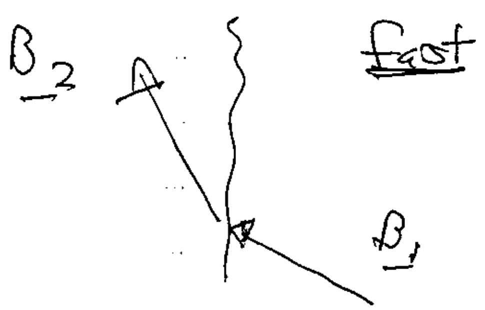
$\Rightarrow$

$$\boxed{\begin{aligned} \frac{P_2}{P_1} &= r + (\gamma-1)\frac{r V_{x_1}^2}{2 c_{s_1}^2}\left[1 - \frac{\cos^2\theta}{r} - \sin^2\theta\left(\frac{V_{x_1}^2 - V_A^2}{V_{x_1}^2 - r V_A^2}\right)\right] \\ V_{y_2}/V_{y_1} &= \frac{V_{x_1}^2 - V_A^2}{V_{x_1}^2 - r V_A^2} = \frac{1}{r}\frac{B_{y_2}}{B_{y_1}} \end{aligned}}$$

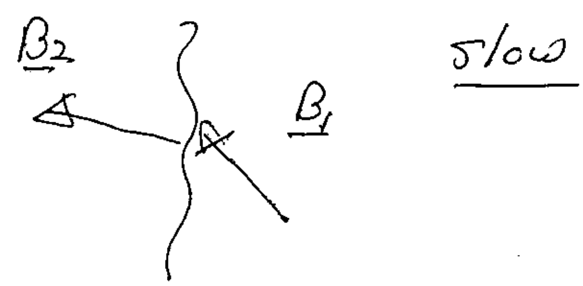
so, the answer (obliquity phenomena?):

$B_{y_2} > B_{y_1}$ if $V_{x_1}^2 > r V_A^2 > V_A^2$ $\rightarrow$ "fast shock"

$B_{y_2} < B_{y_1}$ if $V_{x_1}^2 \leq V_A^2 < r V_A^2$ $\rightarrow$ "slow shock"



refract away normal



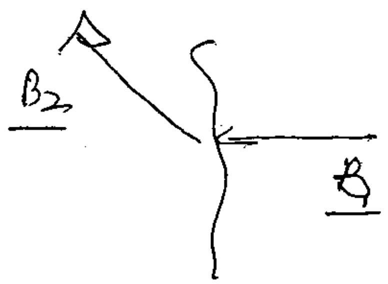
refract toward normal.

what of equalities?

if $\quad V_{x_1}^2 = r V_A^2$

$B_{1y} = 0$

$B_{2y} \neq 0$ $\quad \Rightarrow$ "switch on" shock



no B_y in ①

$B_y \neq 0$ in ②

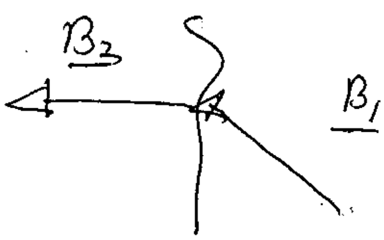
if $\quad V_{x_1}^2 = V_A^2$

$B_{y_1} \neq 0$

$B_{y_2} = 0$ $\quad \Rightarrow$ "switch off" shock

B_y in ①

none in ②



Note:

$\quad$ switch-off $\Leftrightarrow$

- for $\theta = 0$, $(B_{1y} = 0)$ $\quad V_1 = V_A$, $B_{2y} = 0$

$\underline{\text{so}}$ $\quad \underline{B}_1 \parallel \underline{B}_2 \parallel \hat{n}$

$[B_n] = 0 \Rightarrow \quad \underline{B}_1 = \underline{B}_2 \quad \rightarrow$ reduces to parallel and hydro case.

→ What of Intermediate Wave?

⇒ non-shock discontinuity → i.e. no heating, etc.

⇔ rotational discontinuity! (contrast shock.)

→ $j = 0$ (no mass flow thru discontinuity)

→ magnetic field rotates thru an angle in jump

⇒ better understood in context of collisionless shocks

⇒ RD -type structures are seen in solar wind, etc.
(i.e. Ulysseuss)

This brings us to ....

Collisionless Shocks !

Ion-Acoustic Shocks and Solitons — $\left\{\begin{array}{l}\text{simplest form} \\ \text{c-shock}\end{array}\right.$

In quasi-neutral system ($k^2\lambda_{De}^2 \ll 1$)

$$n_e = n_0 \exp\left[|e|\phi/T_e\right]$$

$$\frac{\partial n_i}{\partial t} + v\frac{\partial n_i}{\partial x} = -n_i\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -|e|\frac{\partial \phi}{\partial x}$$

$$\phi = \frac{T_e}{|e|}\ln\left(n_e/n_0\right) = \frac{T_e}{|e|}\ln\left(n_i/n_0\right)$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{T_e}{|e|}\frac{\cancel{n_0}}{n_e}\frac{1}{\cancel{n_0}}\frac{\partial n_i}{\partial x}$$

$$\Rightarrow \quad \frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v) = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{T_e}{n_i}\frac{\partial n_i}{\partial x}$$

→ isomorphic to 1D gas-dynamic equations (albeit isothermal)

→ steepening, shock formation will occur

→ but, as dissipation miniscule, shock limited by dispersion, not dissipation

i.e. isomorphism to gas dynamics ⇒ $k^2\lambda_D^2 \ll 1$ (quasi-neutrality)

Shock structure limited when $L \sim \lambda_{De}$
→ Quasi-neutrality violated!

i.e. allowing for dispersion:

$$n_e = \exp\left[|e|\phi/T_e\right] \qquad \text{Boltzmann Electrons}$$

$$\left.\begin{aligned} &\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v) = 0 \\ &\partial v_i/\partial t + v_i \partial v_i/\partial x = -\frac{|q|}{m_i}\frac{\partial \phi}{\partial x} \end{aligned}\right\} \text{fluid eq'ns}$$

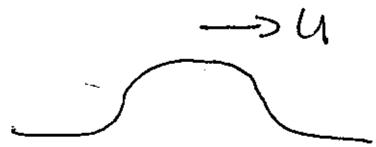
$$\tilde{n}_e = \exp(q\phi/T_e)$$

$\tilde{n}_i$:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i V) = 0$$

$$\frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} = -\frac{q}{m_i}\frac{\partial \phi}{\partial x}$$

$$\left\{ \begin{matrix} n_i \\ V_i \\ \phi \end{matrix} \right\} = F(x-ut)$$



i.e. localized solution, moving at u.

$$-u n_i' + (n_i V)' = 0$$

$$(V-u)V' = -\frac{q}{m_i}\phi'$$

Now, integrating with $\left.\begin{matrix} \phi \to 0 \\ V \to 0 \\ n \to n_0 \end{matrix}\right\}$ $x \to \infty$

$\Rightarrow$

$$-u n_i + n_i V = -u$$

$$\Rightarrow (u-V)n_i = u \quad \to \text{ to ensure } n \to n_0$$

$$n_i = u/(u-V)$$

Similarly,

$$\frac{q\phi}{m_i} = -\frac{1}{2}(u-v)^2 + \frac{u^2}{2} \qquad \text{(to ensure } \phi \to 0\text{)}$$

$$\Rightarrow \left(\frac{1}{2}u^2 - \frac{q}{m_i}\phi\right) = \frac{1}{2}(u-v)^2$$

$$\Rightarrow (u-v) = \left(u^2 - \frac{2q\phi}{m_i}\right)^{1/2}$$

so $$\frac{\partial^2\phi}{\partial x^2} = -4\pi n_0 q\left(\frac{1}{\left(1-\frac{2q\phi}{m_i u^2}\right)^{1/2}} - \exp(q\phi/T_e)\right)$$

$$\phi'\phi'' = -4\pi n_0 q\,\phi'\left(\frac{1}{\left(1-\frac{2q\phi}{m_i u^2}\right)^{1/2}} - \exp\left(\frac{q\phi}{T_e}\right)\right)$$

integrating $\Rightarrow$

$$\frac{1}{2}\phi'^2 + V(\phi) = 0$$

$$\boxed{V(\phi) = -4\pi n_0\left\{mu\left(u^2 - \frac{2q\phi}{m}\right)^{1/2} + T_e e^{q\phi/T_e}\right\} + C}$$

$\hookrightarrow$ Sagdeev Potential

$$\phi'' = dV/d\phi$$

Collisionless shock $\Leftrightarrow$ solitary wave

$\rightarrow$ Ion acoustic soliton reduced to particle orbit pblm!.

Define Mach # $\mathcal{M} = u/c_s$

$$u = \mathcal{M} c_s$$

$\Rightarrow$

$$V(\phi) = -4\pi n_0 \left\{ m \mathcal{M} c_s \left(\mathcal{M}^2 c_s^2 - \frac{2q\phi}{m} \right)^{1/2} + T_e e^{q\phi/T_e} \right\} + C$$

$$= -4\pi n_0 \left\{ T_e \mathcal{M} \left(\mathcal{M}^2 - \frac{2q\phi}{T_e} \right)^{1/2} + T_e e^{q\phi/T_e} \right\} + C$$

Thus:

$\rightarrow$ need $\mathcal{M}^2 > 2q\phi_M/T_e$ for soliton to exist
(reality)

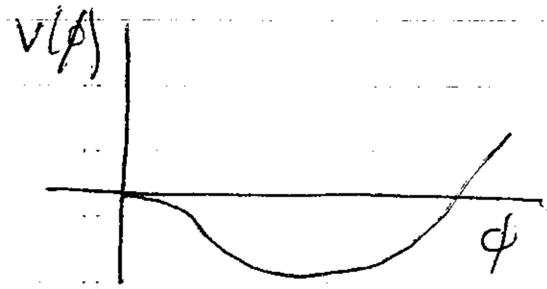
$\frac{u^2}{c_s^2} > 2q\phi_M/T_e$ (critical velocity)
$\rightarrow$ speed $-$ amplitude connection

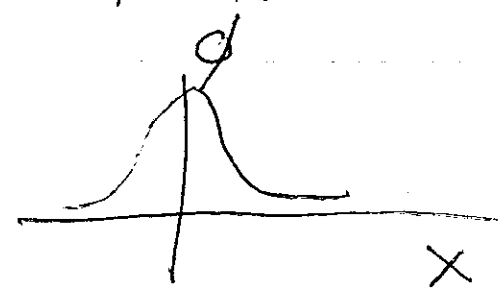
$\rightarrow$ Similarly, for small ϕ,

$$V(\phi) \cong -4\pi n_0 \left\{ T_e \mathcal{M}^2 \left(1 - \frac{q\phi}{\mathcal{M}^2 T_e} - \frac{1}{2} \left(\frac{q\phi}{T_e \mathcal{M}^2} \right)^2 \right) + T_e + q\phi + \frac{T_e}{2} \left(\frac{q\phi}{T_e} \right)^2 \right\}$$

$$\cong -4\pi n_0 \left\{ T_e(1+\mathcal{M}^2) + \cancel{q\phi} - \cancel{q\phi} + \frac{T_e}{2} \left(\frac{q\phi}{T} \right)^2 \left(-\frac{1}{\mathcal{M}^2} + 1 \right) \right\}$$

Now, for soliton $\Rightarrow$ need bound state





Then $V''(\phi)\Big|_{\phi\to 0} < 0 \Rightarrow \mathcal{M}^2 > 1$

So need $\mathcal{M} > 1$ for soliton formation.

$\rightarrow$ Similarly, need $\mathcal{M} \lesssim 1.6$

$\therefore$ for soliton, need $1 < \mathcal{M} < 1.6$
($e\phi/T$ small)

i.e. have

$$V(\phi) = -4\pi n_0 \left\{ m u \left(u^2 - \frac{2e\phi}{m}\right)^{1/2} + T_e\, e^{e\phi/T} \right\}$$

$$= -\phi'^2$$

take ϕ_{max} when $V(\phi) = 0 \rightarrow \phi' \equiv 0$

$\Rightarrow$ defines ϕ_{max}

More Generally:

→ as dissipation miniscule, shock limited by dispersion, not dissipation

i.e. quasi-neutrality ⟺ $k^2\lambda_{De}^2 \lll 1$

when $L_{shock} \sim \lambda_{pe}$ ⇒ quasineutrality violated!

∴ ion-acoustic shock limited by dispersion

i.e. $\omega^2 = k^2 c_s^2 / 1 + k^2\lambda_{De}^2$

→ Generally, sub-classify shocks into:

collisional → old standard hydrodynamics
L_{shock} limited by dissipation

collisionless → a/a' ion-acoustic in plasma
L_{shock} limited by dispersion
⇒ forms soliton

Aside: Some Generic Properties of Solitons

Contrast → Sound wave $\omega = k c_s$
$x = (c_s + V)t + f(v)$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + c_s\frac{\partial V}{\partial x} = 0$$

→ Dispersive Ion Acoustic Wave

$$\omega^2 = k^2 c_s^2 / 1 + k^2 \lambda_D^2$$

$k\lambda_D < 1 \Rightarrow \quad \omega = k c_s \left(1 - k^2 \lambda_D^2 / 2\right)$ $\left(\frac{\omega}{k} \text{ odd in } k\right)$

suggests model equation of form:

$$\frac{\partial \mathcal{E}}{\partial t} + (c_s + \mathcal{E}) \frac{\partial \mathcal{E}}{\partial x} + \frac{c_s \lambda_D^2}{2} \frac{\partial^3 \mathcal{E}}{\partial x^3} = 0$$

of generic form:

$$\frac{\partial \mathcal{E}}{\partial t} + u_0 \frac{\partial \mathcal{E}}{\partial x} + \alpha \mathcal{E} \frac{\partial \mathcal{E}}{\partial x} + \beta \frac{\partial^3 \mathcal{E}}{\partial x^3} = 0$$

$$a = \alpha \mathcal{E}$$
$$y = x - u_0 t$$

$$\Rightarrow \boxed{\frac{\partial a}{\partial t} + a \frac{\partial a}{\partial y} + \beta \frac{\partial^3 a}{\partial y^3} = 0}$$ (Korteweg - deVries Eqn (KdV))

dispersion vs. dissipation

contrast

$$\frac{\partial a}{\partial t} + a \frac{\partial a}{\partial y} - \bar{\nu} \frac{\partial^2 a}{\partial y^2} = 0$$ (Burgers Eqn.)

Burgers → dissipative ($\bar{\nu}$ limits steepening)

$$L_{shock} \sim \bar{\nu}/a$$

KdV → dispersive (ω variation with k ⇒ u variation with k limits steepening — diff'nt scale comp.)

$$L_{soliton} \sim (\beta/a)^{1/2}$$

Solution of KdV Equation:

$$\frac{\partial a}{\partial t} + a\frac{\partial a}{\partial y} + \beta\frac{\partial^3 a}{\partial y^3} = 0$$

$$a = a(y - V_0 t) \qquad \Rightarrow \quad V_{wave} = u_0 + V_0$$

⇒

$$\beta a''' + a a' - V_0 a' = 0$$

Invariant
$a \to a + V$
$V_0 \to V_0 + V$

∴

$$\beta a'' + \frac{1}{2}a^2 - V_0 a = 1/2\, C_1$$

$$2\beta a' a'' + a' a^2 - 2V_0 a' a = C_1 a' \qquad (* \, 2a')$$

$$\Rightarrow \quad \beta a'^2 = -\frac{1}{3}a^3 + V_0 a^2 + C_1 a + C_2$$

+ can reduce to quadrature

Convenient to factorize:

$V_0,\ c_1,\ c_2 \rightarrow a_1,\ a_2,\ a_3$

$$\Rightarrow \beta a'^2 = -\frac{1}{3}(a-a_1)(a-a_2)(a-a_3)$$

where $V_0 = \frac{1}{3}(a_1 + a_2 + a_3)$

For $\rightarrow$ bounded $|a(y)|$
$\rightarrow$ need a_1, a_2, a_3 real
if $a_1 > a_2 > a_3$

$$\Rightarrow a_1 \geq a \geq a_2 \qquad (\beta a'^2 > 0)$$

$\therefore$ $a_3 = 0$ is no loss generality

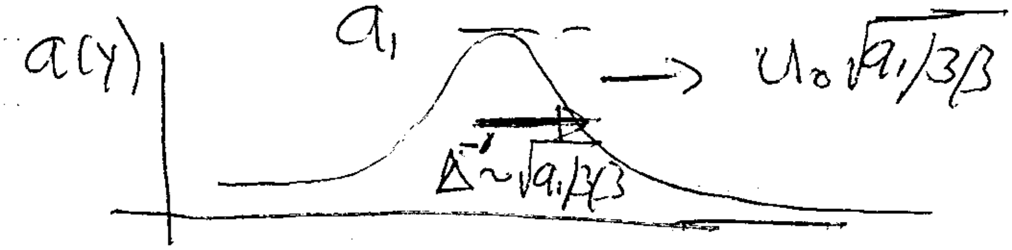
$$\Rightarrow \beta a'^2 = \frac{1}{3}(a_1 - a)(a - a_2)\, a$$

if $a_2 = 0$ Exact solution of NL KdV Eqn.

$$\therefore \boxed{a(y) = a_1 \cosh^{-2}\left(\frac{1}{2} y \sqrt{a_1/3\beta}\right)}$$

$$= a_1 \cosh^{-2}\left(\frac{1}{2}(x - u_0 t)\sqrt{a_1/3\beta}\right)$$

$\Rightarrow$



$\Rightarrow$ $\rightarrow$ soliton has finite width $\Delta \sim \sqrt{3\beta / a_1}$ $\quad \beta \sim \lambda_{De}^2$ for IA

$\Rightarrow \Delta \sim \lambda_D$

$\leftrightarrow$ contrast zero-width shock

$\rightarrow$ Soliton has finite amplitude a_1

with $V \sim u_0 \sqrt{a_1/3\beta}$

$\therefore$ bigger solitons move faster!

Note: $a_2 \neq 0 \Rightarrow$ non-localized, oscillatory solution

General Comments:

$\rightarrow$ Collisional shock $\quad \Delta \sim \nu / a$

Collisionless Shock $\quad \Delta \sim \lambda_D \sqrt{c_s / a}$

$\therefore$ Debye length sets discontinuity scale

$\rightarrow$ Can treat collisionless shock via

$$\nabla^2 \phi = -4\pi n_0 q (\tilde{n}_i - \tilde{n}_e \quad)$$

etc $\Rightarrow$ Sagdeev Potential