Black Holes (Ph 161)

An introduction to General Relativity.

Ph 161 Black Holes Tu - Th 9:30 AM to 10:50 AM

Professor: George M. Fuller
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office hours: Fridays, 2:00 PM to 3:30 PM, 329 SERF

T.A.: J. J. Cherry office hours: Monday 12:30 PM to 1:20 PM
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email: jcherry@physics.ucsd.edu

Discussion Section: TBA

Course Outline:
This is a course on introductory General Relativity, black holes, and relativistic astrophysics.
We will talk about spacetime and the equivalence principle and we will learn how to
think about, and calculate, relevant physical quantities in curved spacetime.
Along the way we will have to develop some mathematical tools and concepts
for handling geometric objects like vectors. I will assume that you have taken the Ph 2 sequence
or its equivalent and that you have had a linear algebra course (this is essential).
We will present some classic solutions for Einstein's field equations: the Schwarzschild solution for a
spherically symmetric, static spacetime; and the Kerr family of metrics for axially symmetric,rotating spacetimes.
Both of these are the foundation for understanding the astrophysics of black holes.

Grading and General Requirements:
There will be (nearly) weakly homework assignments that will be graded and will count for 60% of the final grade.
The other 40% of the grade will be based on a final paper and/or oral presentation (talk) on relevant subject matter.
We will discuss appropriate project subject matter in class.If you opt to write a final paper it must be at least 10 pages
in length with appropriate references. If, instead, you opt for the oral presentation be prepared to give a (rigidly) timed
10 minute talk to the entire class and turn in a short paper (a few pages) summarizing the talk and giving relevant references.
In either the paper or the talk, you should plan on getting across the basic ideas in a concise and readable way.
You will be graded on how well you have understood your topic, how effectively you have integrated into your
paper/presentation basic tools and concepts from class, and how effectively you can communicate your ideas.
I am requiring this because all scientists and engineers must learn these writing and communication skills -
this is the sort of thing that you will be doing frequently in your professional lives.
All students must attend the sessions where we have the talks.
This session will be arranged in discussion in class.

Physics 161
Black Holes and Milky Way Galaxy

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Phone: 858-822-1214
Email: gfuller@ucsd.edu
Office Hours: Fridays, 2:00 PM to 3:30 PM, 329 SERF

Course TA: J. J. Cherry
Phone: 619-985-3433
Email: jcherry@physics.ucsd.edu

Lecture Tuesday/Thursday 9:30 AM to 10:50 AM Ledden Auditorium (2250 HSS)
Discussion Section: Friday, 12-12:50 p.m., Pepper Canyon Hall 122

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All students must attend the sessions where we have the talks. This session will be arranged in discussion in class.

Homework: 60% of grade - This must be your own effort. You can talk with other students but you absolutely must turn in your own work; no copying. I am looking to see That you made an honest attempt to grapple with the problem. In my view, the correct answer is not as important as clear evidence that you understood the concepts from the lecture and reading and have made an effort to apply your understanding.

Final Project: Paper and/or Oral Presentation: 40% of grade -
Either of these can fall into two categories: (1) a review of an interesting topic relating to black holes, general relativity, or cosmology, etc.; or (2) an interesting calculation of your own. This latter category might, *e.g.,* be a “fleshing out” of some calculation or argument that we glossed over in class or in the book, or it might be a computer calculation of, *e.g.,* orbits of photons near a black hole.

Papers should be at least 10 pages with appropriate references. Oral presentations will be 10 minutes and these will be absolutely rigidly timed, just like an APS meeting. To pull this off, it will be best to practice it many times before hand. Oral presentations must be accompanied by a short paper (a few pages), giving the basic point and outline of the talk and containing all relevant and appropriate references.

Books:

Text: “GRAVITY: An Introduction to Einstein’s General Relativity,” James B. Hartle (Addison Wesley; San Francisco, 2003)

Reference: “A First Course in General Relativity,” Bernard F. Schutz (Cambridge University Press, Cambridge, 1990)

Reading assignment Weeks 1 & 2

Hartle: chapters 1, 2, 3, 4, 5

Ph 161 Black Holes

Homework Assignment 1

Due Tuesday, January 23, 2006

This should be your own work; do not copy problem solutions.

(1.) Frame $\bar{\mathcal{O}}$ is boosted along frame $\mathcal{O}$'s x-axis by speed v. By consulting the spacetime diagrams in the lecture slides, give a geometric proof that $\bar{\mathcal{O}}$'s $\bar{x}$-axis (the locus of events *simultaneous* with event $\bar{t} = 0$, $\bar{x} = 0$ in frame $\bar{\mathcal{O}}$) as plotted in frame $\mathcal{O}$ corresponds to the equation $t = v\,x$ (where we have set $c = 1$, of course).

(2.) Consider one observer (frame $\mathcal{O}$) who measures the length of a rod at rest, laying on the x-axis in this frame to be $l = 5\,\mathrm{m}$. Another observer (frame $\bar{\mathcal{O}}$) moves with constant speed $v = 0.9$ (that is, 90% the speed of light) along $\mathcal{O}$'s x-axis. What does he measure for the length of this rod? Illustrate with a spacetime diagram why these observers measure different lengths.

(3.) Describe in words and explain how the Einstein summation convention works. We gave an example of a matrix representation for a simple Lorentz boost along one of the spacelike axes. One should be wary of these matrix representations because they can get you in trouble if you do not pay close attention to how the sums work in the Einstein summation convention. For example, can you come up with a matrix representation for the double sum $ds^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta$?

(4.) Hartle Chapter 4: problem 13.

(5.) Hartle Chapter 4: problem 15.

(6.) Hartle Chapter 5: problem 1.

(7.) Hartle Chapter 5: problem 4.

(8.) Hartle Chapter 5: problem 7.

(9.) Hartle Chapter 5: problem 20.

(10.) A photon has no four-velocity, of course, but it does have a four-momentum $\mathbf{p}$: after all, it carries energy and three-monetum. Give an argument for why $\mathbf{p}$ must be tangent to the photon's world line. What does this imply for the photon's rest mass?

Black Holes

What are they made of?

They are just curved *spacetime*
-- “GRAVITY”--
In this case the
fossil spacetime curvature left
over from an object (or objects)
that has (have) collapsed inside an
event horizon.

How do we know they are there?

Material (gas) can fall into a BH and
in so doing be heated up to the point
where light (radiation) is emitted.
Once inside the event horizon, however,
nothing gets out!

Einstein's General Relativity dates from 1916. It stood more or less aloof from the rest of modern physics until the advent of x-ray astronomy and the discovery of quasars (QSO's - quasi stellar radio sources) in the early 1960's. Then it was realized that the prodigious energy requirements of compact sources could only be obtained from gravitational energy. This sparked a revolution in General Relativity research. There continues to be a symbiotic relationship between the advance of new telescope technology and the advance of physics.

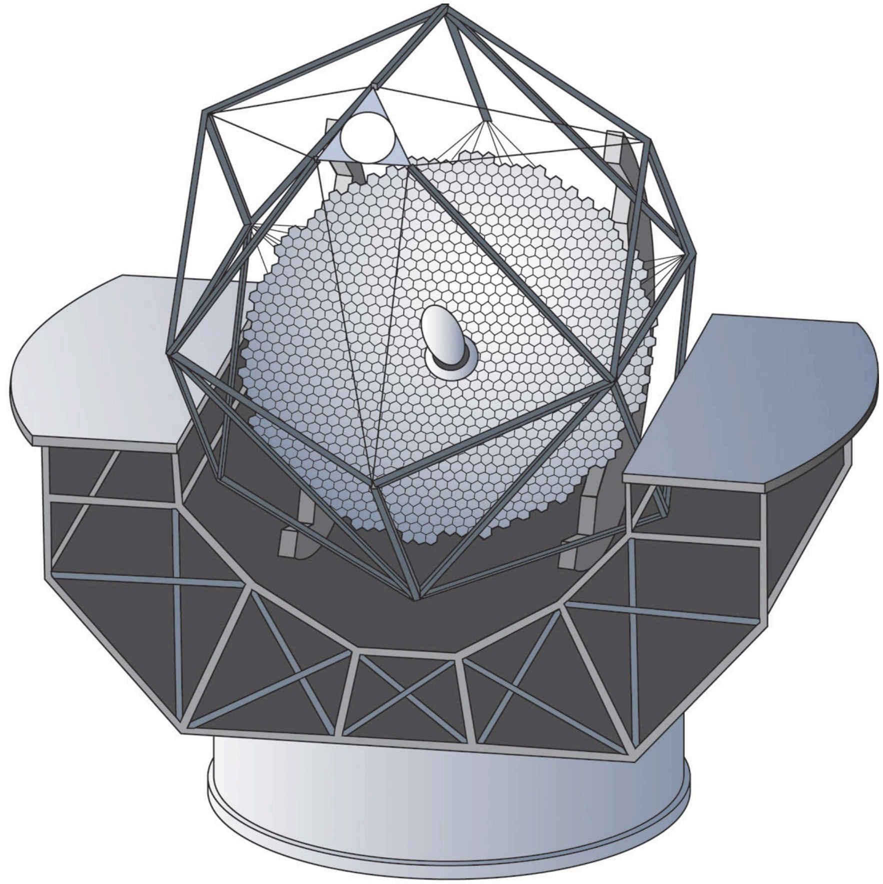
General Relativity is now a key tool of all astrophysicists and particle physicists.

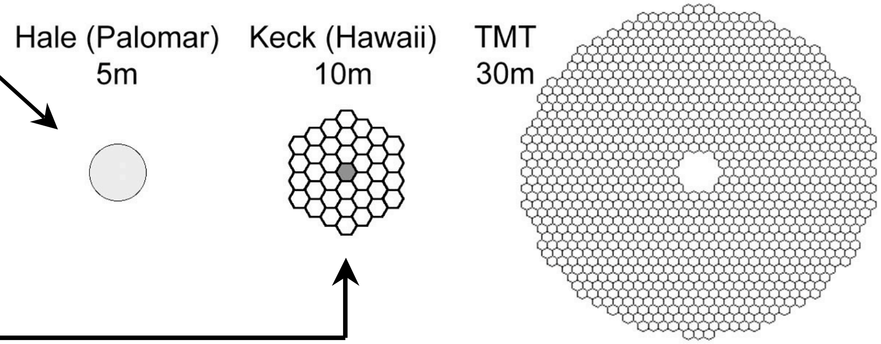
It has a justly deserved reputation for being mathematically difficult -but it's basic ideas are straightforward and accessible to beginning students.

Here we will try to steer a middle course - introduce enough math to get at the essentials and emphasize physical reasoning.

The advent of bigger and bigger mirrors has revolutionized optical astronomy and physics (NOAO)

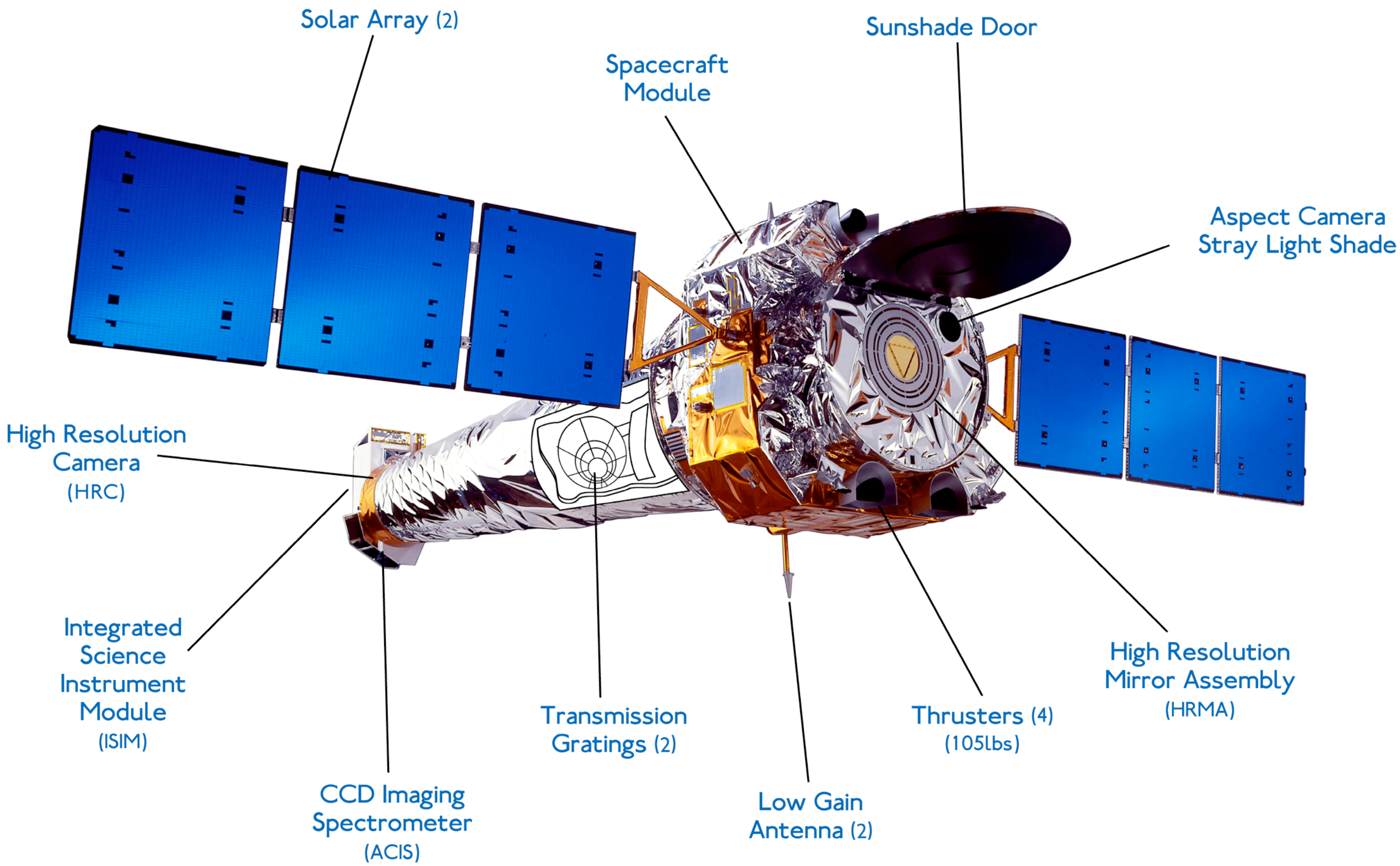
Hale telescope (Palomar)





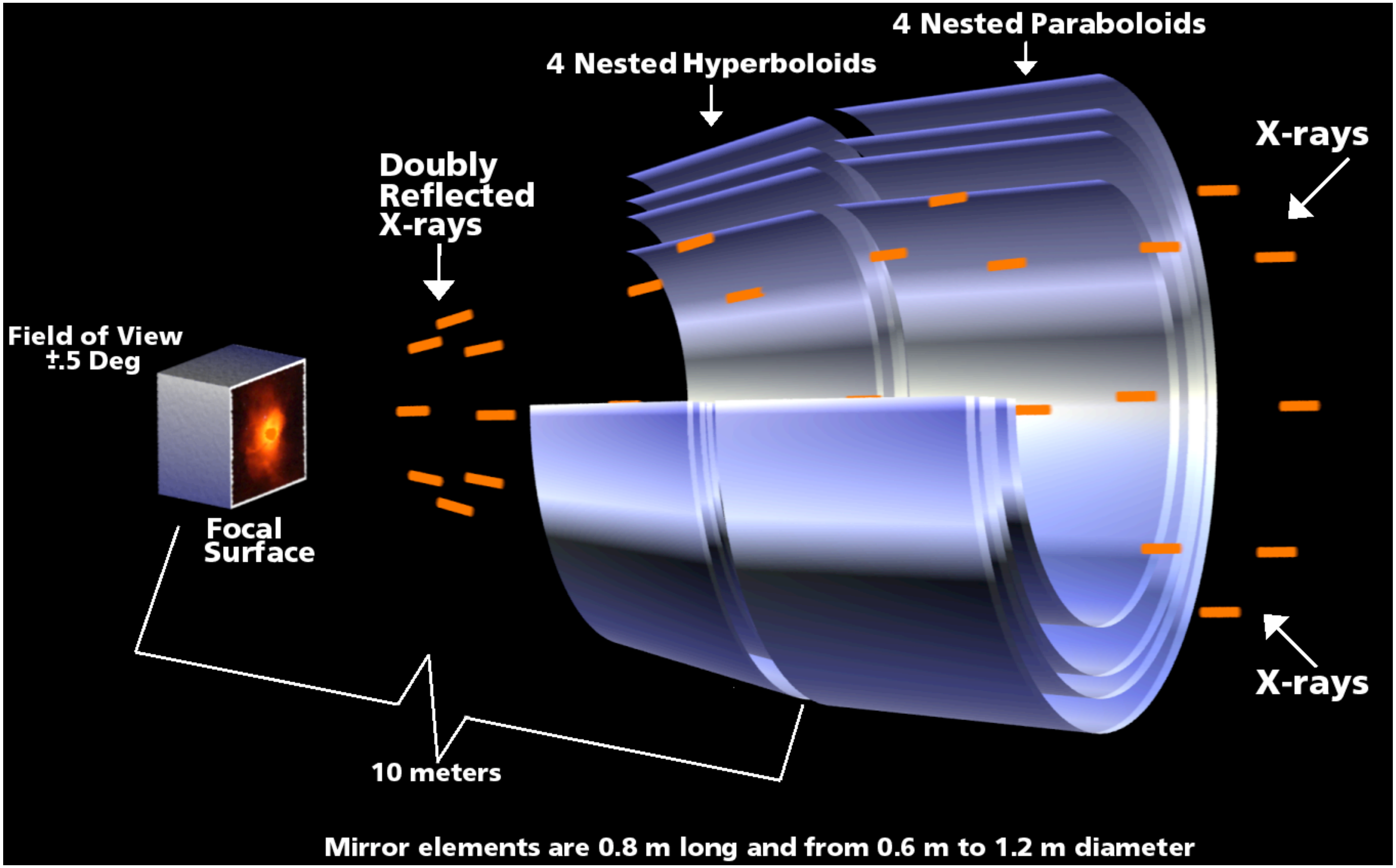
Hubble Space Telescope (HST) NASA

Chandra X-Ray Observatory



Spacecraft currently in orbit - NASA/CXC/SAO

Schematic of Grazing Incidence, X-ray Mirrors

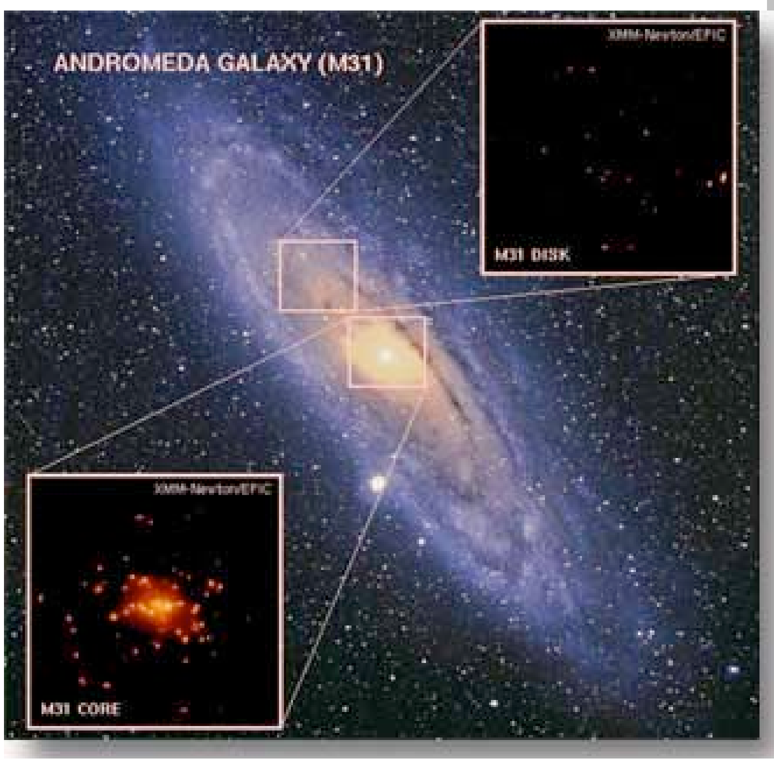


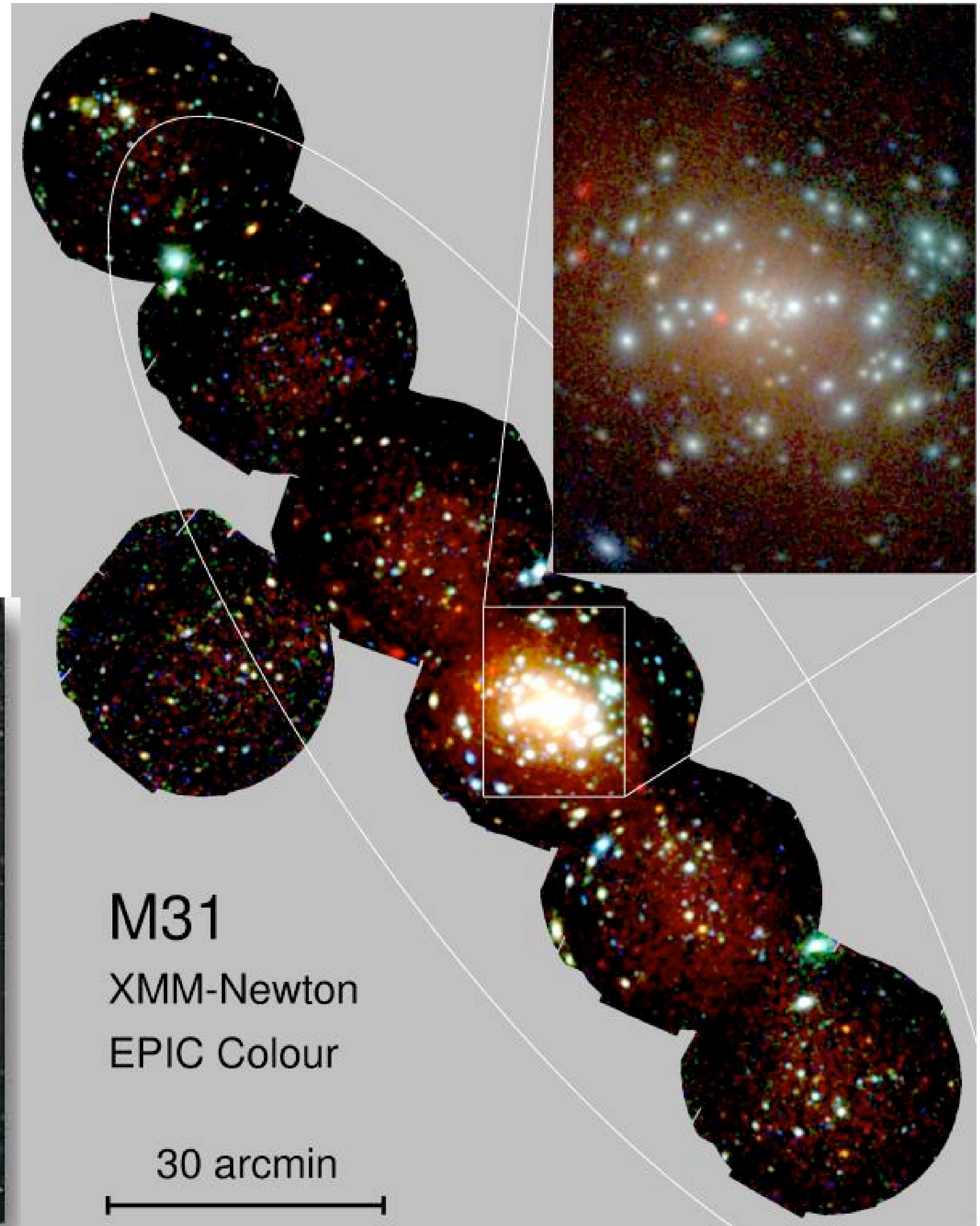
NASA/CXC/SAO

XMM - NEWTON Satellite (X-Ray Observatory)

XMM-Newton
X-ray sources in the
Andromeda galaxy

(image courtesy of W. Pietsch
MPE Garching & ESA)



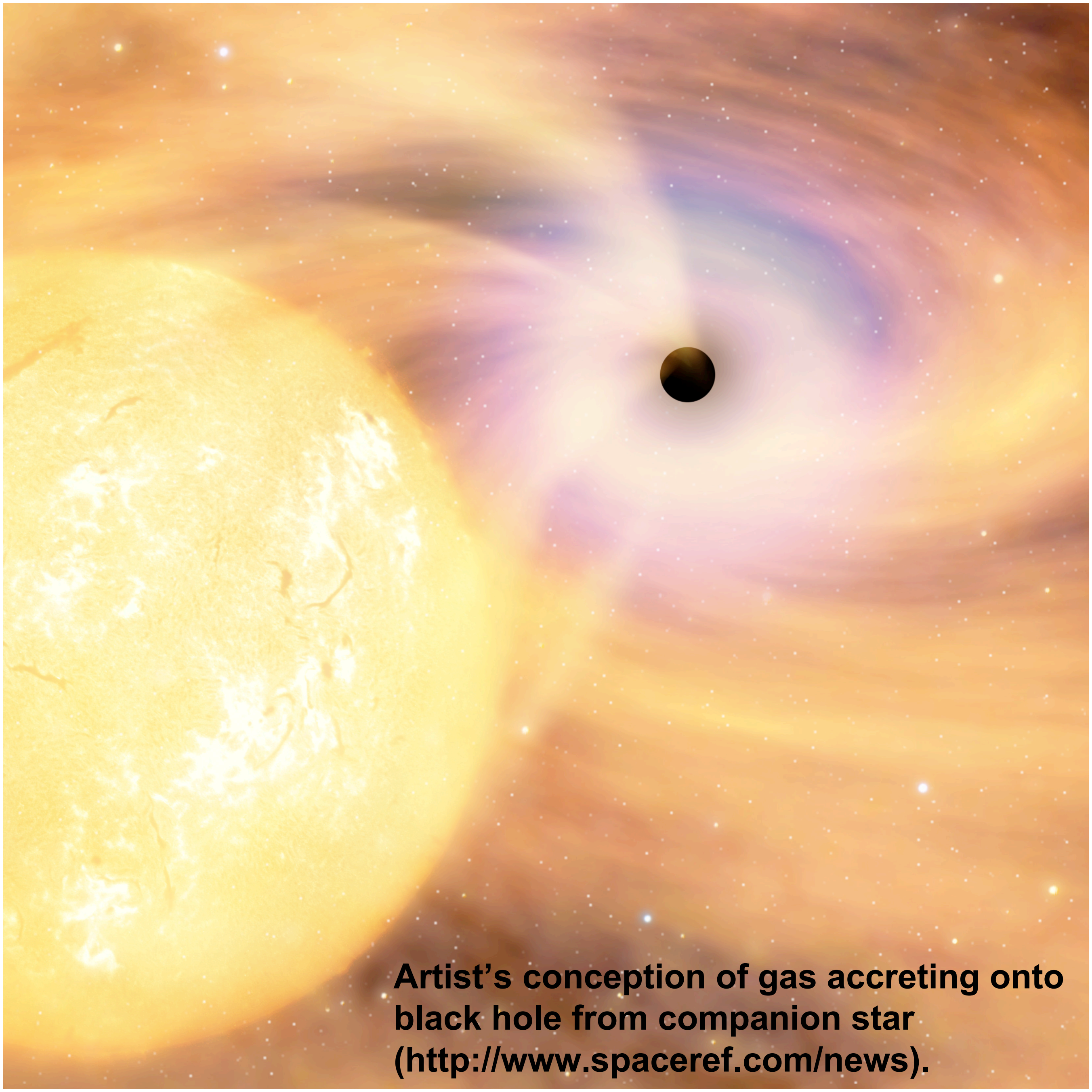


Stellar Mass Black Holes

$$M \Rightarrow \ \sim 1\,\mathrm{M}_\odot \ \text{to} \ \sim 100\,\mathrm{M}_\odot$$

These objects form from the collapse of massive stars
(supernova explosions)
and/or from accretion of material on a neutron star
subsequent to a collapse event.

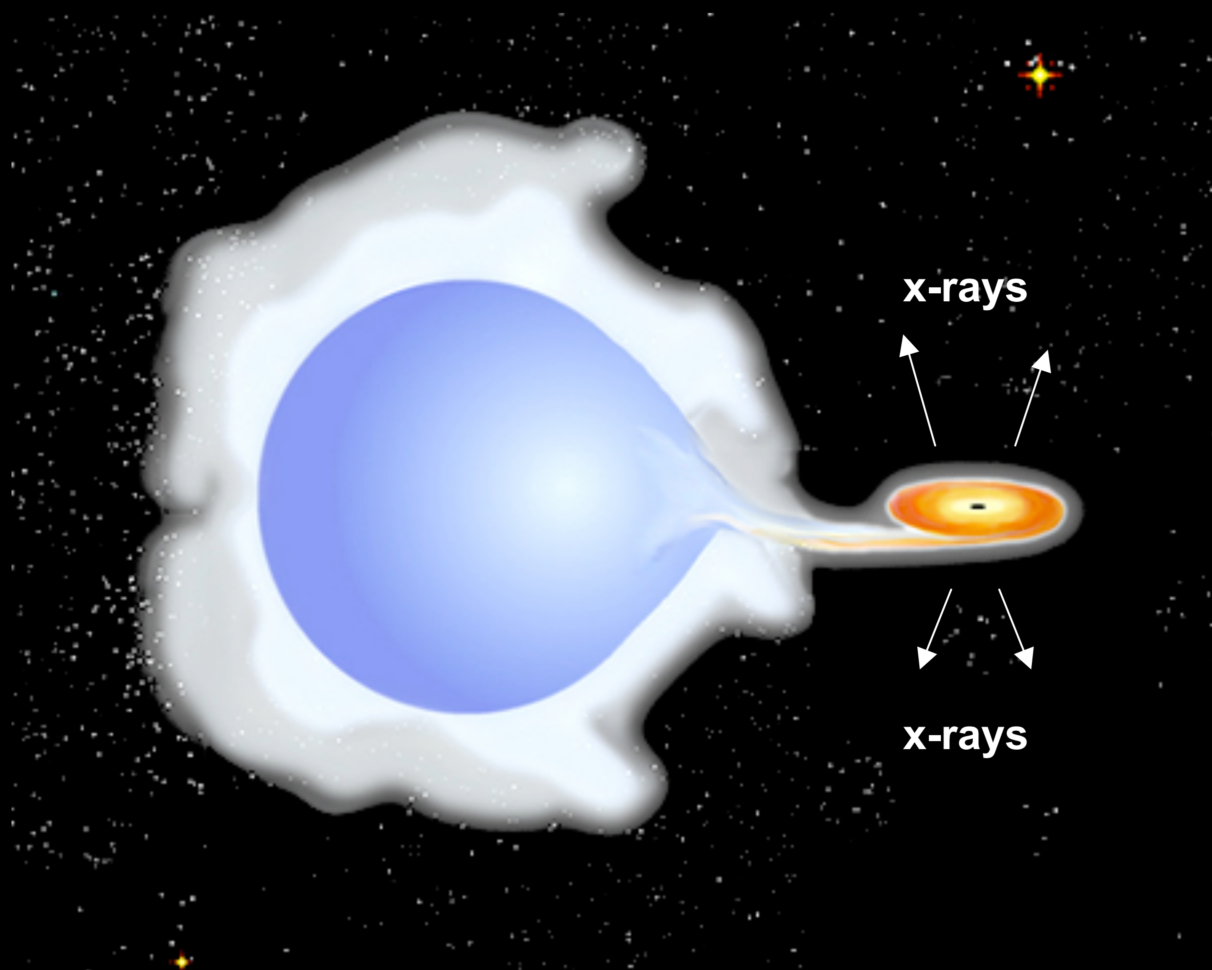
$$\text{mass of sun} \ \ 1\,\mathrm{M}_\odot \approx 1.989 \times 10^{33}\,\mathrm{g}$$

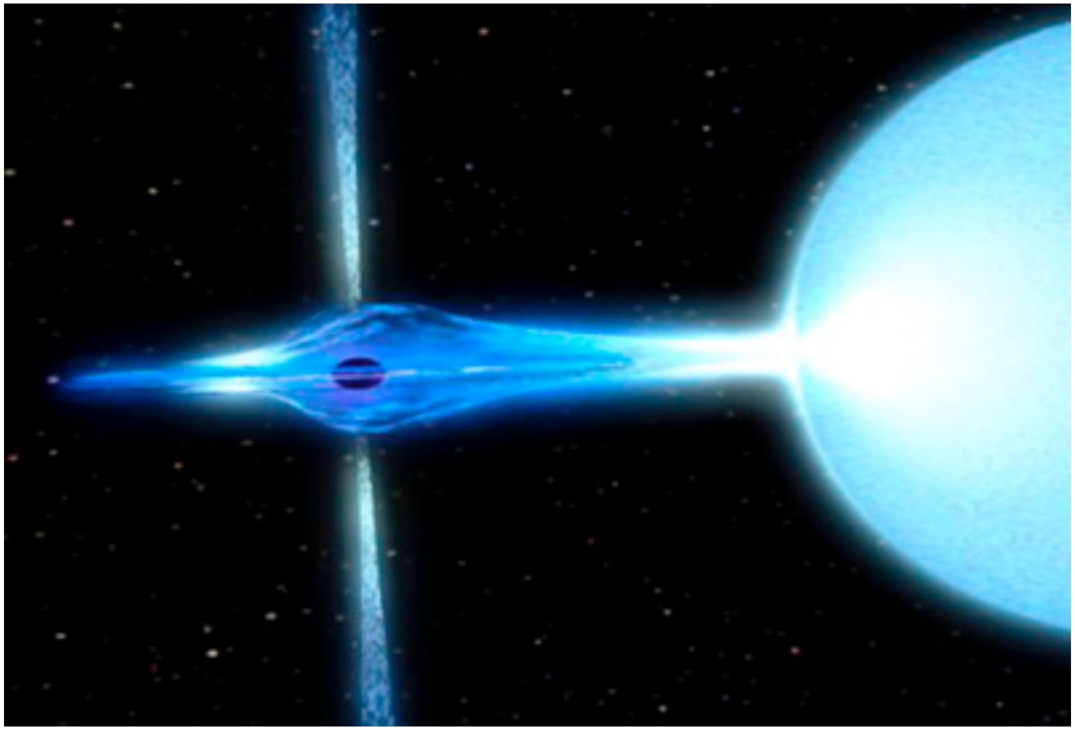


Artist’s conception of gas accreting onto black hole from companion star (http://www.spaceref.com/news).

companion star feeding gas into BH accretion disk

(science.nasa.gov)





Side view of companion star feeding gas into the accretion disk around a rotating (Kerr) black hole. (GSFC, NASA)

Intermediate- and Super-massive Black Holes

intermediate mass $M \Rightarrow \ \sim 300\,\mathrm{M}_\odot \ \mathrm{to} \ \sim 1000\,\mathrm{M}_\odot$

These *may* form from the collapse of very massive or supermassive stars (pair instability and GR instability supernovae, respectively).

supermassive $M \Rightarrow \ \sim 10^4\,\mathrm{M}_\odot \ \mathrm{to} \ \sim 10^9\,\mathrm{M}_\odot$

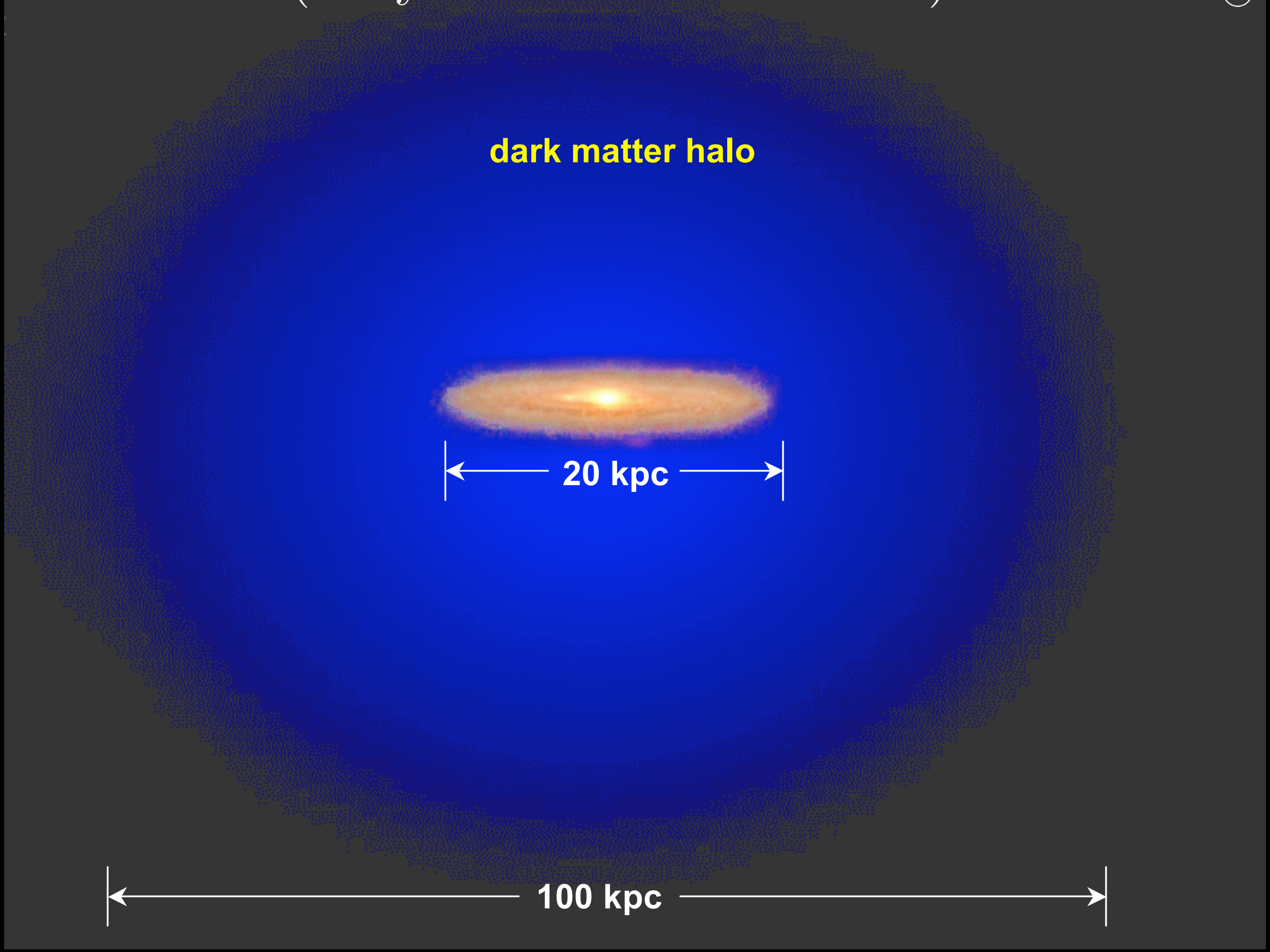
There seems to be a supermassive black hole at the center of nearly every galaxy.
Our galaxy has a hole with mass ~ 10^6 solar masses;
andromeda has one with ~ 3×10^7 solar masses.

QSO's (quasars) and Active Galactic Nuclei (AGN) have monster central black holes.
Accretion of gas on these gives rise to prodigious optical, radio, and x-ray and high energy particle emission.

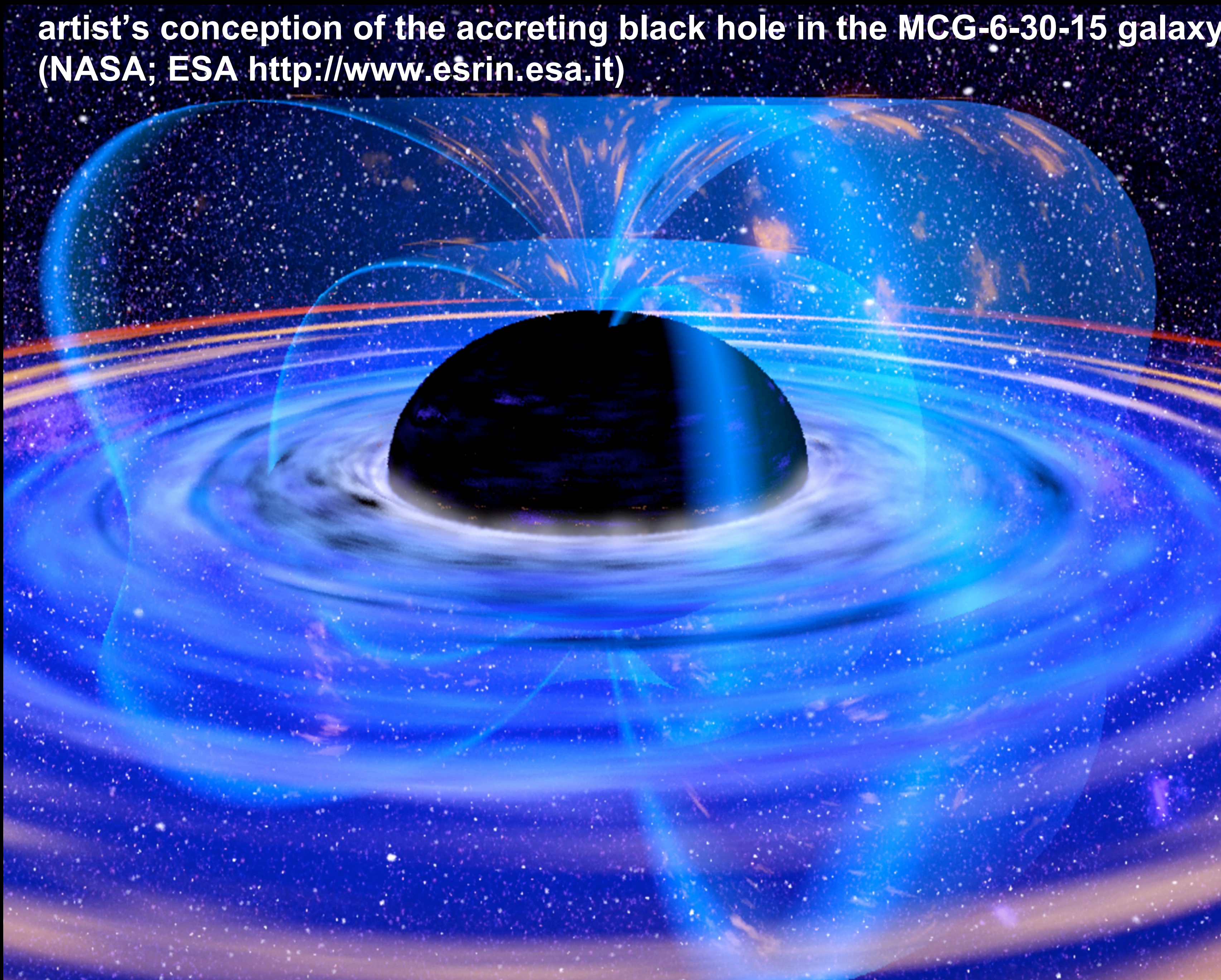
size and composition of a typical galaxy

$$\text{baryonic mass} \sim 10^{11}\,\mathrm{M}_\odot$$

$$\text{total mass (baryons + dark matter)} \sim 10^{12}\,\mathrm{M}_\odot$$

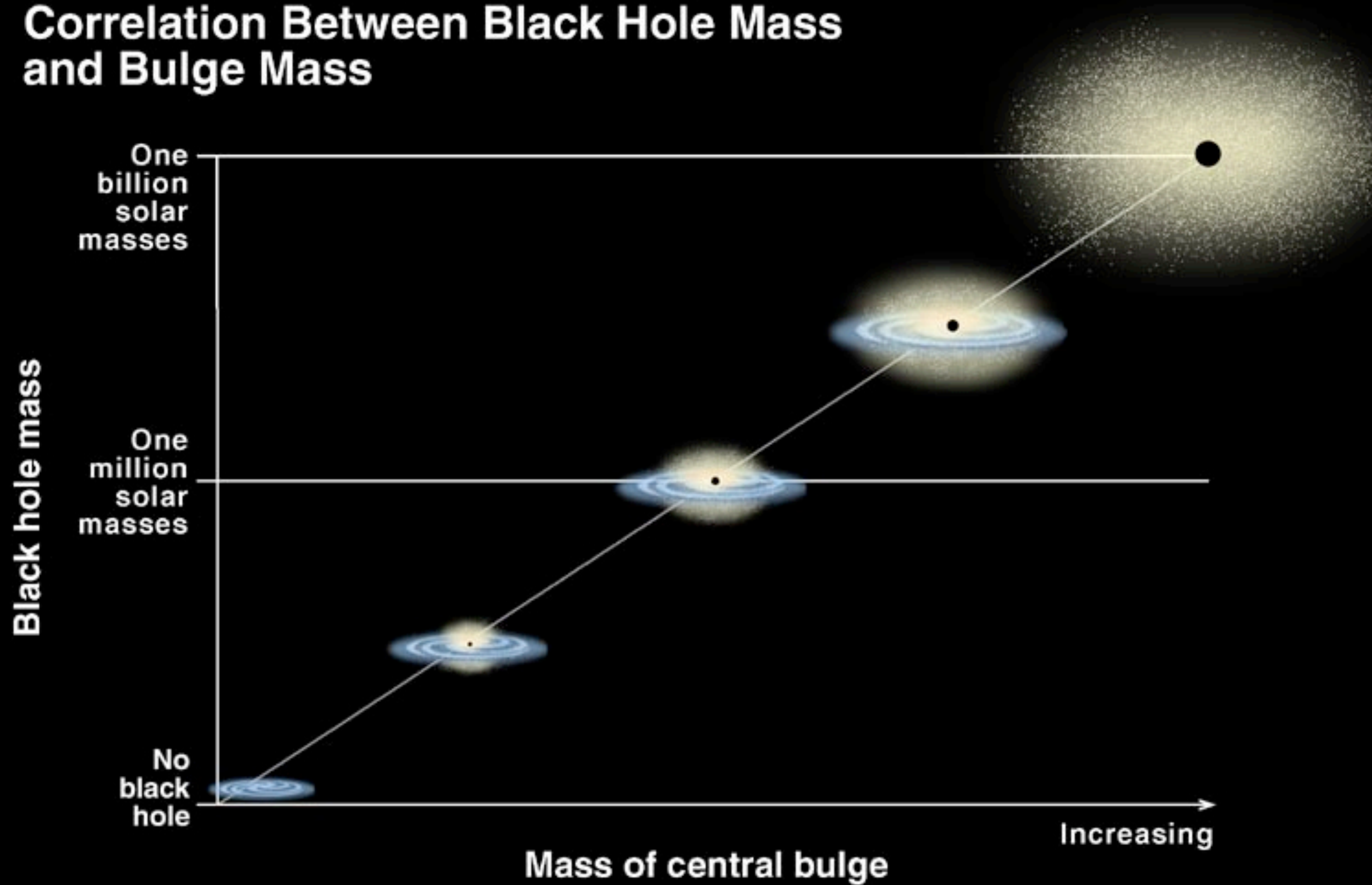


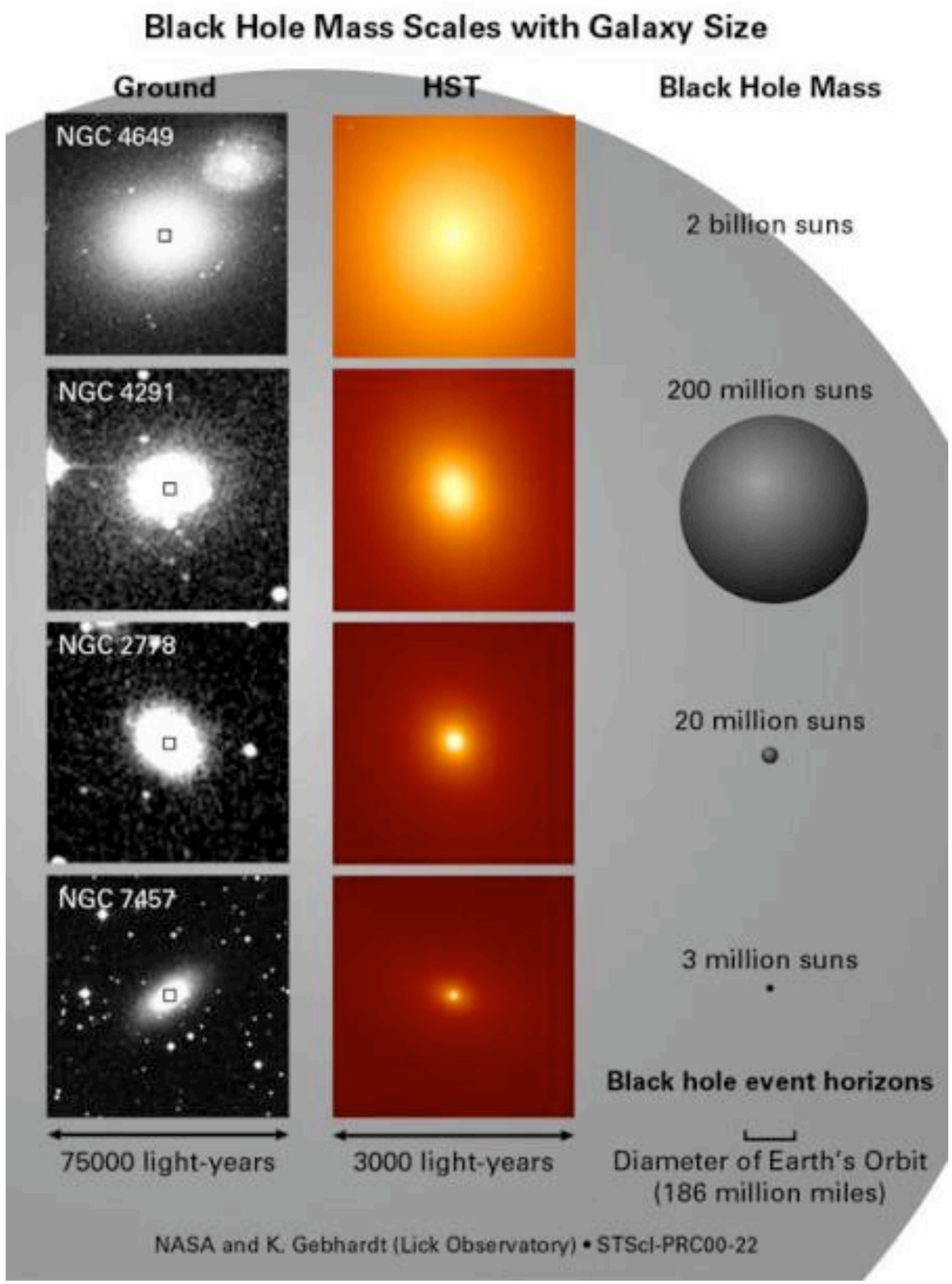
chandra.as.utexas.edu/ ~kormendy/n4216-halo.gif



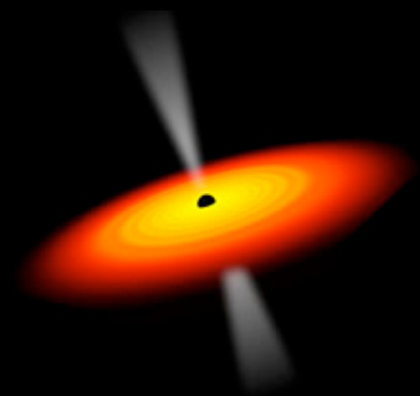
artist's conception of the accreting black hole in the MCG-6-30-15 galaxy
(NASA; ESA http://www.esrin.esa.it)

Correlation Between Black Hole Mass and Bulge Mass



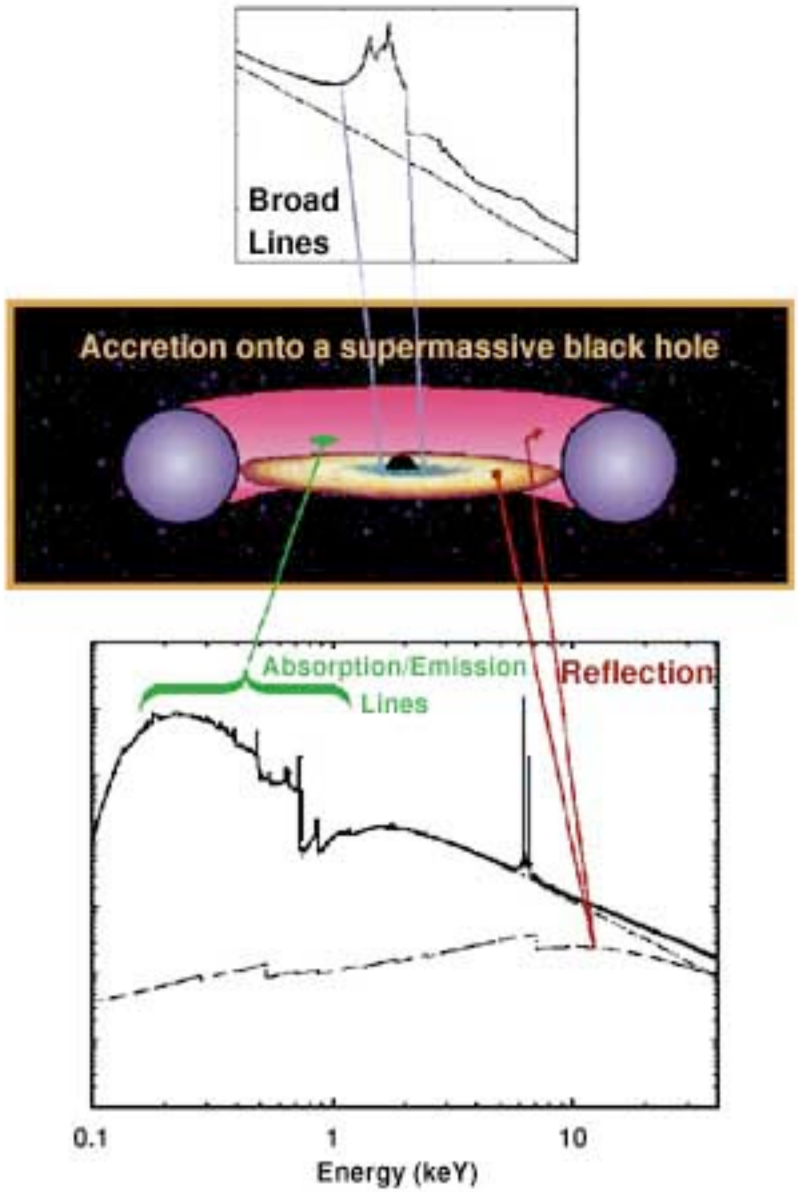

Black Hole Mass Scales with Galaxy Size
Ground
HST
Black Hole Mass
NGC 4649
2 billion suns
NGC 4291
200 million suns
NGC 2778
20 million suns
NGC 7457
3 million suns
Black hole event horizons
75000 light-years
3000 light-years
Diameter of Earth's Orbit
(186 million miles)
NASA and K. Gebhardt (Lick Observatory) • STScI-PRC00-22

The QSO paradigm -
accretion disk and polar jets of relativistic particles around a rotating supermassive black hole.

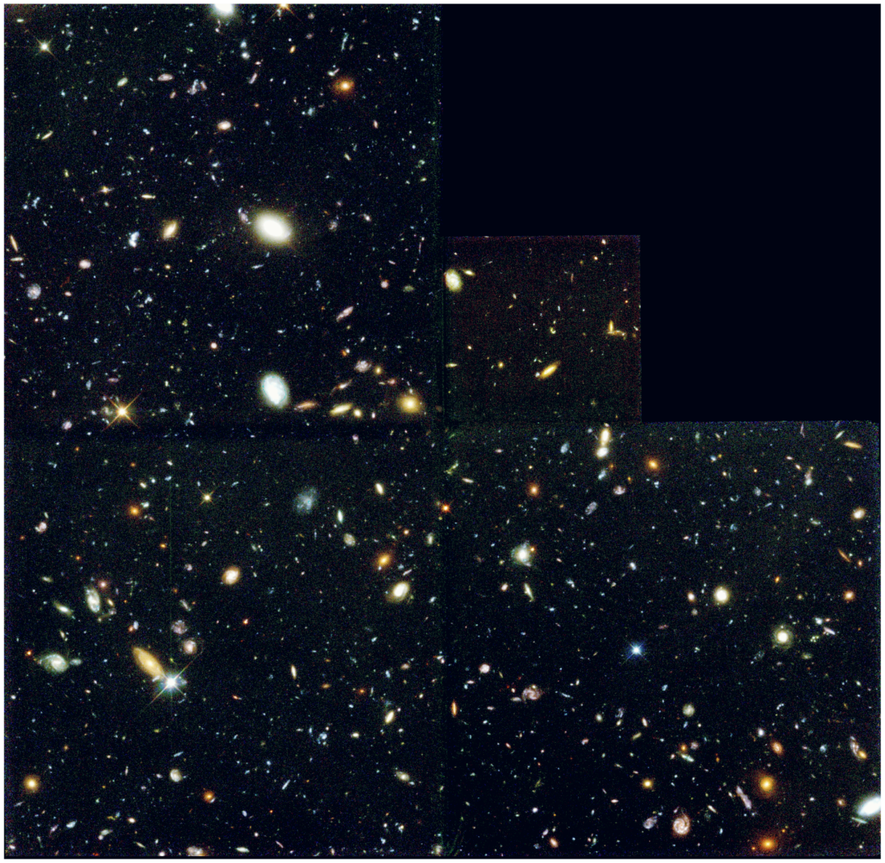


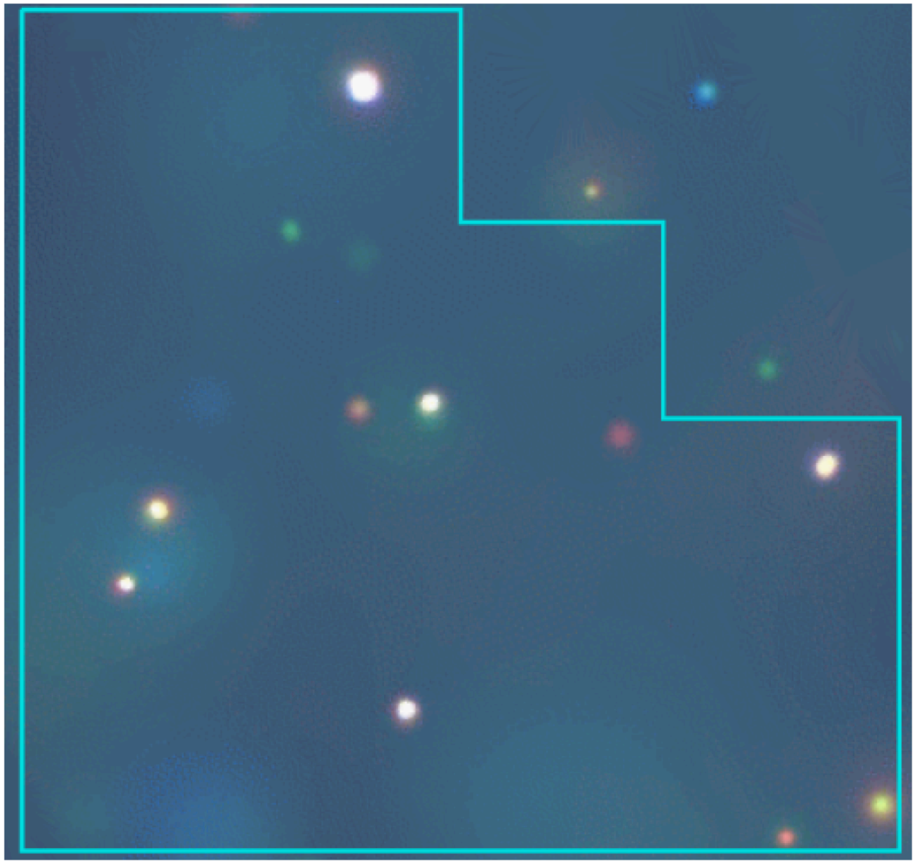
http://www.seasky.or/cosmic/sky7a09.html

accretion disk & associated emission from supermassive BH



http://cossc.gsfc.nasa.gov/epo/brochures/new_win/nw12.html

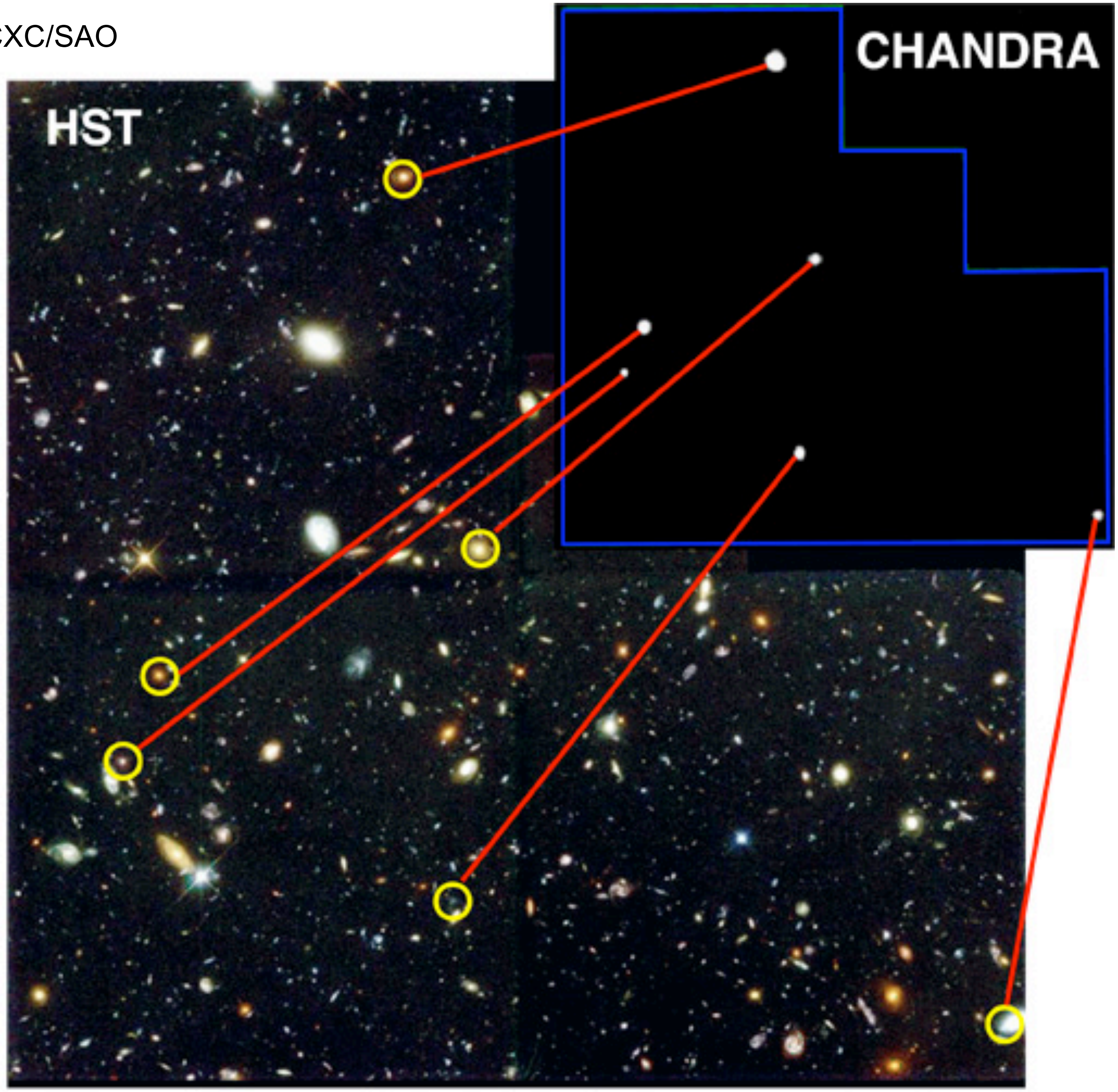




Optical image from the Hubble Space Telescope (HST) of the “Hubble Deep Field” region of the sky

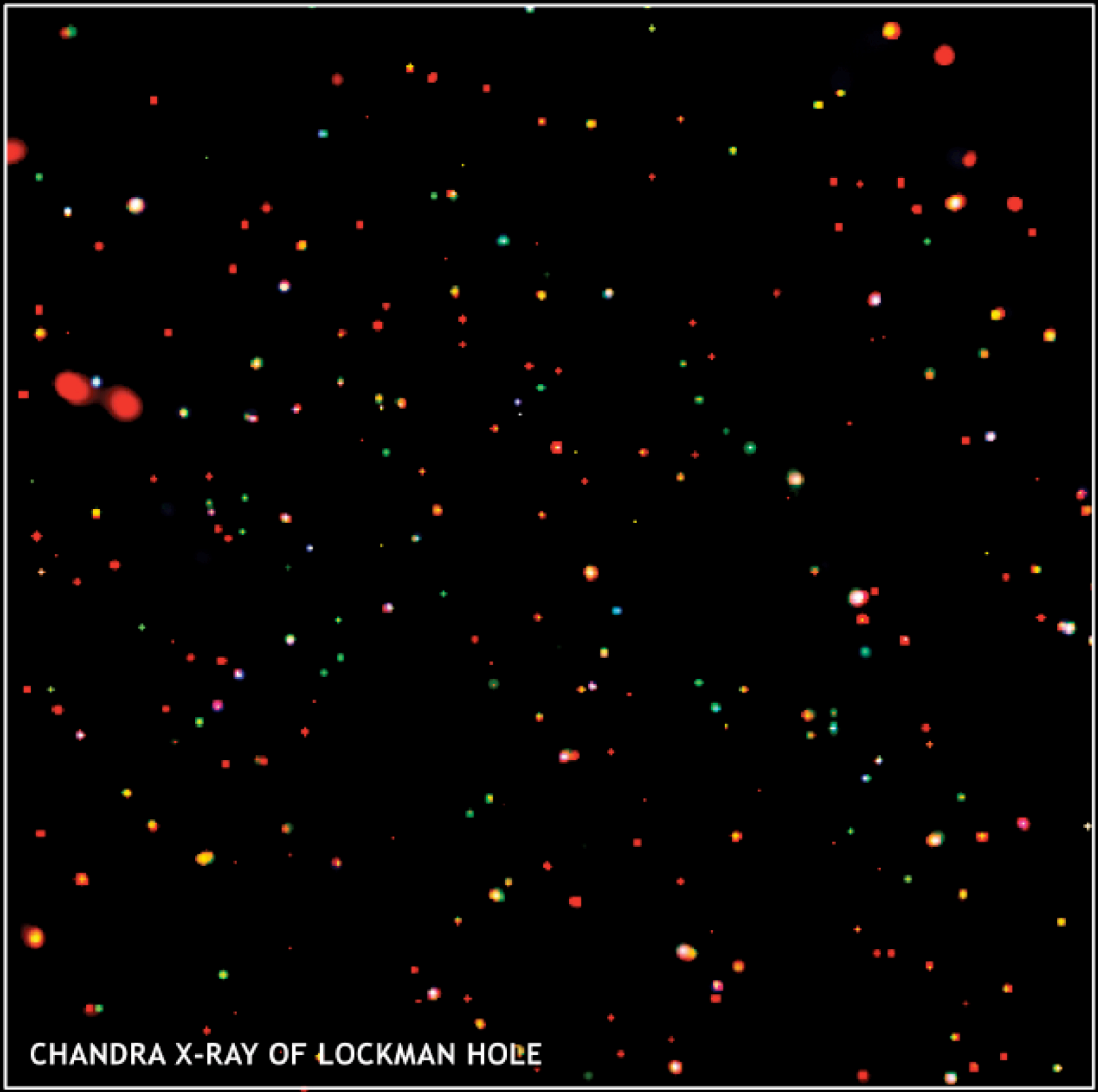
X-Ray image of the same patch of sky by the Chandra X-Ray Observatory (“Chandra Deep Field”)

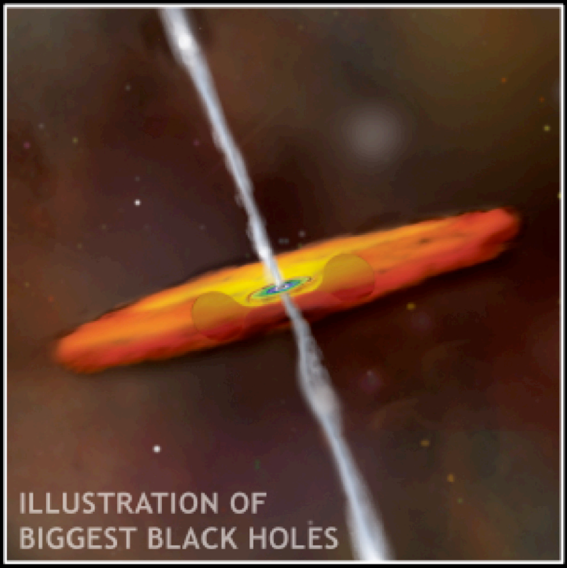
NASA/CXC/SAO

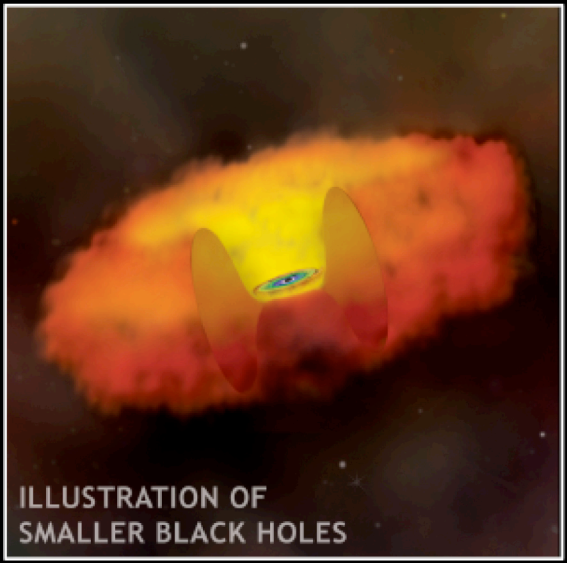


Identifications of optical HST sources and Chandra x-ray sources - these are accreting supermassive BH's.

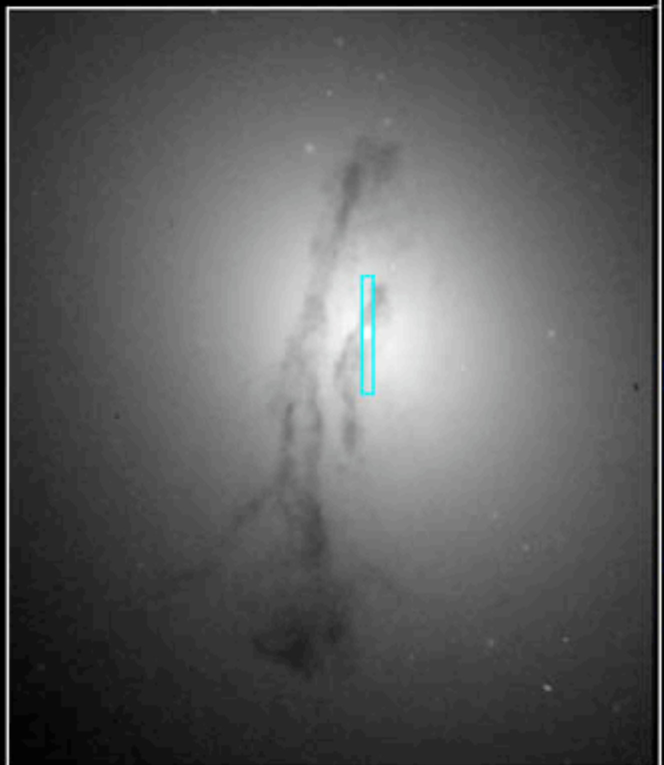
Chandra x-ray image of a region of the sky with little absorption called the Lockman Hole - it reveals many extra-galactic supermassive black holes.
(http://www.science.nasa.gov/headlines/images/blackhole)



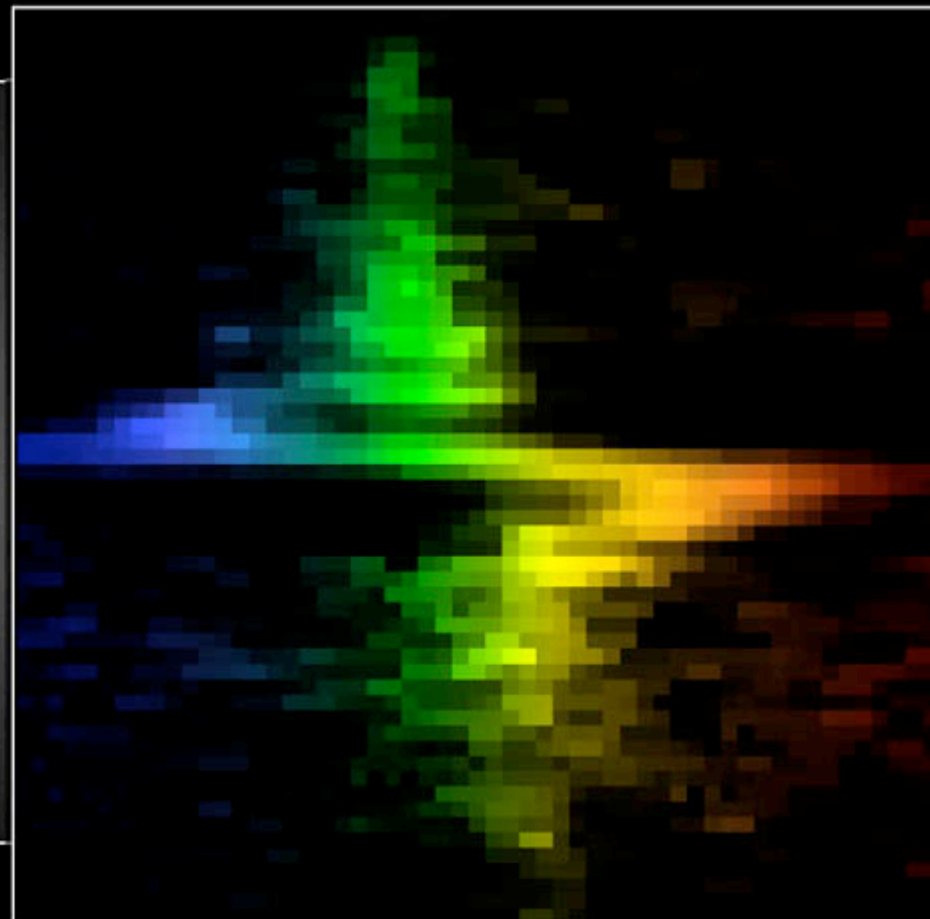




Galaxy M84 Nucleus



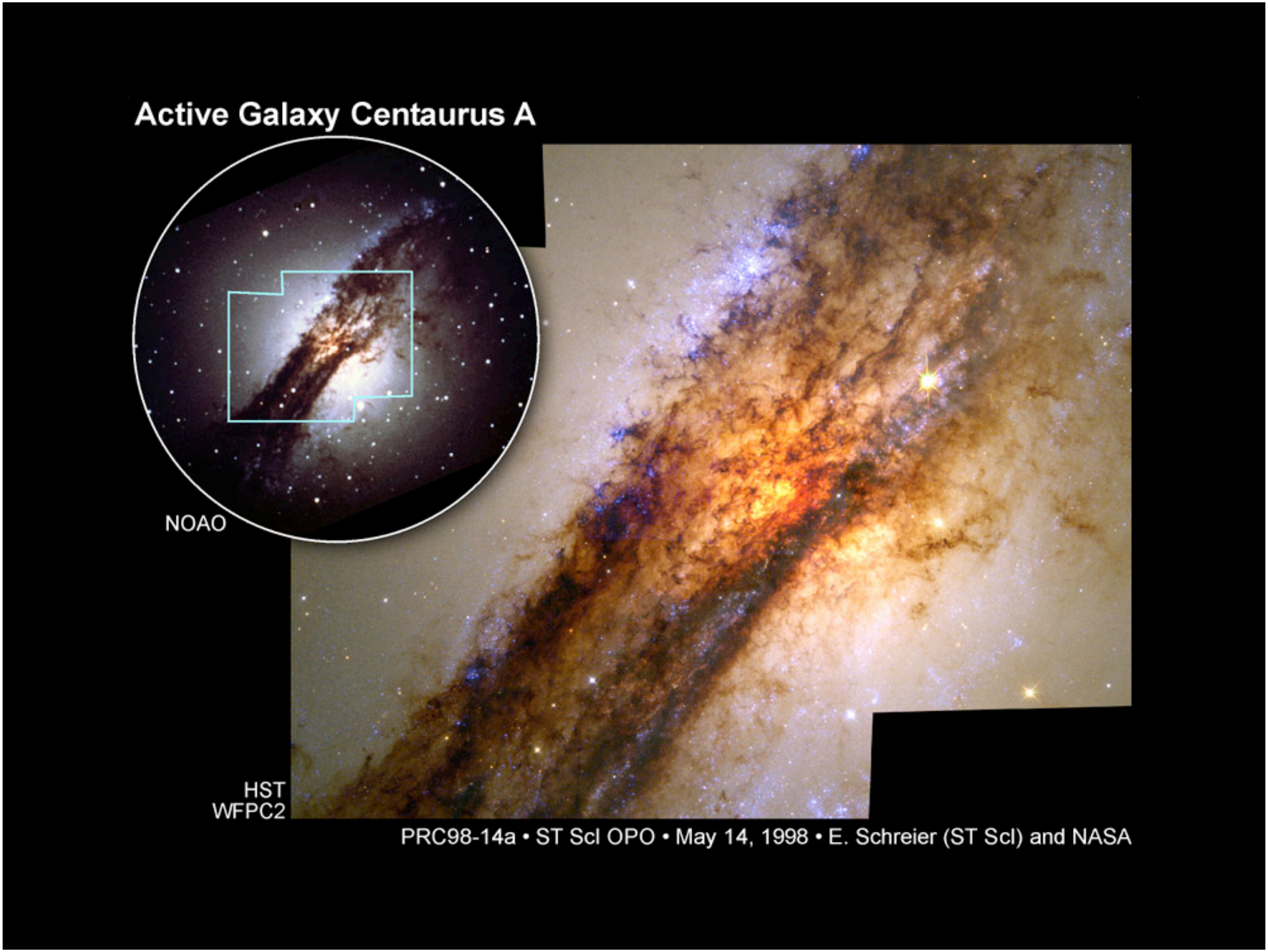
WFPC2

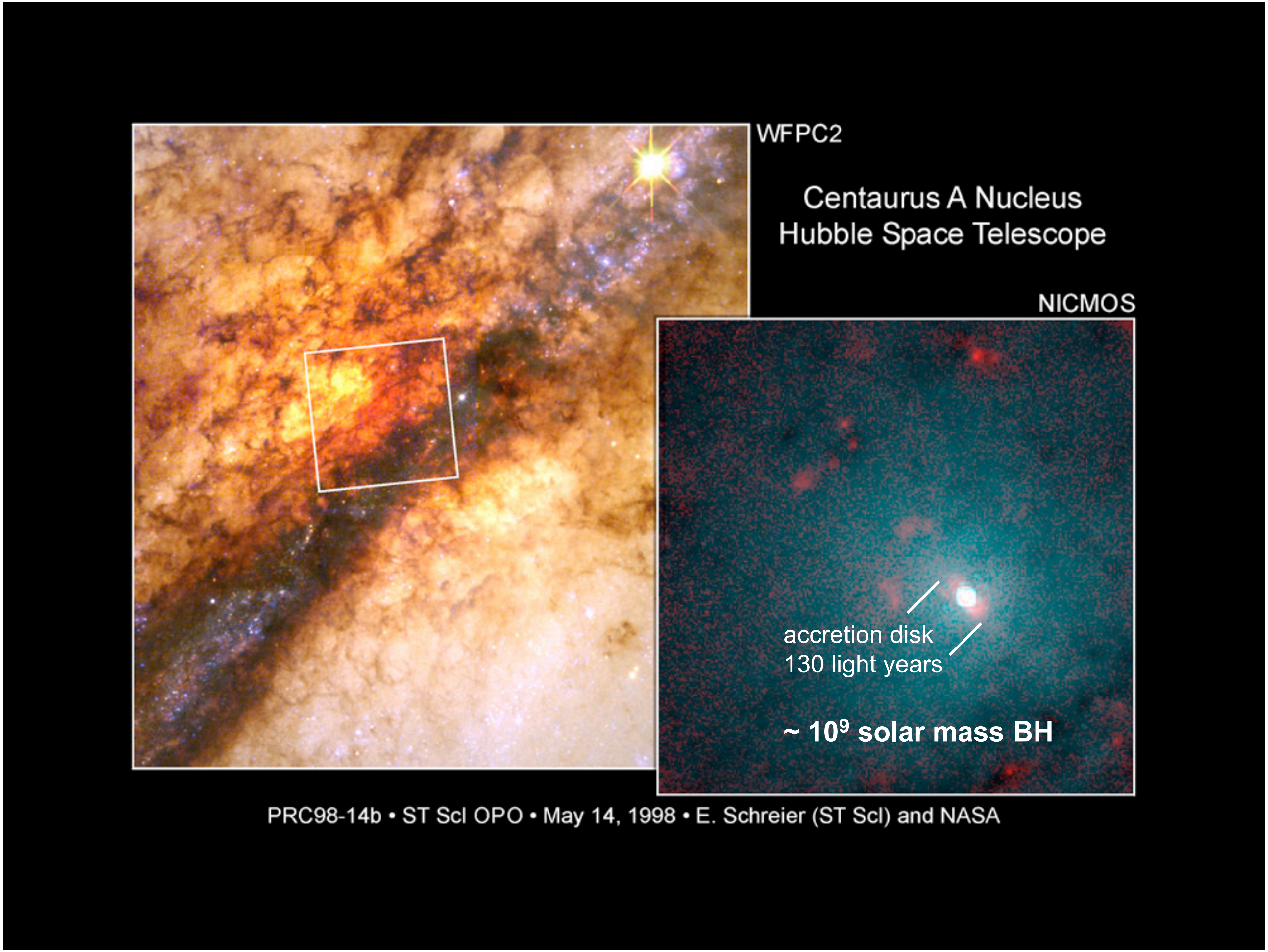


STIS

Hubble Space Telescope

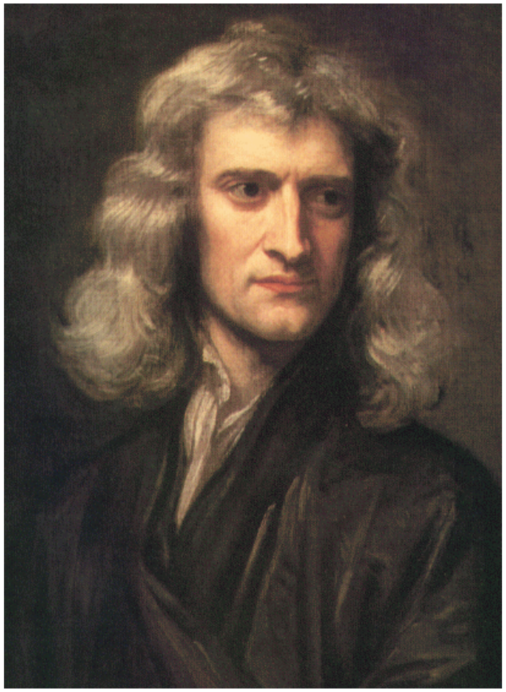
PRC97-12 • ST ScI OPO • May 12, 1997 • B. Woodgate (GSFC), G. Bower (NOAO) and NASA


Active Galaxy Centaurus A
NOAO
HST
WFPC2
PRC98-14a • ST Scl OPO • May 14, 1998 • E. Schreier (ST Scl) and NASA



WFPC2
Centaurus A Nucleus
Hubble Space Telescope
NICMOS
accretion disk
130 light years
~ 10^9 solar mass BH
PRC98-14b • ST ScI OPO • May 14, 1998 • E. Schreier (ST ScI) and NASA

The Greats of Space & Time & Gravitation



Newton

Einstein

Preliminaries

- Cartesian geometry
- Newtonian Mechanics
- executive summary of the difference of the Newtonian and Einsteinian views of time, space, and gravitation.
- spacetime

Cartesian Coordinates in a flat 2-D space

coordinate system (x, y)

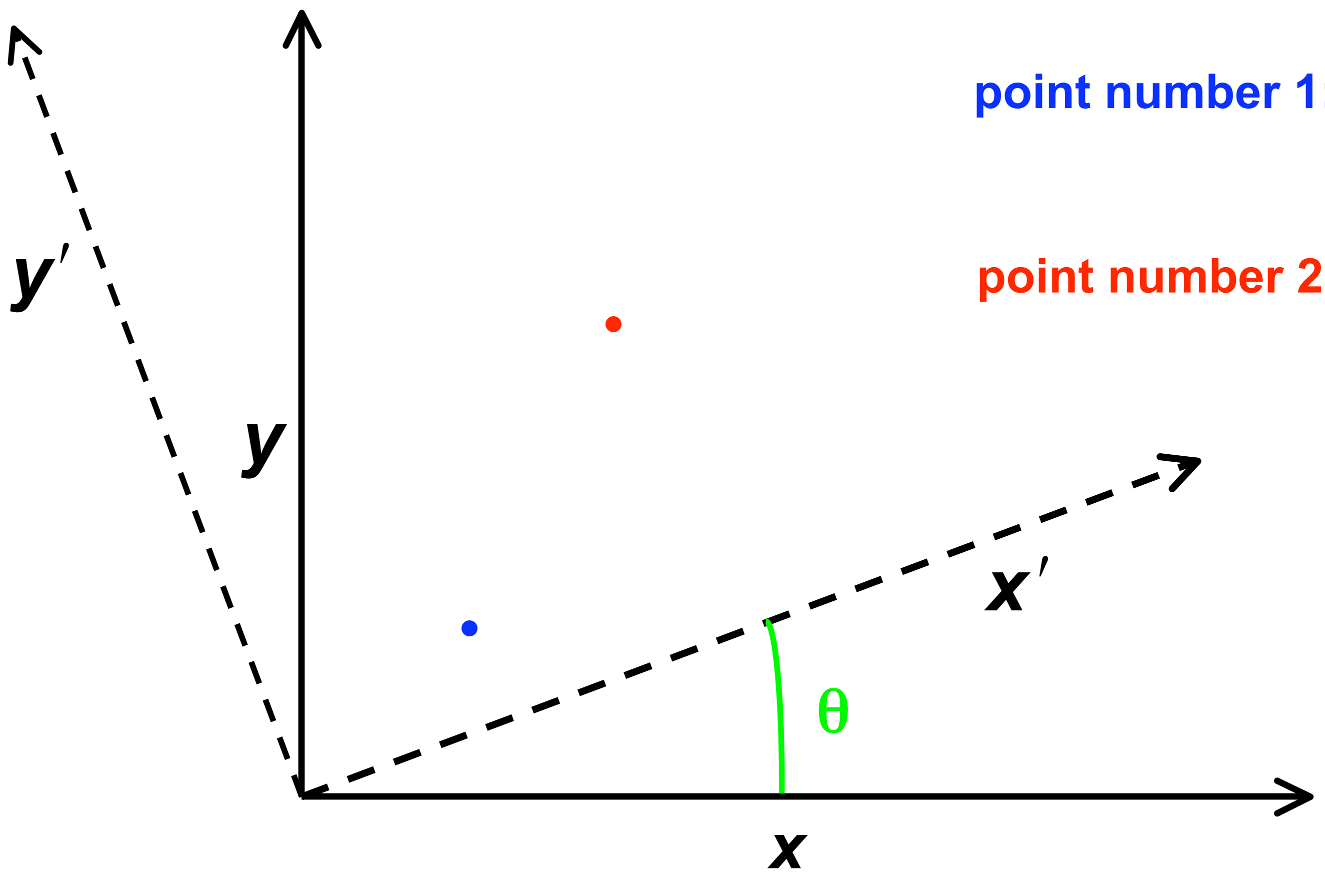
(x', y')

point number 1: (x_1, y_1)

(x_1', y_1')

point number 2: (x_2, y_2)

(x_2', y_2')



invariant *distance-squared* between these points

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2$$

The points in space are real, physical *objects*.
The coordinates that label them depend on a particular set of coordinate axes, the coordinate system.

The relation between the numbers (coordinates) that give the address of point 1 in the two coordinate systems in this case is:

$$\mathrm{x}_1' = \cos\theta\,\mathrm{x}_1 + \sin\theta\,\mathrm{y}_1$$

$$\mathrm{y}_1' = -\sin\theta\,\mathrm{x}_1 + \cos\theta\,\mathrm{y}_1$$

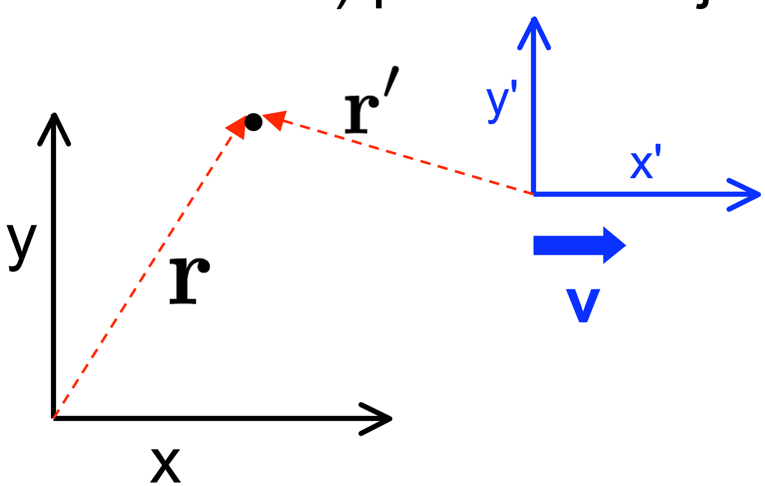
The generalization of these considerations to 3 dimensions is straightforward and obvious. The distance (and distance-squared) between two points is the same when computed in *ANY* coordinate system.

Distance (and distance-squared) is an *invariant*, meaning it is the same in all coordinate systems.

This 3-D Cartesian (Euclidean) space is well known to you. It serves as the frame work for all of Newtonian Mechanics.

Newtonian Dynamics

- View motion and acceleration of particles/objects in ***inertial coordinate systems*** - these are, *e.g.*, Cartesian coordinate systems that move at constant velocity with respect to each other. (They are un-accelerated coordinate systems.)
- This way, all coordinate systems (inertial frames) agree on the accelerations of (and so the forces on) particles/objects.



$$\mathbf{r} = \mathbf{r}' + \mathbf{v}\,t$$

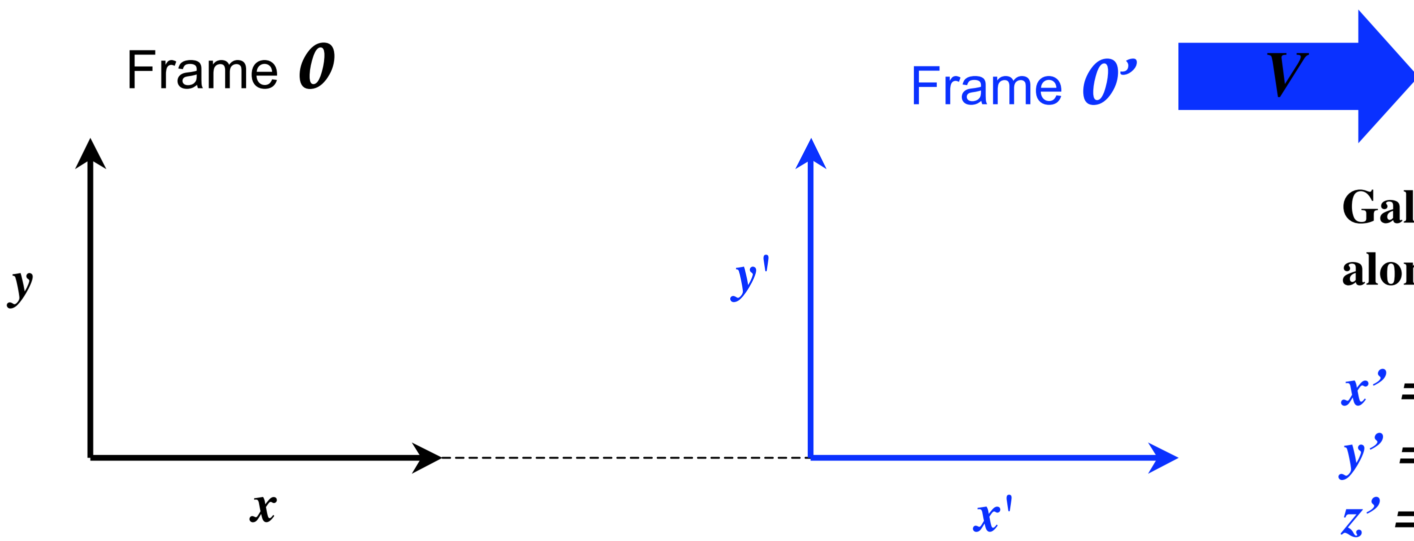
$$m\frac{d^2\mathbf{r}}{dt^2} = m\frac{d^2\mathbf{r}'}{dt^2} = \mathbf{F}$$

“inertial mass” = m

Time *t* is universal here!!!

Galilean Relativity

Given some inertial frame (coordinate system), we can construct another by either rotating the axes (three independent angles in three dimensions) and/or by “boosting” to another set of axes that are moving with respect to the first with some (constant) velocity. See example of a boost on the previous slide, or this one . . .



Galilean boost along x-axis of O

$x' = x - V\,t$

$y' = y$

$z' = z$

Newtonian Gravitation

- Space is filled with inertial coordinate systems, each of which covers all of space (the axes can be extended to infinity in any direction - they are *global*), and each of which has the *SAME* time parameter *t*.

- There are gravitational forces between mass points. From Newton's **F**=m**a** law we can compute the accelerations of particles if we know their gravitational and inertial masses and we use the Newtonian gravitational force law:

$$\mathbf{F}_{ij} = \frac{G\, m_{ig}\, m_{jg}}{r^2}\, \mathbf{e}_{ij}$$

e.g., accleration of particle *i* is $\mathbf{a}_i = \mathbf{F}_{ij}/m_{i\,\mathrm{inert}}$

The inertial and *gravitational* masses need not be equal, only proportional.

Newtonian gravitation is *linear* so that gravitational forces just add up linearly. If the mass density throughout space is just given by ρ then the gravitational potential energy at some point $\mathbf{r}$ is given by . . .

$$\varphi(\mathbf{r}) = -\mathrm{G} \int \frac{\rho(\mathbf{r}')\, d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

. . . and the gravitational force on a point mass with mass m at this location is . . .

$$\mathbf{F} = -m\nabla\varphi$$

Einsteinian Gravitation

There are only *local* inertial coordinate systems (only valid and *inertial* in a small patch of space & time) in a four-dimensional spacetime in which time is not universal, but rather one of the coordinates.

There are NO gravitational forces. Particles move on *locally* straight lines, as determined by the local inertial coordinate systems.

Inertial and gravitational masses are identical - the *Equivalence Principle*.

The orientation and relationship of the neighboring inertial coordinate systems ("curvature") is determined by the local content of mass-energy through the Einstein Field Equation - this is *nonlinear*.

Einstein's Relativity

The speed of light is the same when measured in *ANY* coordinate system.

(Speed of a light beam as measured by any observer, no matter what his state of motion, is always the same value.)

Einstein was led to this by experiment: Maxwell's equations for electricity and magnetism; the Michelson-Morely experiment; the Eotvos experiment.

→ **Special Relativity** (1905)
global inertial coordinate systems; no gravitation

→ **General Relativity** (1916)
local inertial coordinate systems; equivalence principle

Dimensions

We know of four dimensions:
3 *spacelike* dimensions (look around you)
1 *timelike* dimension (look at your watch)

So, we live in a (3 + 1 =) 4 dimensional *spacetime*.

But time and length do not have the same "units," do they? We measure, for example, length in *meters* and time in *seconds*.

Let's agree to measure time and length with the same units.

We shall multiply time *t* by the speed of light in vacuum, *c*, a universal constant.

$$c \approx 2.9979 \times 10^8 \, \mathrm{m\,s}^{-1}$$

But we are going to be a little tricky with this. Whenever you see *t*, we will really mean *c t* .

Now, the slang used by physicists for this convention is to say that "we have set *c* = 1".

Notice that with this convention all speeds and the magnitudes of all velocities are *dimensionless.*

$$v = \frac{x}{t} \equiv \frac{x}{c\,t} = \frac{(x/t)}{c} = \frac{v}{c}$$

With this convention, if something is moving across a coordinate system with speed $v = 1$ it is moving at the speed of light, at $v = 0.5$ it is moving at half the speed of light, at $v = 0.1$ it is moving at a tenth light speed, *etc*.

Einstein's theory of RELATIVITY is the study of the geometry of this 4-dimensional space (*spacetime*).

Einstein's theory of relativity does NOT say that everything is relative! Far from it. In fact, a better name for it would be Einstein's theory of *invariants*, quantities that are the *same* for all "observers," that is, coordinate systems.

"observer" = coordinate system

the invariants = "geometric objects,"
***e.g.,* points (events),**
arrows (vectors),
slices/contours (nested surfaces),
etc.

The events (points) in spacetime are real, physical *objects*. The coordinates that label them depend on a particular set of coordinate axes, the coordinate system.

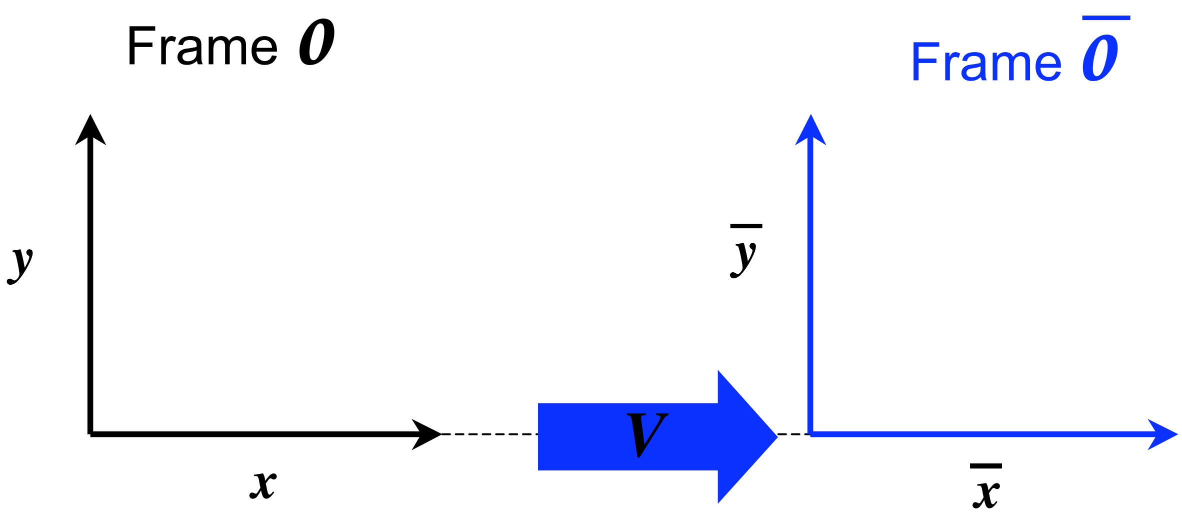
In thinking about spacetime, I have found that it is best to think about all events, those that happened in the "past," as well as those that are to happen in the "future," as the same as those happening "now." They are all just as real, just as physical. All together they form the fabric of spacetime.

For most of us this is an unnatural, sometimes even disturbing viewpoint. So, all those bad things you'd like to forget are not gone! They are just as real as this lecture right "now," and just as real as the good things that haven't happened yet!

Einsteinian Relativity

Given some inertial frame (Minkowski coordinate system), we can construct another by either rotating the spacelike axes and/or by “boosting” to another Minkowski coordinate system moving with respect to the first with some velocity.

But now the coordinates in these systems must be related in a way that keeps the speed of light constant!



Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - V^2}}$$

Lorentz boost along x-axis of O

$$\bar{t} = \gamma\, t - V\gamma\, x$$

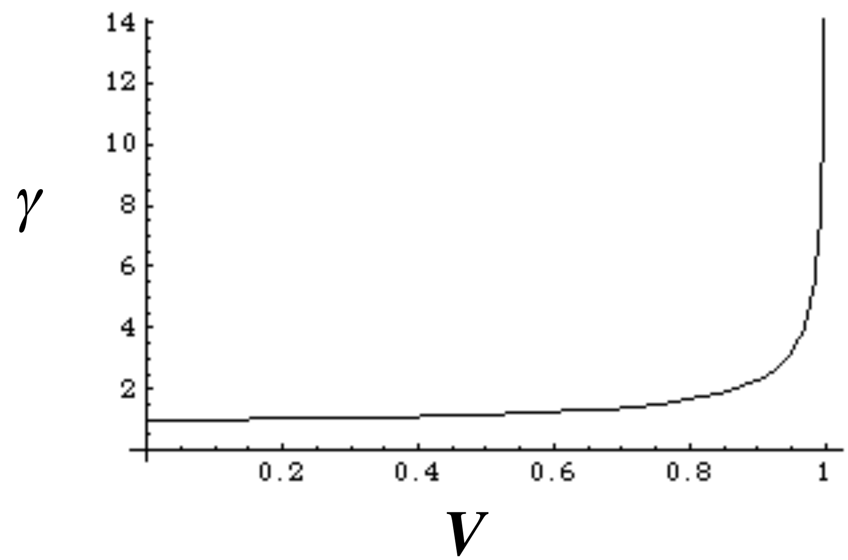
$$\bar{x} = -V\gamma\, t + \gamma\, x$$

$$\bar{y} = y$$

$$\bar{z} = z$$

Lorentz Factor

$$\gamma = \frac{1}{\sqrt{1 - V^2}} \equiv \frac{1}{\sqrt{1 - V^2/c^2}}$$



Spacetime Diagrams

coordinate system (t, x, y, z)

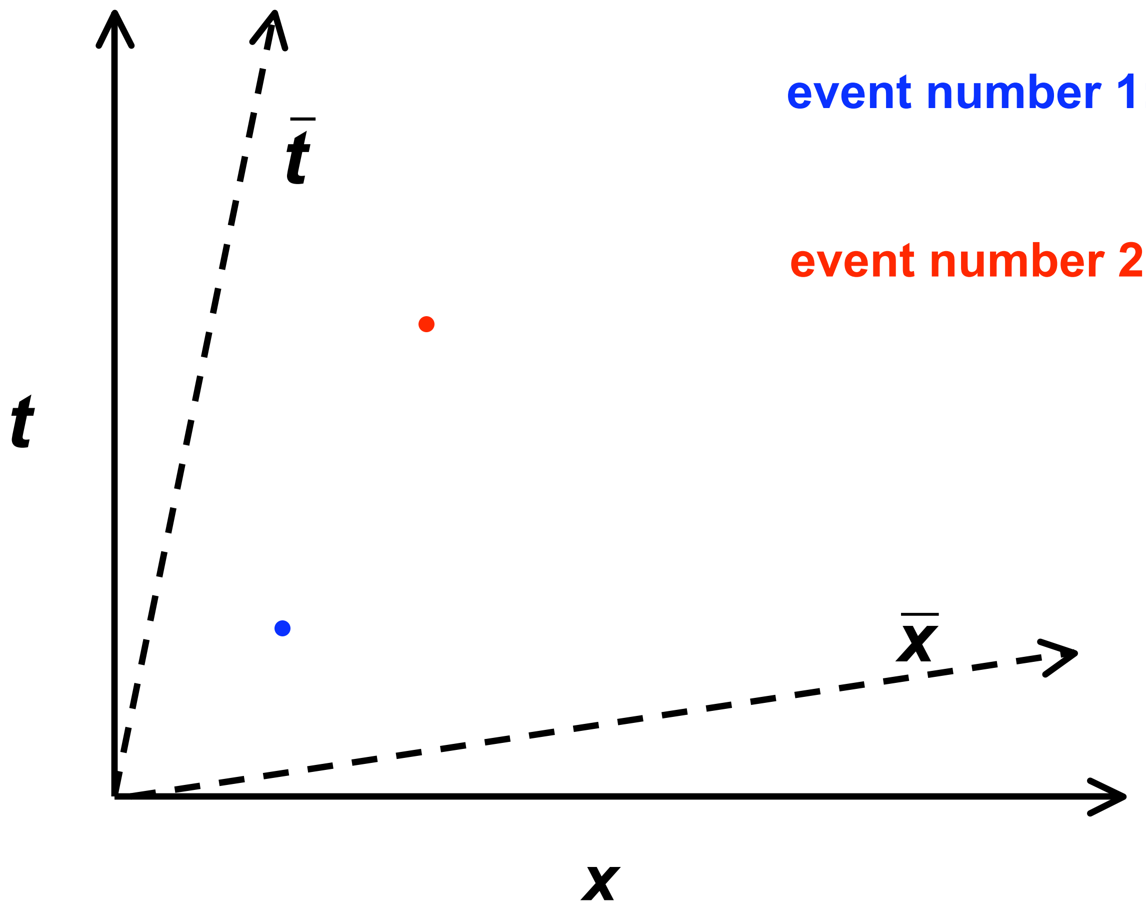
$(\bar{t}, \bar{x}, \bar{y}, \bar{z})$

event number 1: (t_1, x_1, y_1, z_1)

$(\bar{t}_1, \bar{x}_1, \bar{y}_1, \bar{z}_1)$

event number 2: (t_2, x_2, y_2, z_2)

$(\bar{t}_2, \bar{x}_2, \bar{y}_2, \bar{z}_2)$



invariant spacetime interval between these events

$$\Delta s^2 = -(t_2 - t_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = -(\bar{t}_2 - \bar{t}_1)^2 + (\bar{x}_2 - \bar{x}_1)^2 + (\bar{y}_2 - \bar{y}_1)^2 + (\bar{z}_2 - \bar{z}_1)^2$$

The relation between the numbers (coordinates) that give the address of, *e.g.,* point 1 in the two coordinate systems in this case is:

$$
\begin{aligned}
x^{\bar{\mu}} &= \Lambda^{\bar{\mu}}{}_{\nu}\, x^{\nu} \\
&= \Lambda^{\bar{\mu}}{}_{0}\, x^{0} + \Lambda^{\bar{\mu}}{}_{1}\, x^{1} + \Lambda^{\bar{\mu}}{}_{2}\, x^{2} + \Lambda^{\bar{\mu}}{}_{3}\, x^{3}
\end{aligned}
$$

$$
\text{where } t_1 = x^0,\ \ x_1 = x^1,\ \ y_1 = x^2,\ \ z_1 = x^3
$$

All Greek indices take all four spacetime values, 0, 1, 2, 3; all roman indices take only the spacelike values, 1, 2, 3. If an index is repeated (one “up” and one “down” or *vice versa)* then it is summed on - the Einstein summation convention.

$\boxed{\text{the coordinate transformation is the 16 numbers } \Lambda^{\bar{\mu}}{}_{\nu}}$

In the case of global inertial coordinate systems, where we can extend the 4 axes to infinity in each “direction,” the coordinate transformation has constant values everywhere in spacetime and is called a **Lorentz transformation**.

It must be constructed so that the speed of light is the same in any two coordinate systems.

We can show that this transformation will also leave the spacetime interval invariant - the same in any two coordinate systems.

. . . for our example of a boost along x direction of $\boldsymbol{O}$
the Lorentz transformation elements are . . .

$$\Lambda^{\bar{0}}{}_{0} = \gamma \quad \Lambda^{\bar{0}}{}_{1} = -V\gamma \quad \Lambda^{\bar{1}}{}_{0} = -V\gamma \quad \Lambda^{\bar{1}}{}_{1} = \gamma$$

$$\Lambda^{\bar{2}}{}_{2} = 1 \quad \Lambda^{\bar{3}}{}_{3} = 1 \quad \text{all others } 0$$

a matrix representation of this coordinate transformation is

$$\begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma & -V\gamma & 0 & 0 \\ -V\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Suppose we boost along $\boldsymbol{O}$'s y-axis instead of x ?
The relationship between the coordinates in this case is . . .

$$\begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & -V\gamma & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & -V\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

. . . boost along z . . .

$$\begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -V\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -V\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Spacetime Diagrams

coordinate system (t, x, y, z)

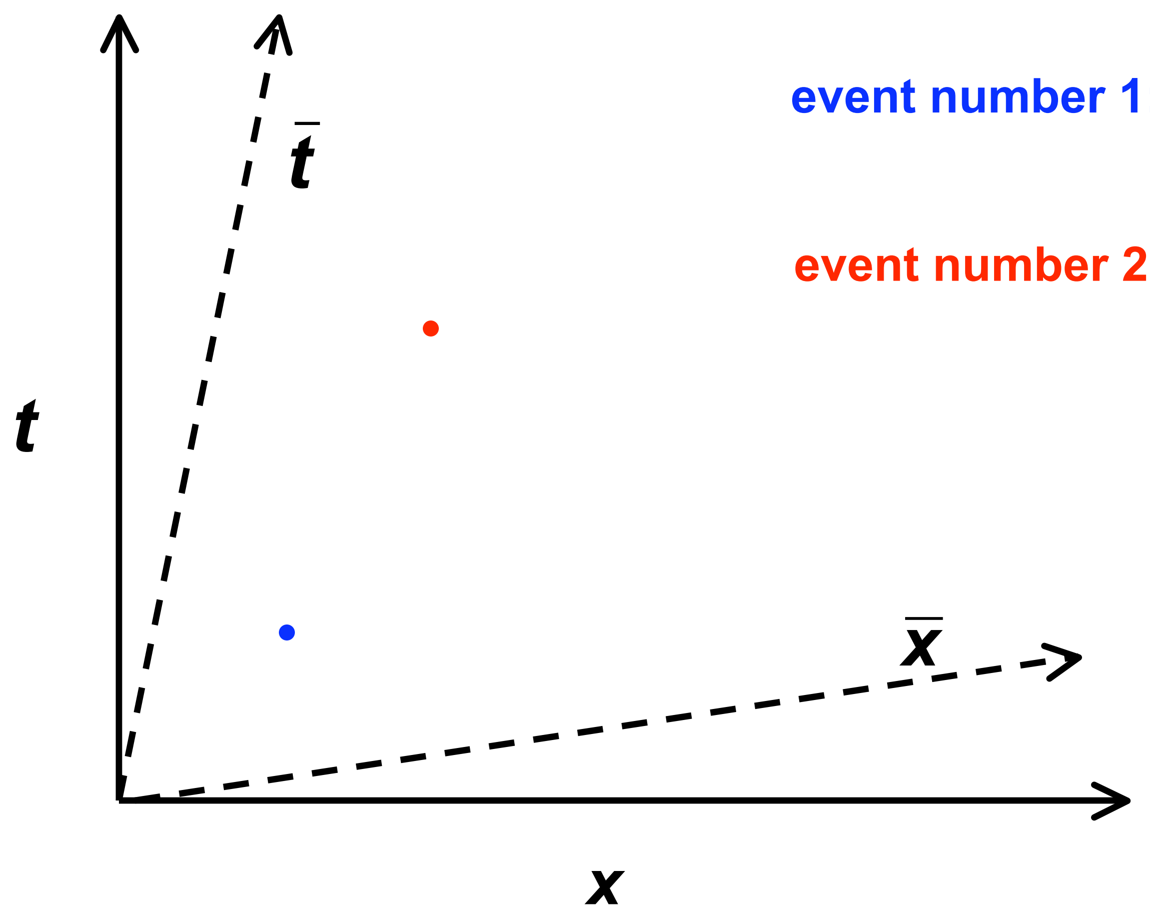
$(\bar{t}, \bar{x}, \bar{y}, \bar{z})$

event number 1: (t_1, x_1, y_1, z_1)

$(\bar{t}_1, \bar{x}_1, \bar{y}_1, \bar{z}_1)$

event number 2: (t_2, x_2, y_2, z_2)

$(\bar{t}_2, \bar{x}_2, \bar{y}_2, \bar{z}_2)$



invariant spacetime interval between these events

$$\Delta s^2 = -(t_2-t_1)^2 + (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 = -(\bar{t}_2 - \bar{t}_1)^2 + (\bar{x}_2 - \bar{x}_1)^2 + (\bar{y}_2 - \bar{y}_1)^2 + (\bar{z}_2 - \bar{z}_1)^2$$

How did we plot the axes of coordinate system (observer) $\bar{O}$?

First note that the ***origin*** of $\bar{O}$ ‘s coordinate system moves along O ‘s x-axis with speed V so its coordinates are . . .

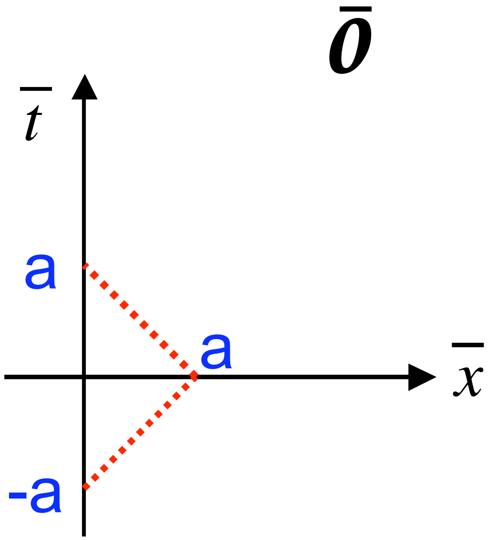
$$x = Vt \quad \text{or} \quad t = \left(\frac{1}{V}\right)x$$

This is $\bar{O}$ ‘s time axis.

Getting $\bar{O}$ ‘s $\bar{x}$-axis is a little trickier. We can get it by noting that the speed of light is the same in *both* . . .

$$t = Vx$$

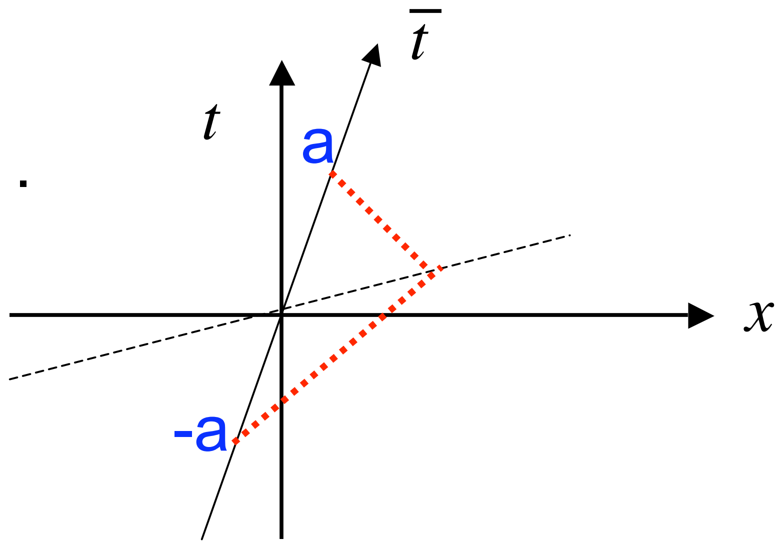
Note that light beams ········ are always straight lines with 45^0 slope in *all* spacetime diagrams - at least for the way we have set them up with $c=1$.

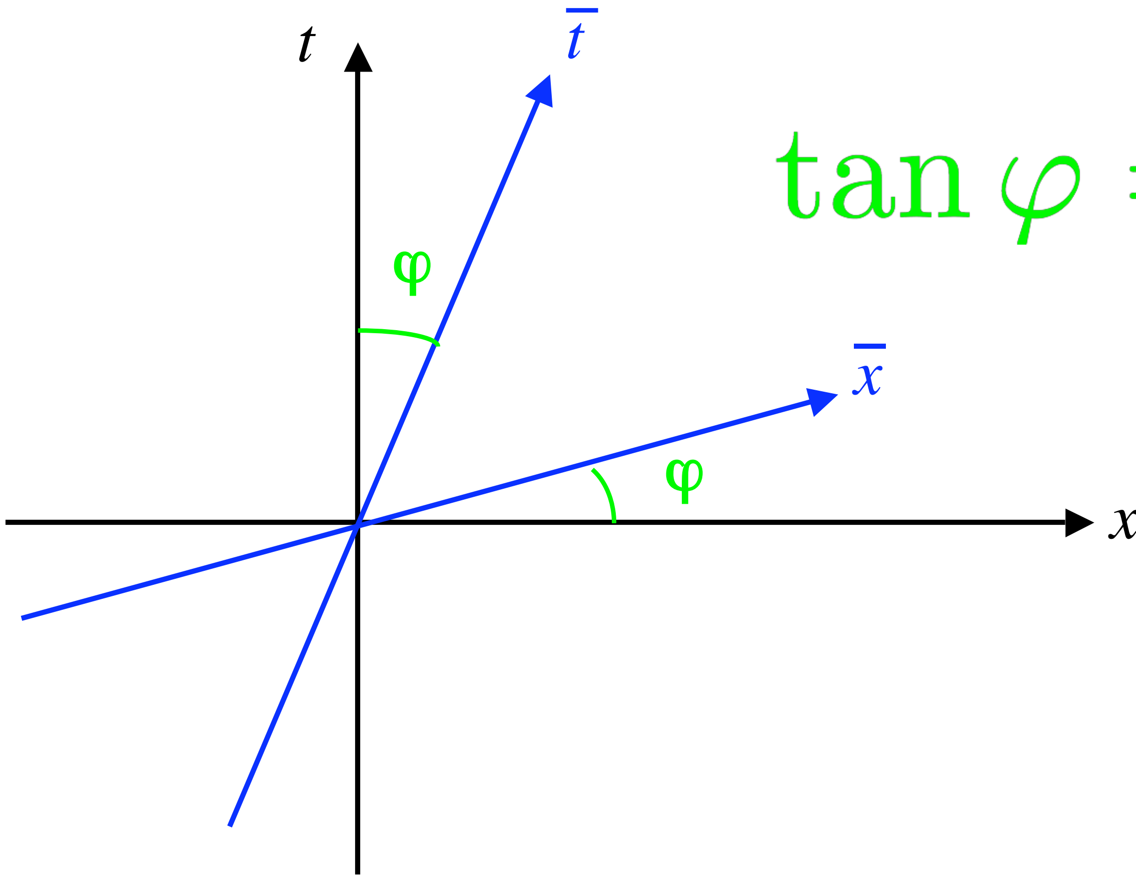


$\bar{x}$-axis : those events which $\bar{\mathcal{O}}$ measures to be *simultaneous* to $\bar{t}=\bar{x}=0$.

Light beam moves out and bounces off mirror at $\bar{x} = \mathrm{a}$.

. . . as “seen” in $\mathcal{O}$ . . .

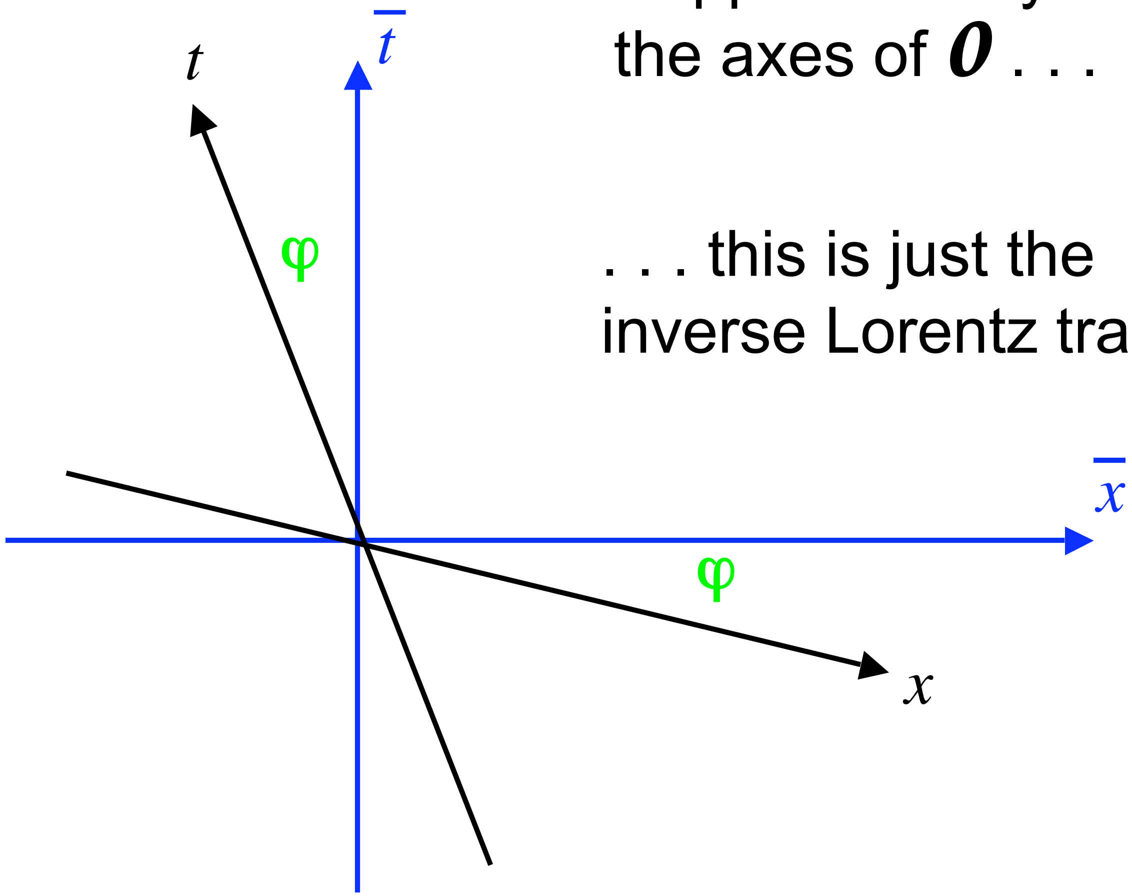




$$\tan\varphi = V$$

. . . boost by V along positive x-axis of $\boldsymbol{O}$ . . .

Suppose now you sit in $\bar{\boldsymbol{O}}$ and plot the axes of $\boldsymbol{O}$ . . .



. . . this is just the inverse Lorentz transformation . . .

$$\tan\varphi = V$$

. . . boost by -V along $\bar{x}$-axis of $\bar{\boldsymbol{O}}$ . . .

A matrix representation of this inverse Lorentz transformation from $\bar{\boldsymbol{O}}$ to $\boldsymbol{O}$ is . . .

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & V\gamma & 0 & 0 \\ V\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}$$

$$x^{\mu} = \Lambda^{\mu}{}_{\bar{\nu}}\ x^{\bar{\nu}}$$

$$x^{\mu} = \Lambda^{\mu}{}_{\bar{\nu}}\; x^{\bar{\nu}}$$

But from before we had that . . . $x^{\bar{\nu}} = \Lambda^{\bar{\nu}}{}_{\beta}\; x^{\beta}$

$$\Rightarrow \quad x^{\mu} = \underbrace{\overset{\text{boost back}}{\Lambda^{\mu}{}_{\bar{\nu}}}\; \overset{\text{boost out}}{\Lambda^{\bar{\nu}}{}_{\beta}}}_{\delta^{\mu}_{\beta}}\; x^{\beta}$$

$$\delta^{\mu}_{\beta} = \begin{cases} = 1 \text{ if } \mu = \beta \\ = 0 \text{ if } \mu \neq \beta \end{cases}$$

Note that coordinate differentials transform under Lorentz transformations just as the coordinates themselves do . .

$$\text{e.g.,} \quad \Delta x^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\alpha} \, \Delta x^{\alpha}$$

This goes for *infinitesimal* coordinate intervals as well . . .

$$\text{e.g.,} \quad dx^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\alpha} \, dx^{\alpha}$$

Failure of Simultaneity

coordinate system (t, x, y, z)

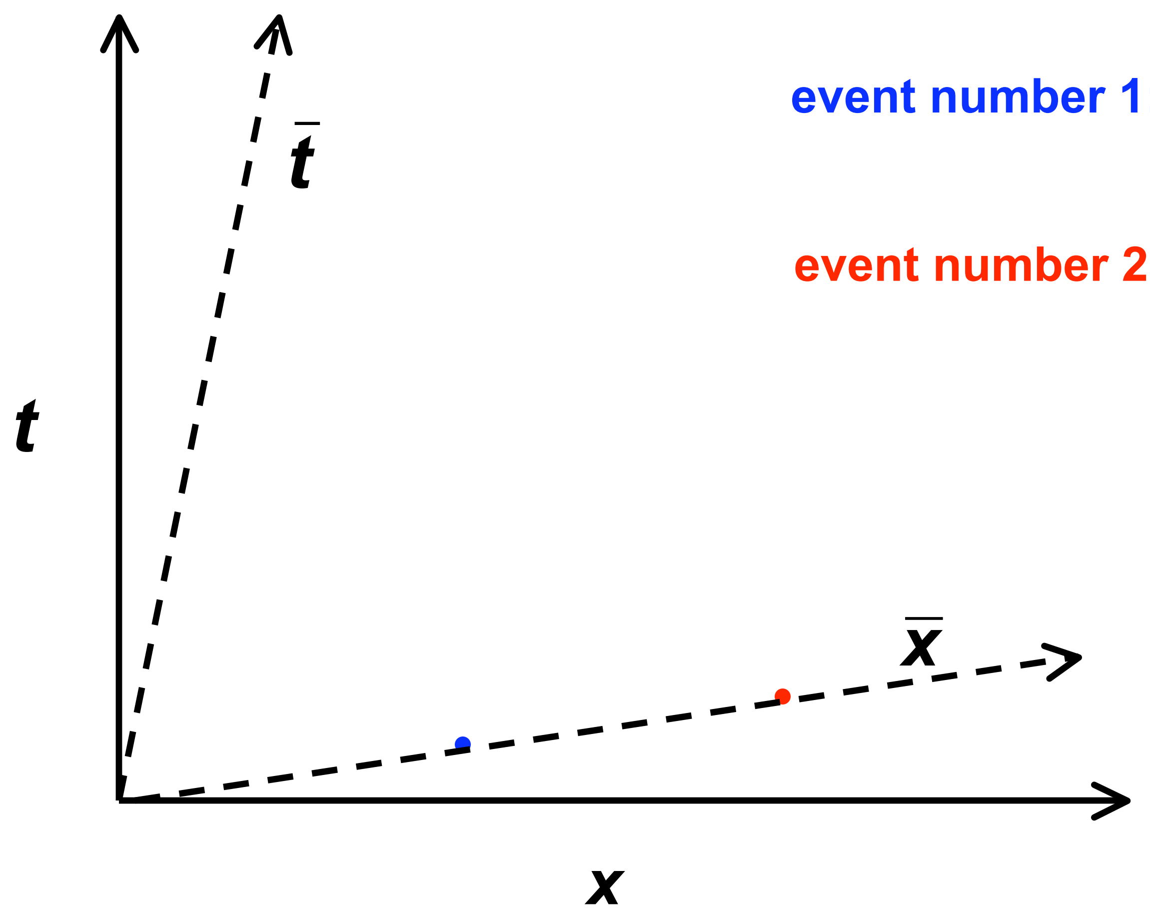
$(\bar{t}, \bar{x}, \bar{y}, \bar{z})$

event number 1: (t_1, x_1, y_1, z_1)

$(\bar{t}_1, \bar{x}_1, \bar{y}_1, \bar{z}_1)$

event number 2: (t_2, x_2, y_2, z_2)

$(\bar{t}_2, \bar{x}_2, \bar{y}_2, \bar{z}_2)$



These events are simultaneous in $\bar{O}$ but not in O

$t_2 - t_1 > 0$ but $\bar{t}_2 - \bar{t}_1 = 0$

Spacetime Diagrams

coordinate system (t, x, y, z)

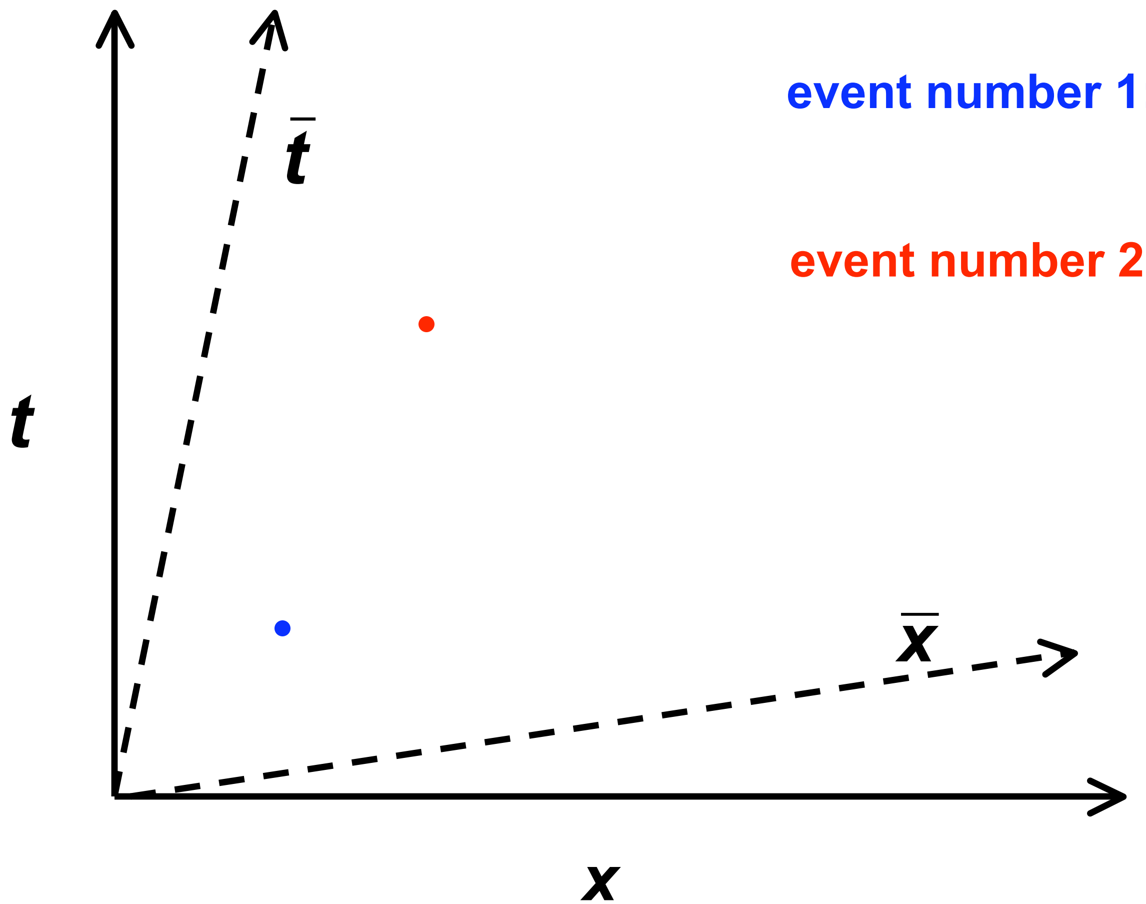
$(\bar{t}, \bar{x}, \bar{y}, \bar{z})$

event number 1: (t_1, x_1, y_1, z_1)

$(\bar{t}_1, \bar{x}_1, \bar{y}_1, \bar{z}_1)$

event number 2: (t_2, x_2, y_2, z_2)

$(\bar{t}_2, \bar{x}_2, \bar{y}_2, \bar{z}_2)$



invariant spacetime interval between these events

$$\Delta s^2 = -(t_2-t_1)^2+(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2 = -(\bar{t}_2-\bar{t}_1)^2+(\bar{x}_2-\bar{x}_1)^2+(\bar{y}_2-\bar{y}_1)^2+(\bar{z}_2-\bar{z}_1)^2$$

Spacetime Interval

$$\begin{aligned}\Delta s^2 &= -\left(\Delta x^0\right)^2 + \left(\Delta x^1\right)^2 + \left(\Delta x^2\right)^2 + \left(\Delta x^3\right)^2 \\ &= -\left(\Delta x^{\bar{0}}\right)^2 + \left(\Delta x^{\bar{1}}\right)^2 + \left(\Delta x^{\bar{2}}\right)^2 + \left(\Delta x^{\bar{3}}\right)^2\end{aligned}$$

. . . always the same in any coordinate system . . .

here, e.g., $\Delta x^0 = t_2 - t_1$, while $\Delta x^{\bar{0}} = \bar{t}_2 - \bar{t}_1$, etc.

The spacetime interval between two events is . . .

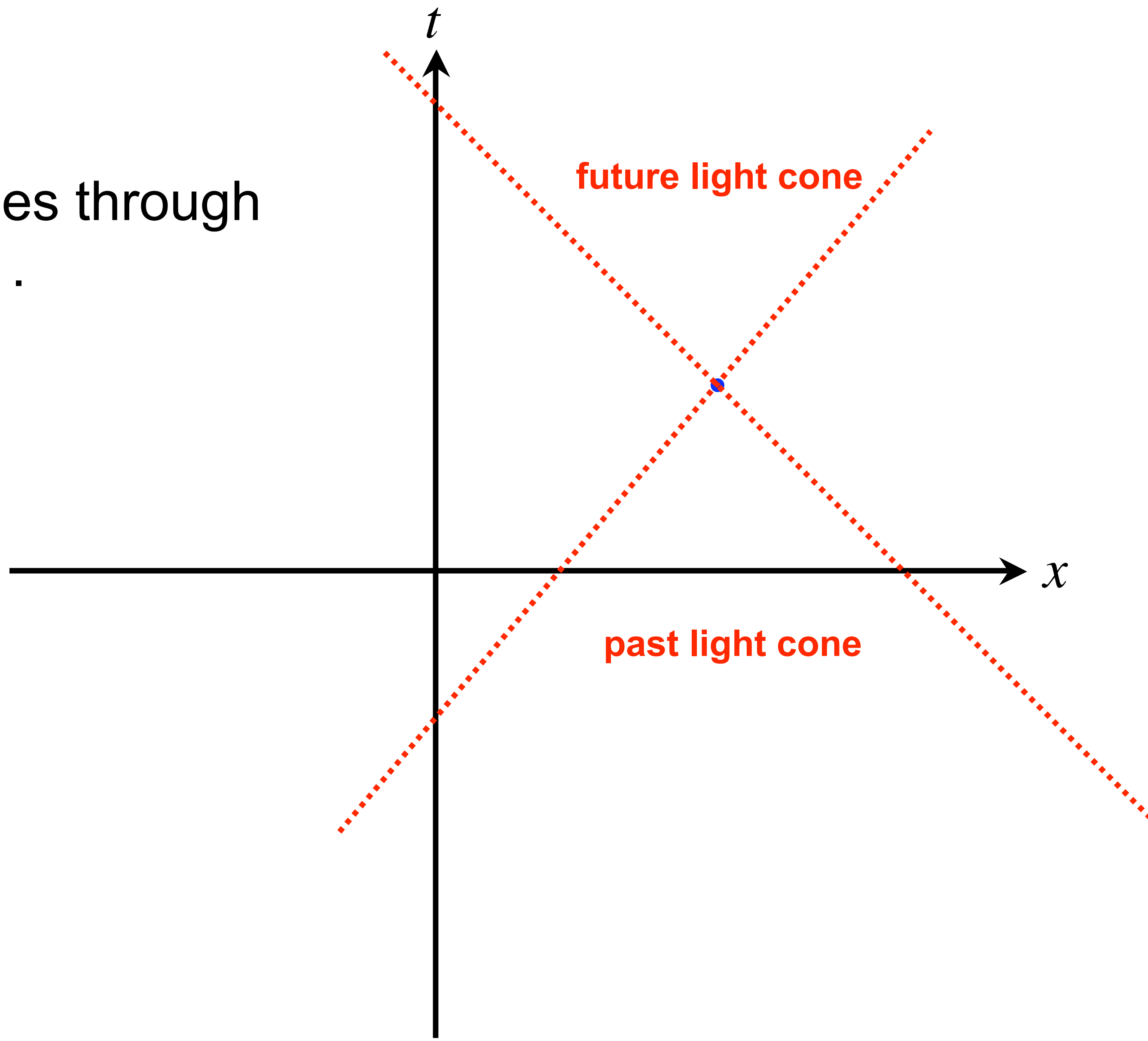
$$\text{spacelike if} \quad \Delta s^2 > 0$$

$$\text{timelike if} \quad \Delta s^2 < 0$$

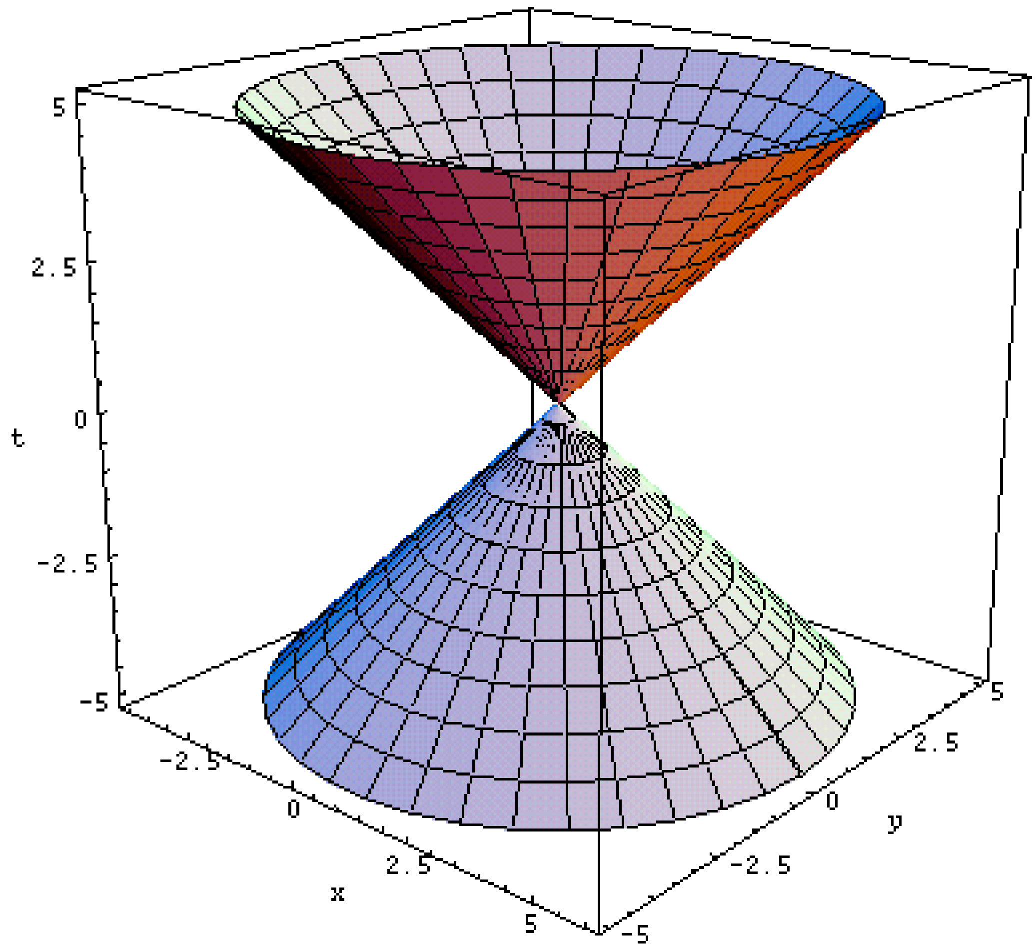
$$\text{null (or lightlike) if} \quad \Delta s^2 = 0$$

This is weird: the spacetime interval, the invariant quantity that replaces "distance" in 2-D and 3-D space, can be positive, zero, and even negative!

add light lines through an event . . .



Light cones - example



www.theory.caltech.edu/people/patricia/lcone2.html

Spacetime Interval

another way to write it . . . $\Delta s^2 = \eta_{\mu\nu}\,\Delta x^\mu\,\Delta x^\nu$

. . . for an *infinitesimal* spacetime interval . . .

$$ds^2 = \eta_{\mu\nu}\,dx^\mu\,dx^\nu$$

a matrix representation of the Minkowski metric is . . .

$$[\eta_{\mu\nu}] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The *components* of the Minkowski metric are . . .

$$\eta_{00} = -1 \qquad \eta_{0i} = \eta_{i0} = 0 \qquad \eta_{ij} = \delta_{ij}$$

The invariance of the spacetime interval means that the Minkowski metric has the same components in all Minkowski coordinate systems . And this means . . .

$$\begin{aligned}\Delta s^2 = \eta_{\mu\nu}\,\Delta x^\mu\,\Delta x^\nu &= \eta_{\mu\nu}\,\left(\Lambda^\mu{}_{\bar\alpha}\,\Delta x^{\bar\alpha}\right)\,\left(\Lambda^\nu{}_{\bar\beta}\,\Delta x^{\bar\beta}\right)\\ &= \left(\eta_{\mu\nu}\,\Lambda^\mu{}_{\bar\alpha}\,\Lambda^\nu{}_{\bar\beta}\right)\Delta x^{\bar\alpha}\,\Delta x^{\bar\beta}\\ &= \eta_{\bar\alpha\bar\beta}\,\Delta x^{\bar\alpha}\,\Delta x^{\bar\beta}\end{aligned}$$

$$\boxed{\eta_{\bar\alpha\bar\beta} = \eta_{\mu\nu}\,\Lambda^\mu{}_{\bar\alpha}\,\Lambda^\nu{}_{\bar\beta}}$$

Proper Time

For *timelike* spacetime intervals we can define the lapse of Proper Time as . . .

$$\Delta\tau \equiv \sqrt{-\Delta s^2}$$

$$d\tau \equiv \sqrt{-ds^2} = \sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}$$

If you are in the rest frame of a clock, the lapse of proper time, an invariant interval between the “ticks” (events) of the clock, is the same as the lapse of coordinate time. Why?

We will define a **vector** (sometimes called a *4-vector*) to be a *geometric object* that in a given coordinate system can be represented by *four numbers* - the components in that coordinate system. We will define these components to transform under Lorentz transformations just the same way coordinate intervals transform.

$$\mathbf{A} \underset{\mathcal{O}}{\Rightarrow} \{A^0, A^1, A^2, A^3\} \Rightarrow A^\mu$$

$$\underset{\bar{\mathcal{O}}}{\Rightarrow} \{A^{\bar{0}}, A^{\bar{1}}, A^{\bar{2}}, A^{\bar{3}}\} \Rightarrow A^{\bar{\alpha}}$$

$$A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\mu} A^\mu$$

A vector like **A** is a real, physical, geometric object which exists independently of any coordinate system.

We can, however, choose a coordinate system and project out components. *Different* coordinate systems will in general have *different* sets of components for this vector - but the vector remains the same.

The rule we have for the way the components transform ensures that the *magnitude* of a vector is frame invariant - because the spacetime interval is invariant.

$$
\begin{aligned}
\mathbf{A} \cdot \mathbf{A} &\equiv \eta_{\mu\nu} \, A^{\mu} \, A^{\nu} \\
&= \eta_{\mu\nu} \, \left(\Lambda^{\mu}{}_{\bar{\alpha}} \, A^{\bar{\alpha}}\right) \, \left(\Lambda^{\nu}{}_{\bar{\beta}} \, A^{\bar{\beta}}\right) \\
&= \left(\eta_{\mu\nu} \, \Lambda^{\mu}{}_{\bar{\alpha}} \, \Lambda^{\nu}{}_{\bar{\beta}}\right) \, A^{\bar{\alpha}} \, A^{\bar{\beta}} \\
&= \eta_{\bar{\alpha}\bar{\beta}} \, A^{\bar{\alpha}} \, A^{\bar{\beta}}
\end{aligned}
$$

The vector $\mathbf{A}$ is . . .

$$\text{timelike if} \quad \mathbf{A} \cdot \mathbf{A} < 0$$

$$\text{spacelike if} \quad \mathbf{A} \cdot \mathbf{A} > 0$$

$$\text{null if} \quad \mathbf{A} \cdot \mathbf{A} = 0$$

Consider a vector $\mathbf{A}$ and a vector $\mathbf{B}$. We can define their inner product in terms of their components in some coordinate system as . . .

$$\mathbf{A} \cdot \mathbf{B} \equiv \eta_{\mu\nu} \, A^{\mu} \, B^{\nu}$$

This is frame invariant, so, *e.g.,*

$$\mathbf{A} \cdot \mathbf{B} = \eta_{\mu\nu} \, A^{\mu} \, B^{\nu} = \eta_{\bar{\alpha}\bar{\beta}} \, A^{\bar{\alpha}} \, B^{\bar{\beta}}$$

Why? Consider the vector $\mathbf{C} = \mathbf{A} + \mathbf{B}$. Clearly, $\mathbf{C}\cdot\mathbf{C}$ is invariant and so $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ is as well!

Two vectors are *orthogonal*
if their inner product vanishes.

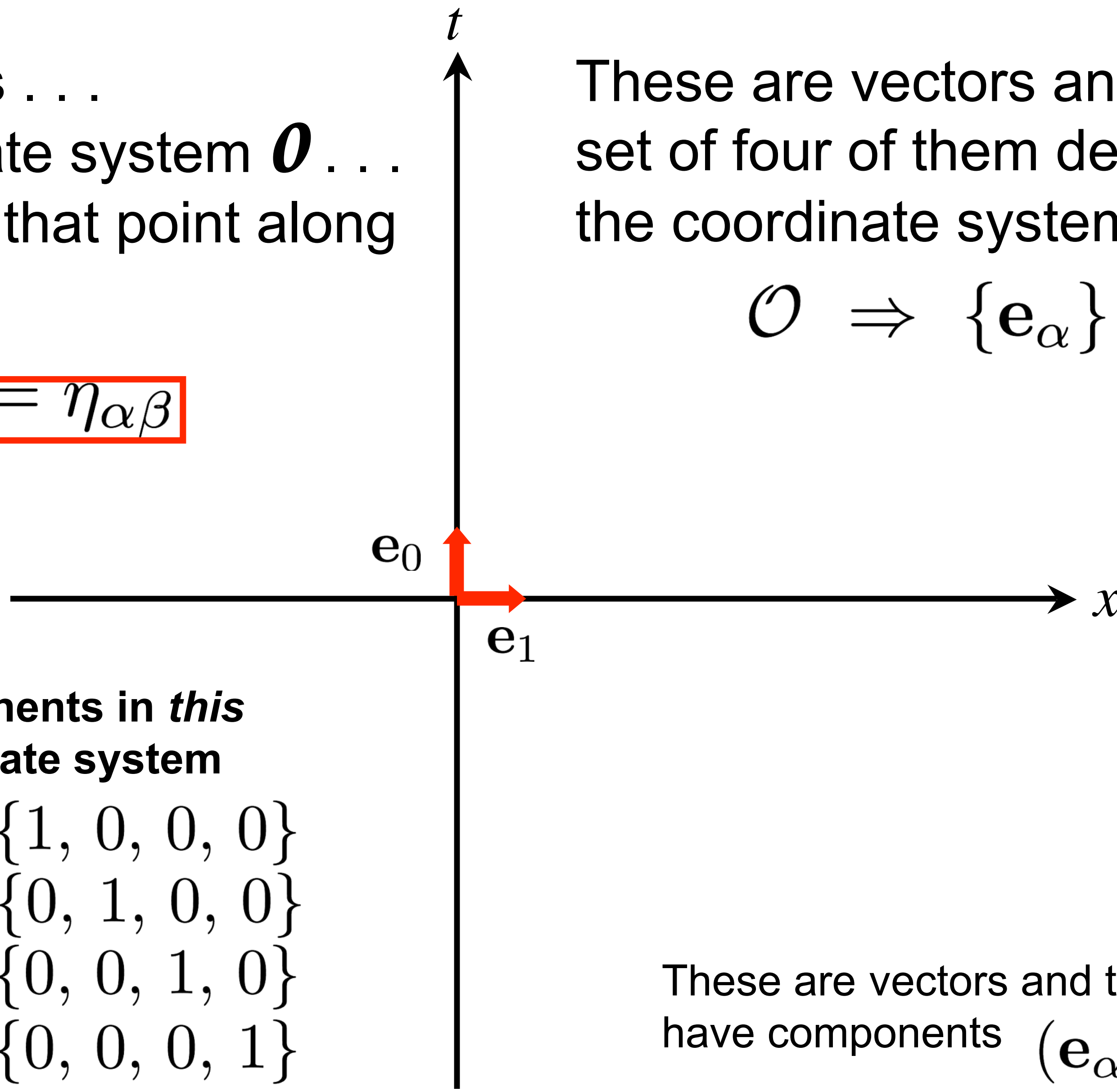
By this definition a null vector
is orthogonal to itself!

Basis Vectors . . .
for a coordinate system ***O*** . . .
"unit" vectors that point along axes

$$\boxed{\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \eta_{\alpha\beta}}$$

These are vectors and the set of four of them define the coordinate system ***O***

$$\mathcal{O} \;\Rightarrow\; \{\mathbf{e}_\alpha\}$$



Components in *this* coordinate system

$$\mathbf{e}_0 \;\Rightarrow\; \{1,\ 0,\ 0,\ 0\}$$
$$\mathbf{e}_1 \;\Rightarrow\; \{0,\ 1,\ 0,\ 0\}$$
$$\mathbf{e}_2 \;\Rightarrow\; \{0,\ 0,\ 1,\ 0\}$$
$$\mathbf{e}_3 \;\Rightarrow\; \{0,\ 0,\ 0,\ 1\}$$

These are vectors and they have components $(\mathbf{e}_\alpha)^\mu$

$$e.g.,\ \ (\mathbf{e}_0)^1 = 0,\ \ (\mathbf{e}_1)^1 = 1,\ \ (\mathbf{e}_3)^3 = 1$$

Note the utility of these basis vectors . . .
Any vector can be written as a linear combination of the basis vectors . . .vectors are geometric objects

$$\mathbf{A} = A^{\mu}\,\mathbf{e}_{\mu} = A^{\bar{\alpha}}\,\mathbf{e}_{\bar{\alpha}}$$

$$= \left(\Lambda^{\mu}{}_{\bar{\alpha}}\,A^{\bar{\alpha}}\right)\mathbf{e}_{\mu} = A^{\bar{\alpha}}\,\left(\Lambda^{\mu}{}_{\bar{\alpha}}\,\mathbf{e}_{\mu}\right)$$

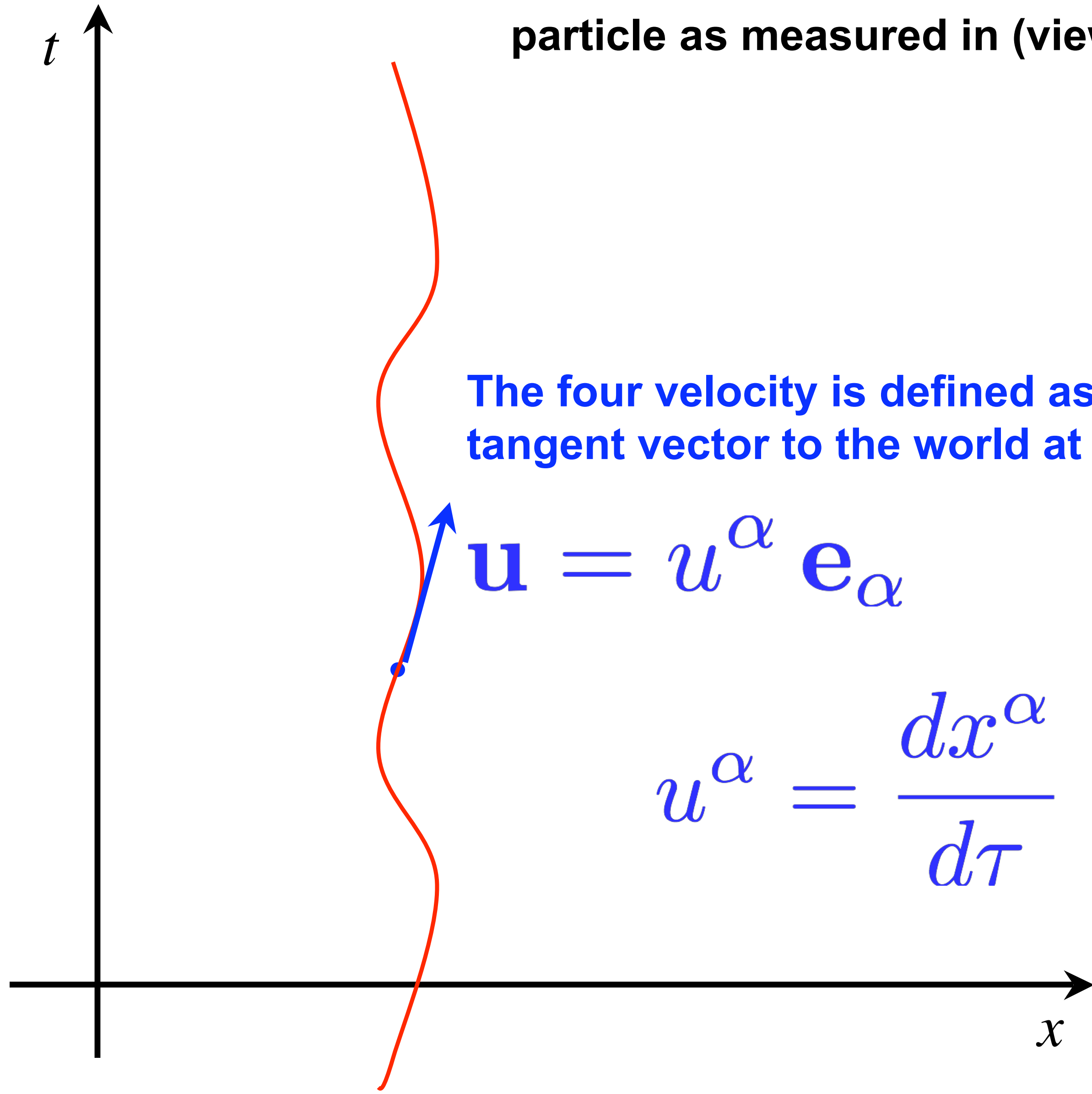
We conclude that the basis vectors transform under a Lorentz Transformation this way . . .

$$\boxed{\mathbf{e}_{\bar{\alpha}} = \Lambda^{\mu}{}_{\bar{\alpha}}\,\mathbf{e}_{\mu}}$$

This just writes one vector as a linear combination of the ***O*** coordinate system's basis vectors.

Consider the world line swept out by a moving particle as measured in (viewed by) frame *O*

O



The four velocity is defined as the tangent vector to the world at this event:

$$\mathbf{u} = u^{\alpha}\,\mathbf{e}_{\alpha}$$

$$u^{\alpha} = \frac{dx^{\alpha}}{d\tau}$$

Note the magnitude of the four velocity is always -1

$$\mathbf{u} \cdot \mathbf{u} = \eta_{\alpha\beta}\, u^\alpha\, u^\beta = \eta_{\alpha\beta}\, \frac{dx^\alpha}{d\tau}\, \frac{dx^\beta}{d\tau}$$

$$= \frac{-(-\eta_{\alpha\beta}\, dx^\alpha\, dx^\beta)}{d\tau^2} = -\frac{d\tau^2}{d\tau^2} = -1$$

$$\text{since } d\tau = \sqrt{-\eta_{\alpha\beta}\, dx^\alpha\, dx^\beta}$$

$$= \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$

$$= dt\sqrt{1 - v_x^2 - v_y^2 - v_z^2} = \frac{1}{\gamma}\, dt$$

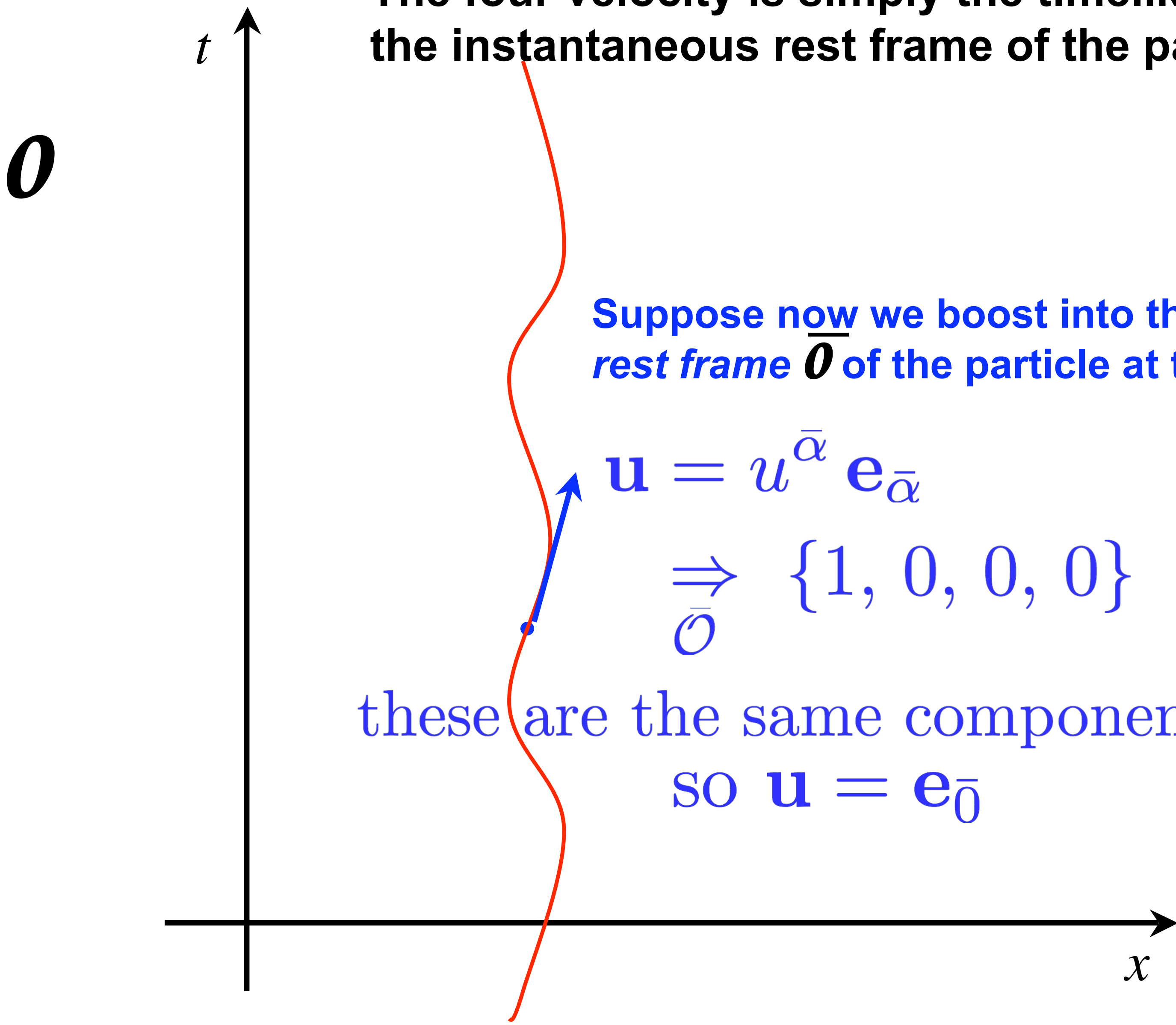
So, we can write the components of the four velocity in frame ***O*** as

$$u^\alpha = \frac{dx^\alpha}{d\tau} \;\Rightarrow\; \{\gamma,\, \gamma\, v_x,\, \gamma\, v_y,\, \gamma\, v_z\}$$

$$\mathbf{u}\cdot\mathbf{u} = -\gamma^2 + \gamma^2\, V^2 = -\gamma^2\left(1 - V^2\right) = -1$$

$$\text{where } V^2 = v_x^2 + v_y^2 + v_z^2 \text{ and, } \; e.g., \;\; v_x = \frac{dx}{dt}$$

The four velocity is simply the timelike basis vector in the instantaneous rest frame of the particle!



Suppose now we boost into the instantaneous *rest frame* $\bar{\mathcal{O}}$ of the particle at this event.

$$\mathbf{u} = u^{\bar\alpha}\,\mathbf{e}_{\bar\alpha}$$

$$\underset{\bar{\mathcal{O}}}{\Rightarrow}\ \{1,\,0,\,0,\,0\}$$

these are the same components as $\mathbf{e}_{\bar 0}$

so $\mathbf{u} = \mathbf{e}_{\bar 0}$

Note that physical particles always travel on timelike world lines.

Note, however, that photons have no four velocity -
it is not defined.

The character (timelike, spacelike, or null) of a curve at any event is the same as the tangent vector at that event.

The energy-momentum four-vector for a particle

(the momentum vector)

$$\mathbf{p} \equiv m\,\mathbf{u}$$

(here m is a scalar quantity called the *rest mass)*

$$\mathbf{p}\cdot\mathbf{p} = m^2\,\mathbf{u}\cdot\mathbf{u} = -m^2$$

$$\mathbf{p} \underset{\mathcal{O}}{\Rightarrow} \{p^0,\, p^1,\, p^2,\, p^3\} = \{E,\, p^x,\, p^y,\, p^z\}$$

$$\underset{\mathcal{O}}{\Rightarrow} \{\gamma m,\, \gamma m v_x,\, \gamma m v_y,\, \gamma m v_z\}$$

Note that in the *instantaneous rest frame* of the particle, $E = m = mc^2$, and $p^x = p^y = p^z = 0$.

"energy" is just a component - different observers will not agree on its value.

$$\mathbf{p} \cdot \mathbf{p} = -E^2 + (p^x)^2 + (p^y)^2 + (p^z)^2 = -m^2$$

$$E = \sqrt{(p^x)^2 + (p^y)^2 + (p^z)^2 + m^2}$$

What about dimensions ("units") here? Since we have set *c* = *1*, energy and momentum have the same units - the same units as rest mass - which we will take to be the same as ENERGY.

$$m = m\,c^2$$

If a particle has 4-momentum $\mathbf{p}$, an observer with 4-velocity $\mathbf{u}_{\text{obs}}$ will measure its energy to be . . .

$$E = -\mathbf{p} \cdot \mathbf{u}_{\text{obs}}$$

Why? Because the observer's 4-velocity *is* his timelike basis vector, and in his rest frame it has components (1, 0, 0, 0). The inner product of this vector with the particle's 4-momentum will pick-off the timelike component - the *energy.*

Remember that the inner product is frame invariant - it does not matter which frame you evaluate in, so choose the frame where it is easiest!

Example: suppose we view the world from some coordinate system and in that system we measure a particle with mass *m* to be at rest.

We also measure the origin of another observer's coordinate system to be moving with 3-velocity W along x .

What energy does this second observer measure the particle to have?

components of momentum vector in first observer's frame $\mathbf{p} \;\Rightarrow\; \{m,\, 0,\, 0,\, 0\}$

components of 4-velocity of 2nd observer in first observer's frame $\mathbf{u}_{\mathrm{obs}\,2} = \mathbf{e}_{\bar{0}} \;\Rightarrow\; \{\gamma,\, \gamma W,\, 0,\, 0\}$

energy as measured by 2nd observer

$$E_{\mathrm{obs}\,2} = -\mathbf{p}\cdot\mathbf{u}_{\mathrm{obs}\,2} = m\gamma$$
$$= \frac{m}{\sqrt{1-W^2}}$$