Black Holes (Ph 161)

An introduction to General Relativity.

Lecture II

Ph 161 Black Holes

Homework Assignment 4

Due Tuesday, February 28, 2006

This should be your own work; do not copy problem solutions.

(1.) Go over your notes from class and Hartle's book and show that if the metric functions for a given coordinate system do not depend on one of the coordinates then the corresponding *covariant* component of four-momentum is conserved for a freely falling particle. By "conserved" we mean that it is constant along the geodesic, the curve swept out by the freely falling particle. The covariant components of four-momentum are the functions $p_\mu \equiv g_{\mu\nu} p^\nu$. (Hint: we did all of this in class!)

(2.) Discuss why we do not have global "energy" conservation in General Relativity. (Of course, energy and momentum are always *locally* conserved.) How does symmetry replace this idea? (Hint: This is what problem (1.) is all about.)

(3.) Hartle Chapter 9: problem 6.

Read Chapter 9 of Hartle's Book

A convenient coordinate system for
***weak* & *static* (no time dependence) gravitational fields**
is given by the following coordinate system/metric:

$$ds^2 = -(1 + 2\varphi)dt^2 + (1 - 2\varphi)\left(dx^2 + dy^2 + dz^2\right)$$

This would be a decent description of the spacetime geometry and gravitational effects around the earth, the sun, and white dwarf stars, but not near the surfaces of neutron stars.

We will explore this metric with variational principles later.

The Schwarzschild Metric

(spherically symmetric, static spacetime)

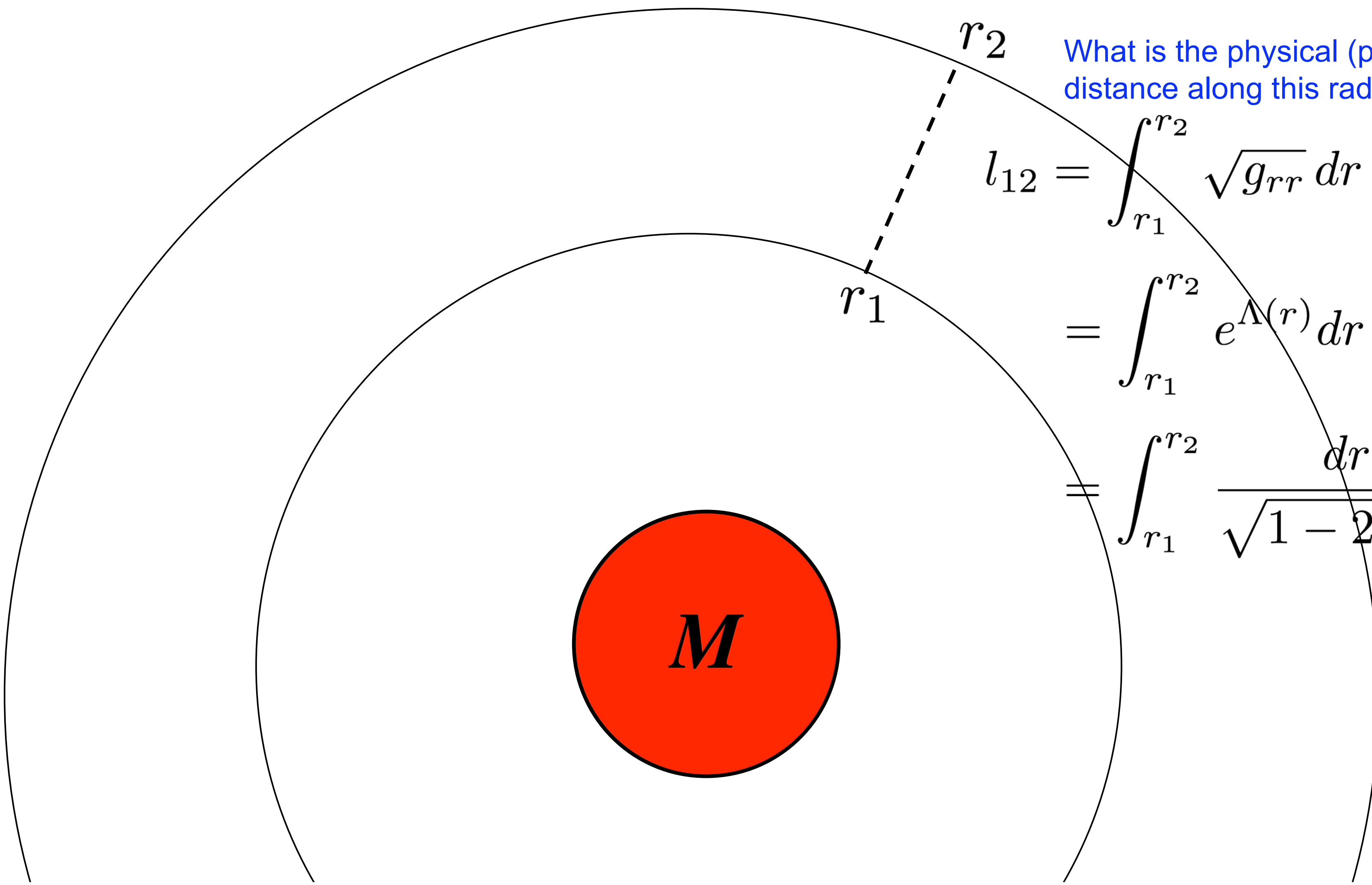
$$ds^2 = -e^{2\phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

Schwarzschild coordinates (t, r, θ, φ)

Functions of radial coordinate r to be determined
by particular spherically symmetric, static distribution
of mass-energy: $\phi(r)$ and $\Lambda(r)$

Schwarzschild metric in vacuum outside a spherical, static distribution of mass M is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$



What is the physical (proper) distance along this radial line?

$$l_{12} = \int_{r_1}^{r_2} \sqrt{g_{rr}}\, dr$$

$$= \int_{r_1}^{r_2} e^{\Lambda(r)} dr$$

$$= \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - 2M/r}}$$

Remember that the metric functions are dimensionless:

$$\frac{2M}{r} = \frac{2GM}{r} = \frac{2M\hbar c}{m_{\rm pl}^2\, r}$$

$$m_{\rm pl} \approx 1.221 \times 10^{22}\,\text{MeV}$$

$$\hbar c \approx 197.33\,\text{Mev fm} \quad \text{and} \quad 1\,\text{fm} = 10^{-13}\,\text{cm}$$

$$1\,\text{M}_\odot \approx 10^{60}\,\text{MeV}$$

Newtonian Potential: $\varphi = -\dfrac{M}{r} = -\dfrac{GM}{r} = -\dfrac{M\hbar c}{m_{\rm pl}^2\, r}$

Schwarzschild Radius

$$r_s \equiv 2M = 2GM = \frac{2M\hbar c}{m_{\rm pl}^2}$$

Schwarzschild Metric: conserved quantities

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

Note that none of the metric functions depend on the timelike coordinate t

This means that the timelike covariant component of the four-momentum of a freely falling particle will be conserved along this particle's world line (a geodesic).

covariant components: $p_\mu \equiv g_{\mu\nu}\, p^\nu$

timelike covariant component:

$$p_0 \equiv g_{0\nu}\, p^\nu = g_{00}\, p^0 = -e^{2\phi(r)} p^0$$

Photon emitted at r_1 with energy $E_{\rm em}$. What is its energy when it gets to r_2 ?

$E_{\rm em} = -\mathbf{p}\cdot\mathbf{u}_{\rm obs}$ In freely falling coordinates: $\mathbf{u}_{\rm obs} \Rightarrow \{1,0,0,0\}$

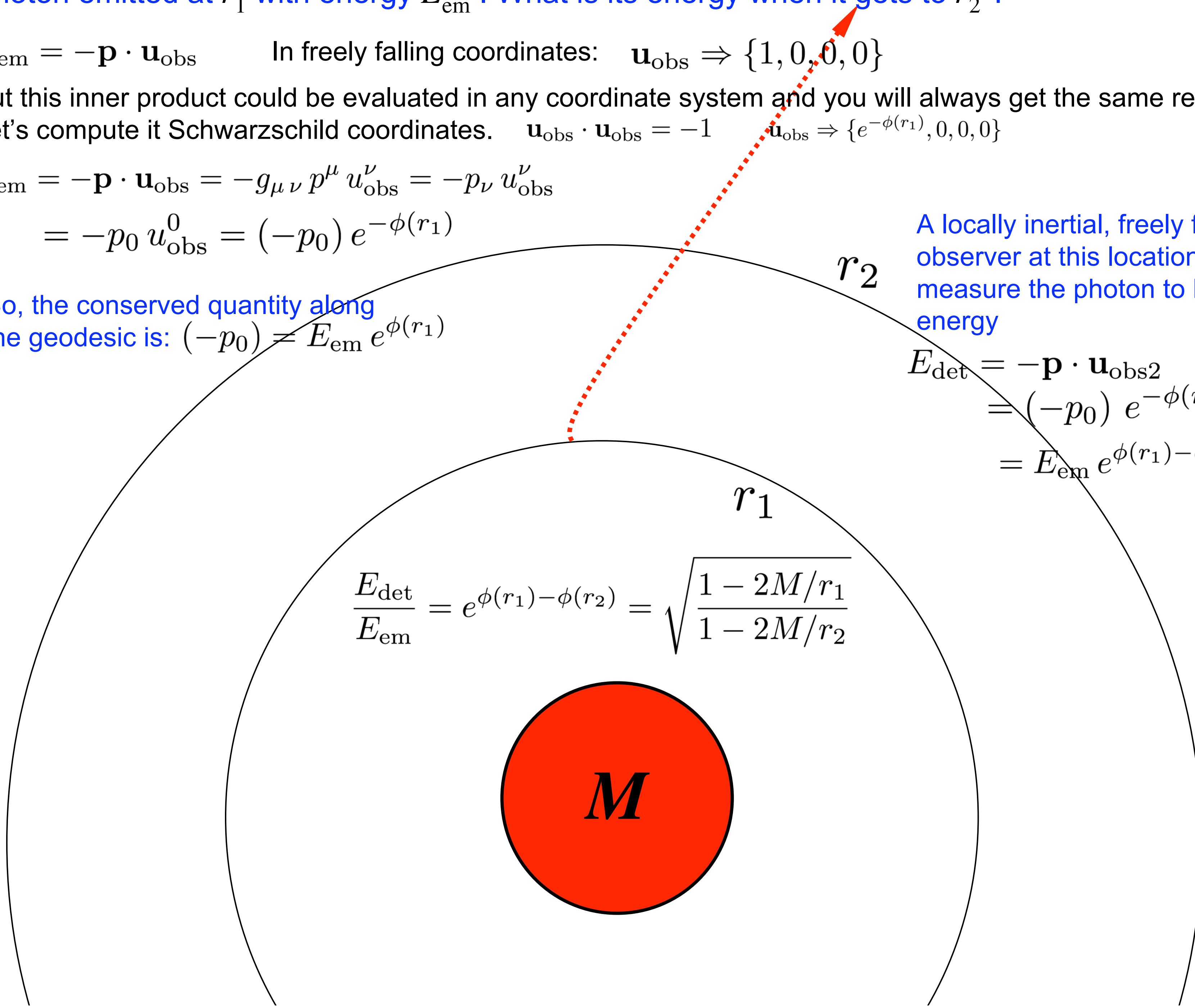
But this inner product could be evaluated in any coordinate system and you will always get the same result.
Let's compute it Schwarzschild coordinates. $\mathbf{u}_{\rm obs}\cdot\mathbf{u}_{\rm obs} = -1$ $\quad \mathbf{u}_{\rm obs} \Rightarrow \{e^{-\phi(r_1)},0,0,0\}$

$$E_{\rm em} = -\mathbf{p}\cdot\mathbf{u}_{\rm obs} = -g_{\mu\,\nu}\,p^{\mu}\,u^{\nu}_{\rm obs} = -p_{\nu}\,u^{\nu}_{\rm obs}$$
$$= -p_0\,u^0_{\rm obs} = (-p_0)\,e^{-\phi(r_1)}$$

So, the conserved quantity along the geodesic is: $(-p_0) = E_{\rm em}\,e^{\phi(r_1)}$

A locally inertial, freely falling observer at this location will measure the photon to have energy

$$E_{\rm det} = -\mathbf{p}\cdot\mathbf{u}_{\rm obs2}$$
$$= (-p_0)\ e^{-\phi(r_2)}$$
$$= E_{\rm em}\,e^{\phi(r_1)-\phi(r_2)}$$



$$\frac{E_{\rm det}}{E_{\rm em}} = e^{\phi(r_1)-\phi(r_2)} = \sqrt{\frac{1-2M/r_1}{1-2M/r_2}}$$

This is the gravitational redshift:

$$\frac{h\nu_{\rm det}}{h\nu_{\rm em}} = \frac{E_{\rm det}}{E_{\rm em}} = e^{\phi(r_1)-\phi(r_2)} = \sqrt{\frac{1-2M/r_1}{1-2M/r_2}}$$

Redshift is defined as : $$z \equiv \frac{\lambda_{\rm det} - \lambda_{\rm em}}{\lambda_{\rm em}}$$

$$1 + z = \frac{E_{\rm em}}{E_{\rm det}} = \sqrt{\frac{1-2M/r_2}{1-2M/r_1}}$$

Fermions (particles of half-integral spin)

No two of these particles can occupy the same quantum state.

Examples: electrons, neutrinos, quarks, neutrons, protons, ^{3}He atoms

$$dn_f = \frac{g}{2\pi^2(\hbar c)^3}\left(\frac{d\Omega}{4\pi}\right)\frac{pE\,dE}{e^{E/T-\eta}+1}$$

$$g = 2s+1$$

$$E = \sqrt{p^2c^2+m^2c^4}$$

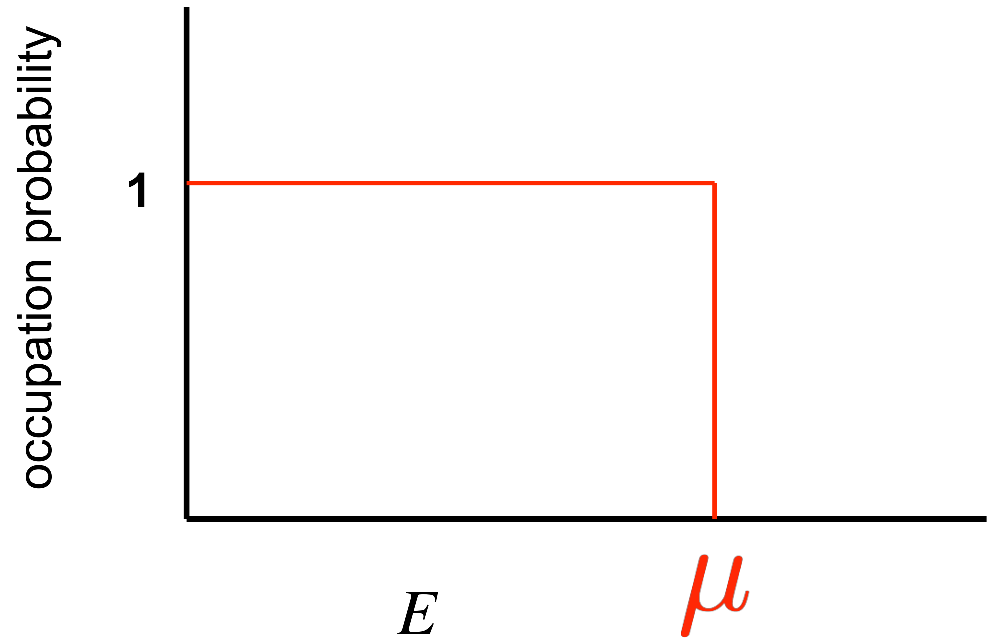
pencil of directions:

$$d\Omega = \sin\theta\, d\theta\, d\varphi$$

$$\eta = \frac{\mu}{T} \text{ where } \mu \equiv \left.\frac{\partial E_{\text{tot}}}{\partial N}\right|_{S}$$

The chemical potential: the amount of energy required to add a particle to the system, keeping the entropy constant. When this is positive it is called the Fermi energy.

degenerate fermions: large chemical potential, small T



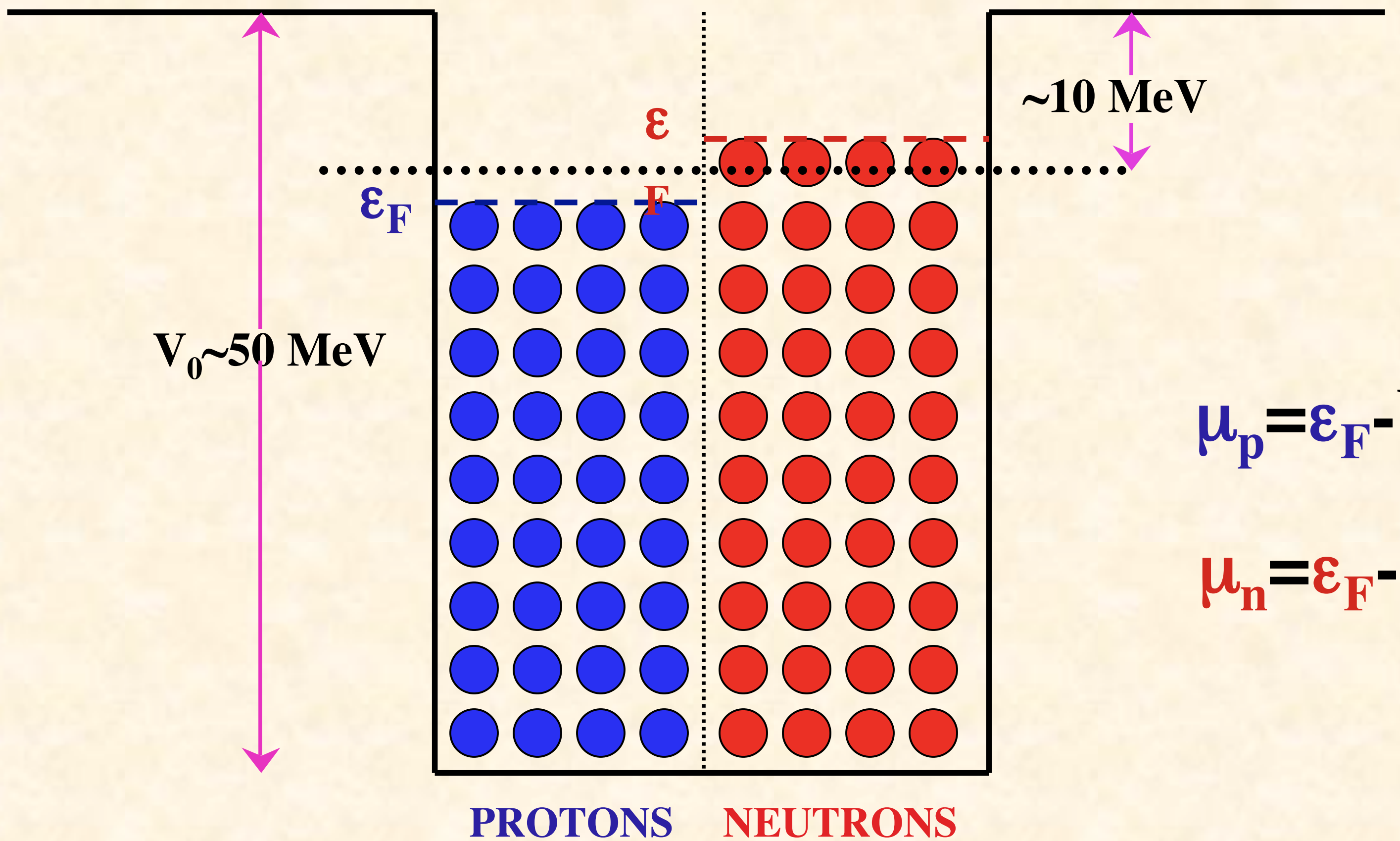
for degenerate electrons:

$$Y_e \equiv \frac{n_{e^-} - n_{e^+}}{n_{\rm baryons}}$$

$$\mu_e \approx 11.1\,{\rm MeV}\left(\frac{\rho\, Y_e}{10^{10}\,{\rm g\,cm^{-3}}}\right)^{1/3}$$

Schematic "Nucleus"

(ignore Coulomb potential for protons)



$\mu_p = \varepsilon_F - V_0$

$\mu_n = \varepsilon_F - V_0$

The man who discovered how stars shine has made many other fundamental contributions in particle, nuclear, and condensed matter physics, as well as astrophysics.

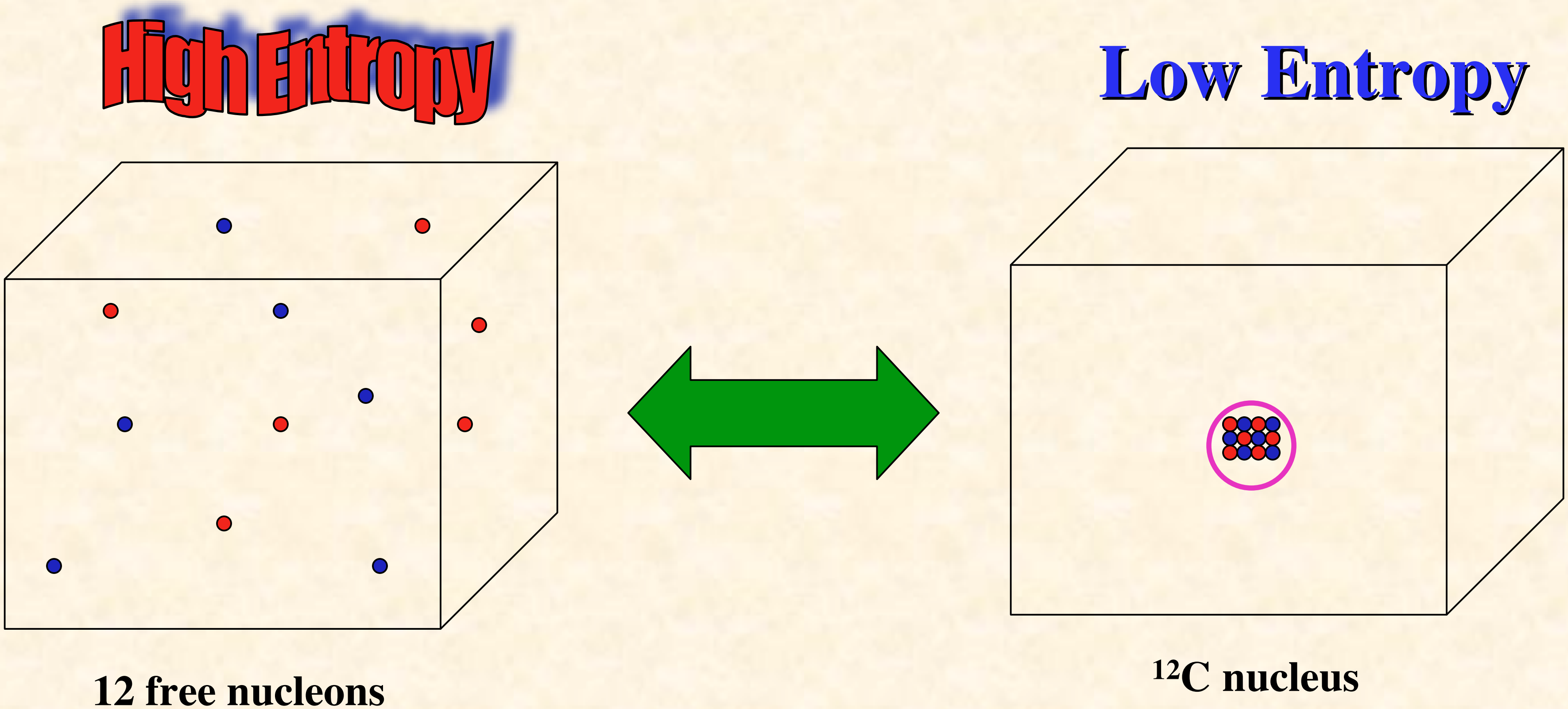
In particular, Hans Bethe completely changed the way astrophysicist's think about equation of state and nucleosynthesis issues with his 1979 insight on the role of entropy.

Hans Bethe

Entropy

$$S = k \log \Gamma$$

a measure of a system's disorder/order



Entropy

entropy per baryon (in units of Boltzmann's constant k) of the air in this room $s/\mathrm{k} \sim 10$

entropy per baryon (in units of Boltzmann's constant k) characteristic of the sun $s/\mathrm{k} \sim 10$

entropy per baryon (in units of Boltzmann's constant k) for a 10^6 solar mass star $s/\mathrm{k} \sim 1000$

entropy per baryon (in units of Boltzmann's constant k) of the universe $s/\mathrm{k} \sim 10^{10}$

total entropy of a black hole of mass M

$$S/\mathrm{k} = 4\pi\left(\frac{M}{m_{\mathrm{pl}}}\right)^2 \approx 10^{77}\left(\frac{M}{\mathrm{M}_{\mathrm{sun}}}\right)^2$$

where the gravitational constant is $G = \frac{1}{m_{\mathrm{pl}}^2}$

and the Planck mass is $m_{\mathrm{pl}} \approx 1.221 \times 10^{22}$ MeV

Gravitational Collapse of Stars to Neutron Stars or Black Holes Releases Huge Amounts of Energy, Most of it as Neutrinos of all kinds.

$\text{Stellar Mass Range: } \sim 10\,M_\odot \text{ to } \sim 100\,M_\odot$

$\rightarrow$ collapse of $1.4\,M_\odot$ iron core

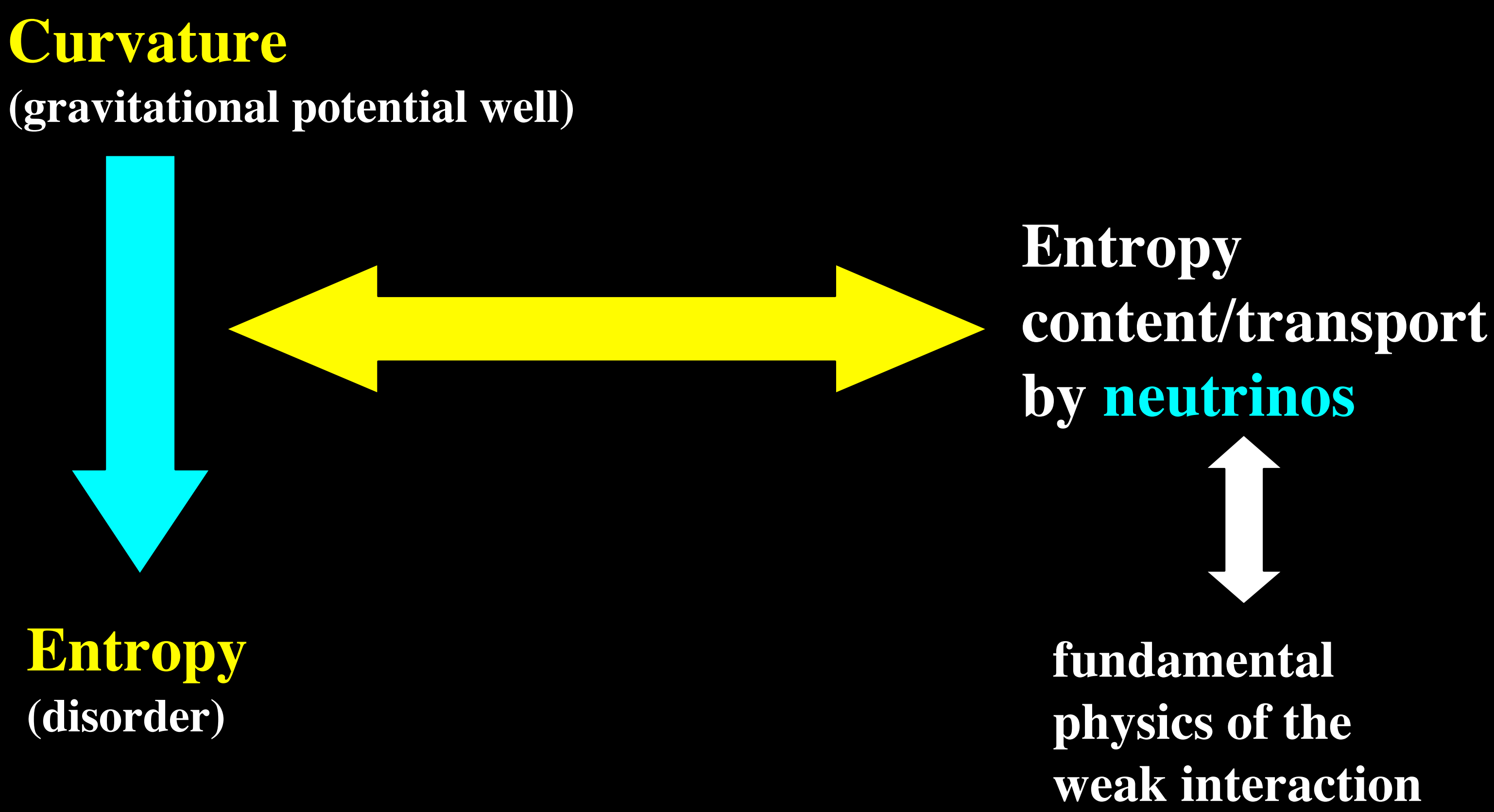
$\rightarrow$ 10% of core rest mass radiated as neutrinos

Very Massive Objects $\sim 10^2\,M_\odot$ to $10^4\,M_\odot$

Supermassive Objects: $\sim 10^4\,M_\odot$ to $10^8\,M_\odot$

collapse to black hole, ~ 5% of rest mass radiated in neutrinos

There is a deep connection between spacetime curvature and entropy (and neutrinos)



Freeze-Out from Nuclear Statistical Equilibrium (NSE)

In NSE the reactions which build up and tear down nuclei have equal rates, and these rates are large compared to the local material expansion rate.

$$Z\,p + N\,n \longleftrightarrow A(Z,N) + \gamma$$

nuclear mass A is the sum of protons and neutrons A=Z+N

$$Z\,\mu_p + N\,\mu_n = \mu_A + Q_A$$

Binding Energy of Nucleus A

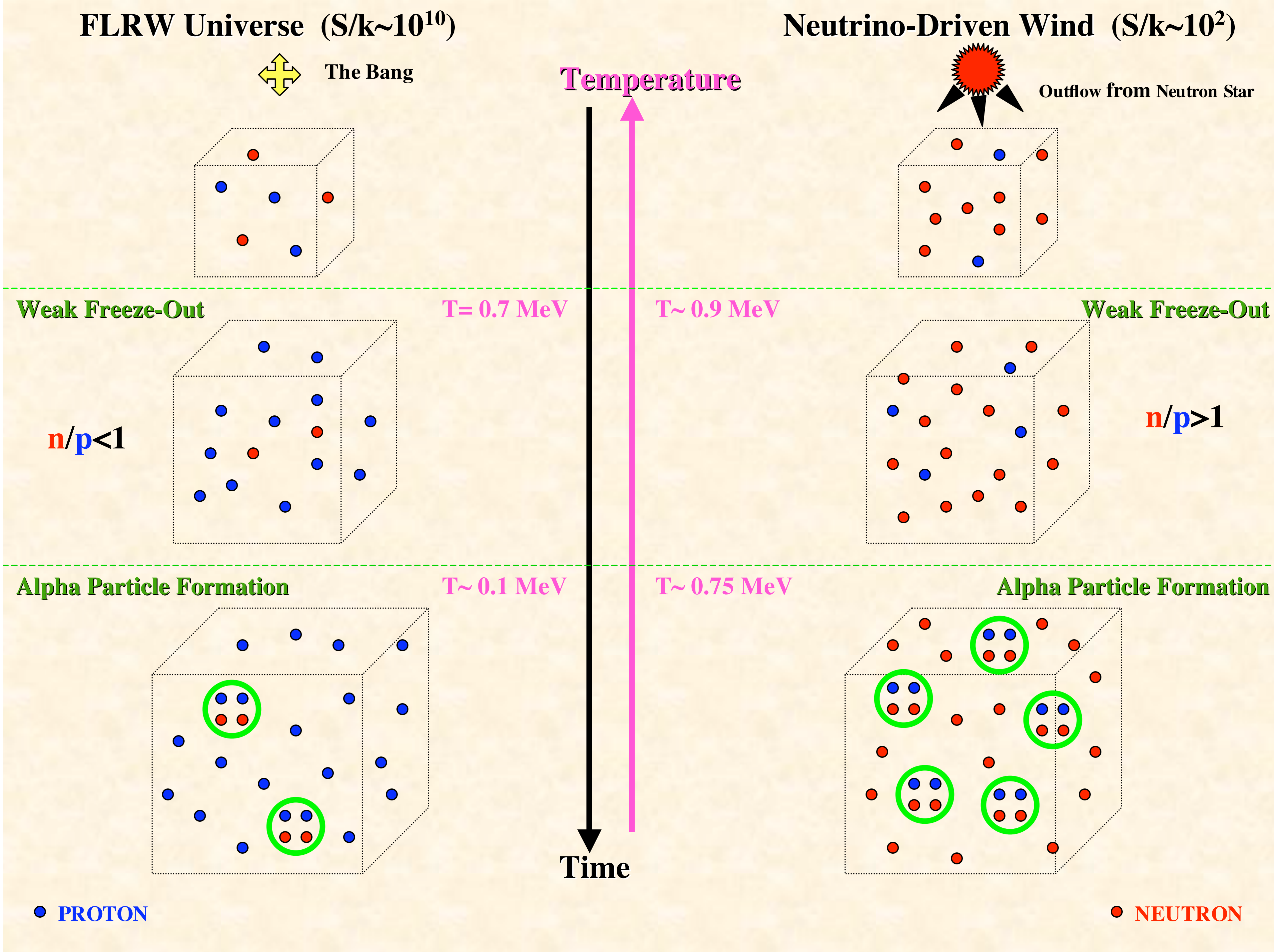
Saha Equation

$$Y_{A(Z,N)} \approx \left[S^{1-A}\right] G \pi^{\frac{7}{2}(A-1)} 2^{\frac{1}{2}(A-3)} A^{3/2} \left(\frac{T}{m_b}\right)^{\frac{3}{2}(A-1)} Y_p^Z Y_n^N e^{Q_A/T}$$

Typically, each nucleon is bound in a nucleus by ~ 8 MeV.

For alpha particles the binding per nucleon is more like 7 MeV.

**But alpha particles have mass number A=4,
and they have almost the same
binding energy per nucleon as heavier nuclei
so they are favored whenever there is a competition
between binding energy and disorder (high entropy).**


FLRW Universe (S/k~10^10)
The Bang
Temperature
Neutrino-Driven Wind (S/k~10^2)
Outflow from Neutron Star
Weak Freeze-Out
T= 0.7 MeV
T~ 0.9 MeV
Weak Freeze-Out
n/p<1
n/p>1
Alpha Particle Formation
T~ 0.1 MeV
T~ 0.75 MeV
Alpha Particle Formation
Time
PROTON
NEUTRON

Neutron-to-proton ratio and energy deposition largely determined by these processes:

$$\nu_e + n \longrightarrow p + e^-$$

$$\bar{\nu}_e + p \longrightarrow n + e^+$$

R. Wagoner, W. A. Fowler, & F. Hoyle

(from D. Clayton's nuclear astrophysics photo archive at Clemson University)

The nuclear and weak interaction physics of primordial nucleosynthesis (or Big Bang Nucleosynthesis, BBN) was first worked out self consistently in 1967 by Wagoner, Fowler, & Hoyle.

This has become a standard tool of cosmologists. Coupled with the observed ^{2}H abundance its has led to a determination of the baryon content of the universe. Coupled with a better understanding of the primordial ^{4}He abundance it could tell us about the lepton numbers and new neutrino physics.

BBN is the paradigm for all nucleosynthesis processes which involve a freeze-out from nuclear statistical equilibrium (NSE).

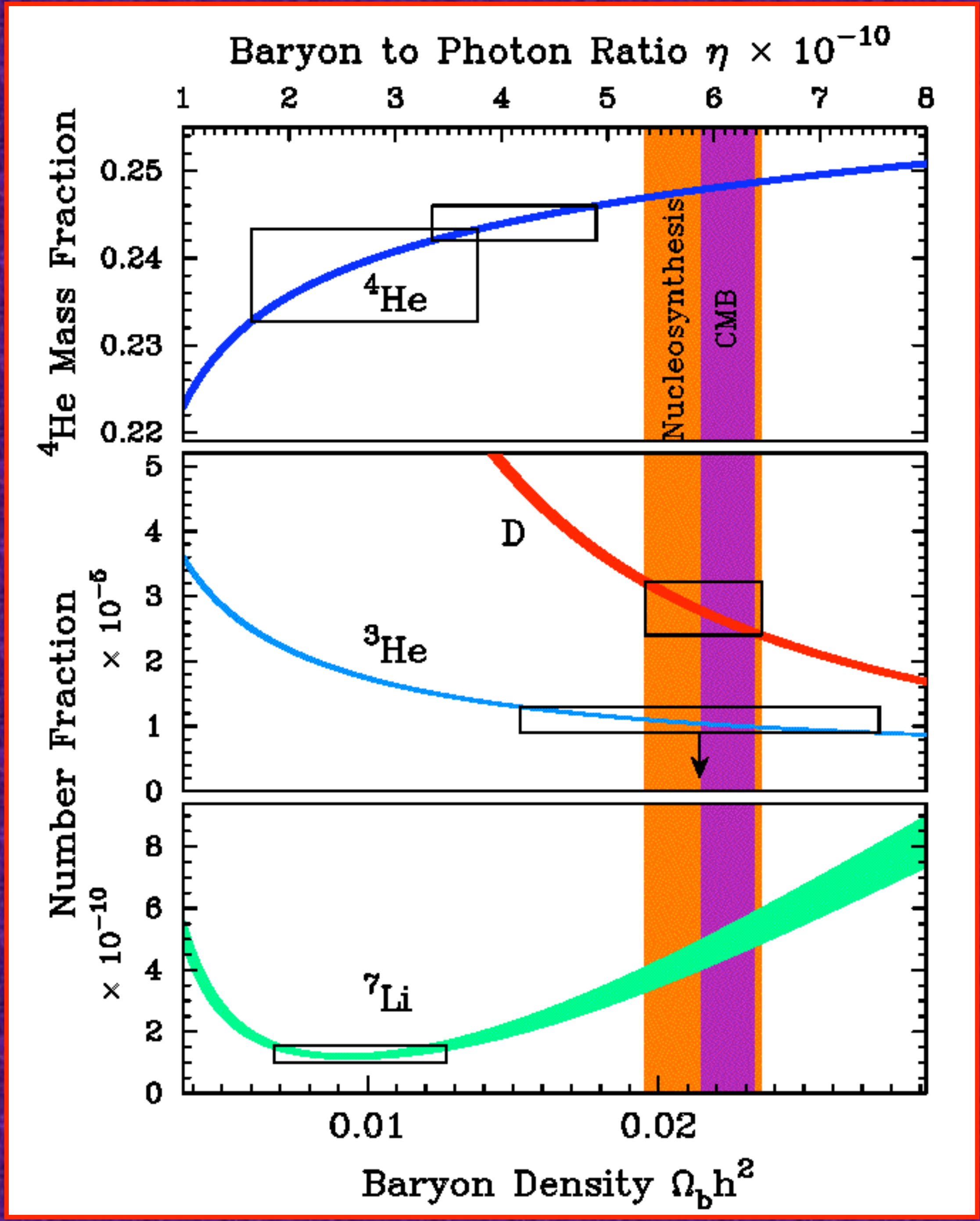
The "baryon number" is defined to be the ratio of the net number of baryons to the number of photons:

$$\eta = \frac{n_b - n_{\bar{b}}}{n_\gamma}$$

The "baryon number," or baryon-to-photon ratio, η is a kind of "inverse entropy per baryon," but is not a co-moving invariant.

$$\eta \approx \frac{2\pi^4}{45}\frac{1}{\zeta(3)}\frac{g_{total}}{g_\gamma}S^{-1}$$

Observations of the isotope-shifted line of deuterium along the lines of sight to high redshift QSO's (Tytler group) provide an accurate determination of the baryon-to-photon ratio η. **(CMB acoustic peak ratios give results consistent with these, as do considerations of large scale structure.)**



This completely alters the way we look at BBN.

The "baryon number" is defined to be the ratio of the net number of baryons to the number of photons:

$$\eta = \frac{n_b - n_{\bar{b}}}{n_\gamma}$$

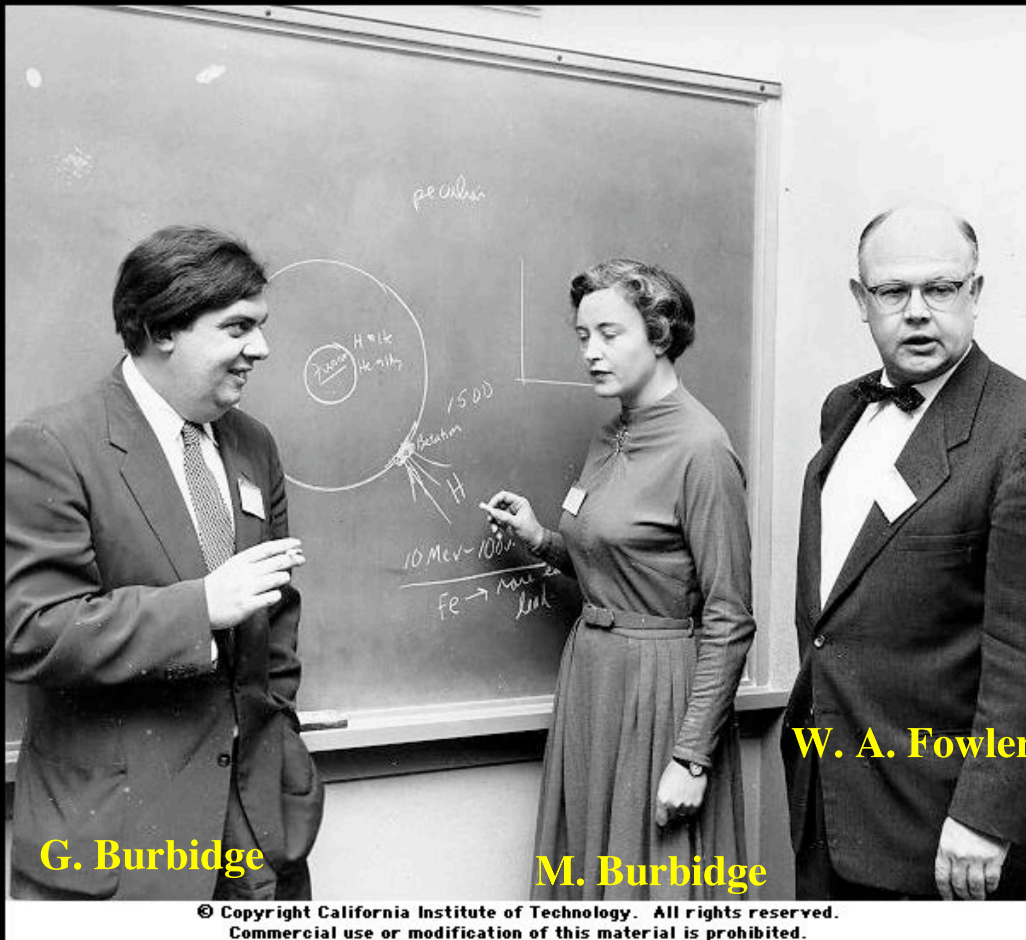
Dave Schramm pioneered the use
of primordial nucleosynthesis
considerations as a probe of
particle physics and cosmology.

In particular, he and his co-workers
pushed to use the
observationally-inferred helium
abundance to determine
the number of flavors of neutrinos.

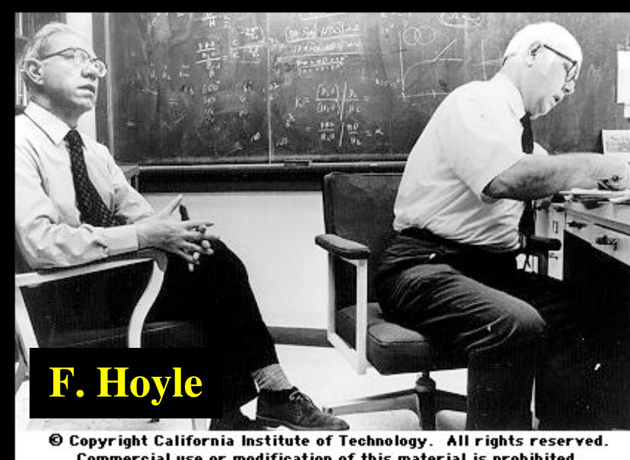
David N. Schramm

So where are the nuclei heavier than deuterium, helium, and lithium made ???

Stars make most of the elements



B^2FH (1957) outlined the basic processes in which the intermediate and heavy elements are cooked in stars.



Core Collapse Supernovae (Types II, Ib, Ic)

I. Collapse and Bounce Epoch

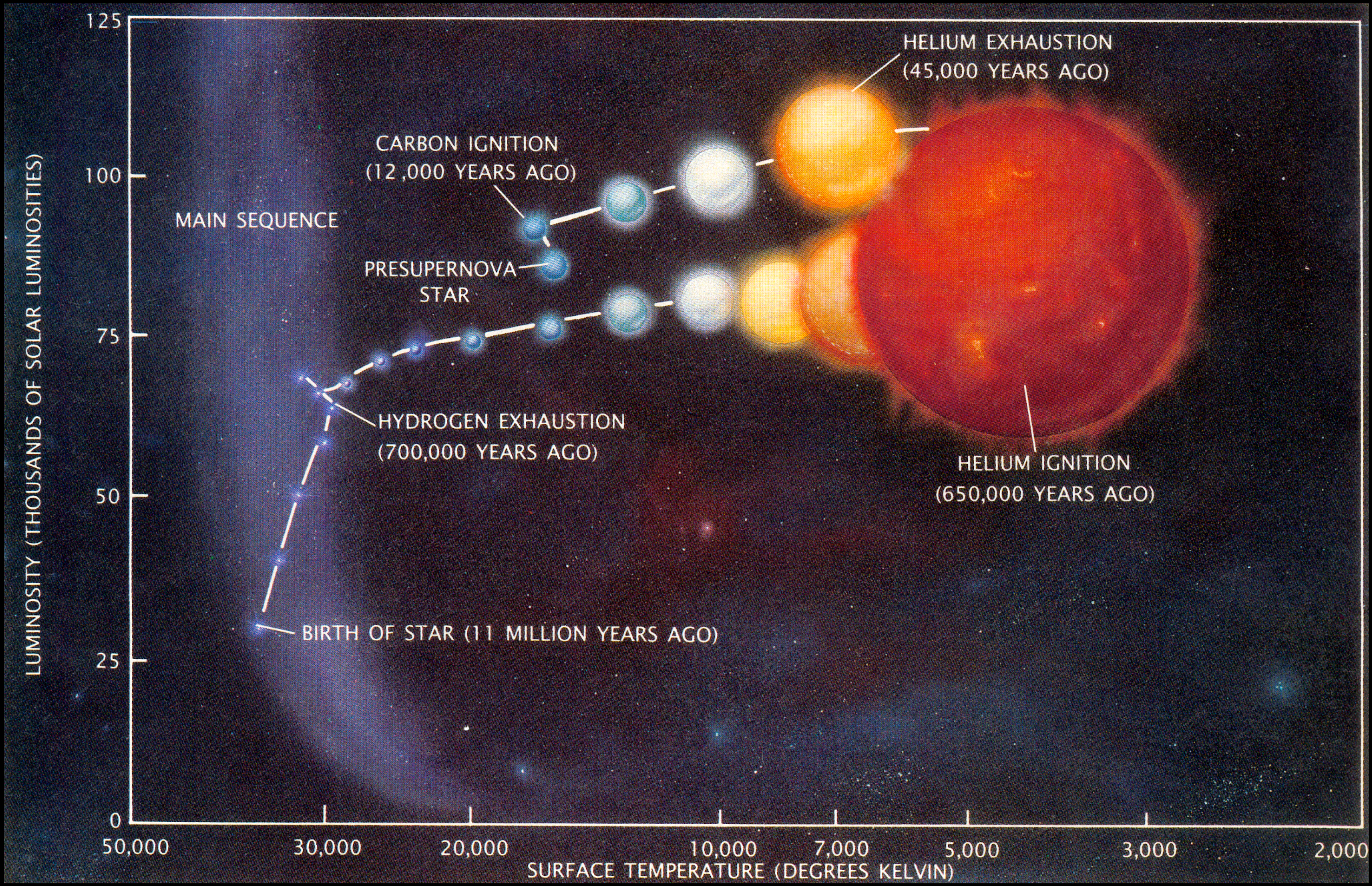
- Massive star (>10 solar masses) evolves in millions of years
- Forms "Fe"-core of 1.4 to 1.6 solar masses
- Core goes dynamically unstable
- Collapse duration of order 1 sec
- Entropy-per-baryon S/k of order 1 (really "Cold")
- Shock generated at core bounce (at edge of homologous core)
- Shock energy subsequently degraded by photo-dissociation of nuclei

II. Shock Re-Heating Epoch

- Time "post-bounce," $t_{pb,}$ from 0.1 s to 0.6 s
- Neutrino processes re-energize shock, drive convection
- Entropy-per-baryon S/k of order 40

III. Hot Bubble/r-Process Epoch

- t_{pb} from 1 s to 20 s
- Entropy-per-baryon S/k of order 70 to 500
- Neutrino-driven "wind"



Weaver & Woosley, *Sci Am*, 1987

Nuclear Burning Stages of a 25 M_{sun} Star

Burning Stage	Temperature	Density	Time Scale
Hydrogen	5 keV	5 g cm^{-3}	7 X 10^6 years
Helium	20 keV	700 g cm^{-3}	5 X 10^5 years
Carbon	80 keV	2 X 10^5 g cm^{-3}	600 years
Neon	150 keV	4 X 10^6 g cm^{-3}	1 year
Oxygen	200 keV	10^7 g cm^{-3}	6 months
Silicon	350 keV	3 X 10^7 g cm^{-3}	1 day
Core Collapse	700 keV ↓	4 X 10^9 g cm^{-3} ↓	~ seconds of order the free fall time
"Bounce"	~ 2 MeV	~10^{15} g cm^{-3}	~milli-seconds
Neutron Star	< 70 MeV initial ~ keV "cold"	~10^{15} g cm^{-3}	initial cooling ~ 15-20 seconds ~ thousands of years

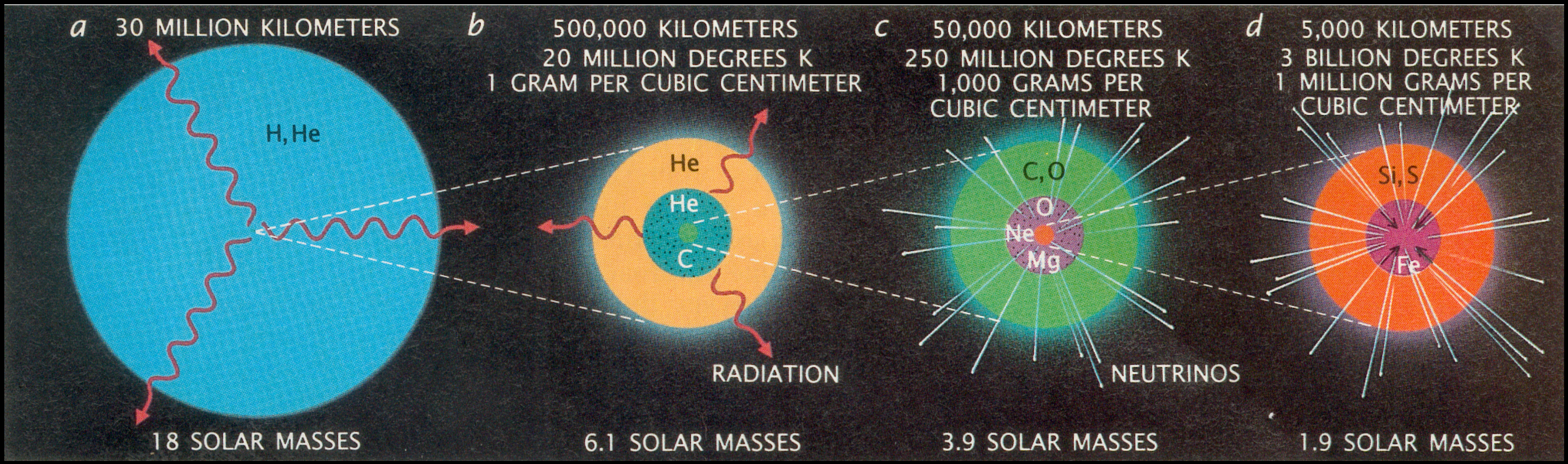
Massive Stars are Giant Refrigerators

From core carbon/oxygen burning onward
the neutrino luminosity exceeds the photon luminosity.

Neutrinos carry energy/entropy away from the core!

Core goes from S/k~10 on the Main Sequence (hydrogen burning)
to a thermodynamically cold S/k ~1 at the onset of collapse!

e.g., the collapsing core of a supernova can be a
frozen (Coulomb) crystalline solid with a
temperature ~1 MeV!



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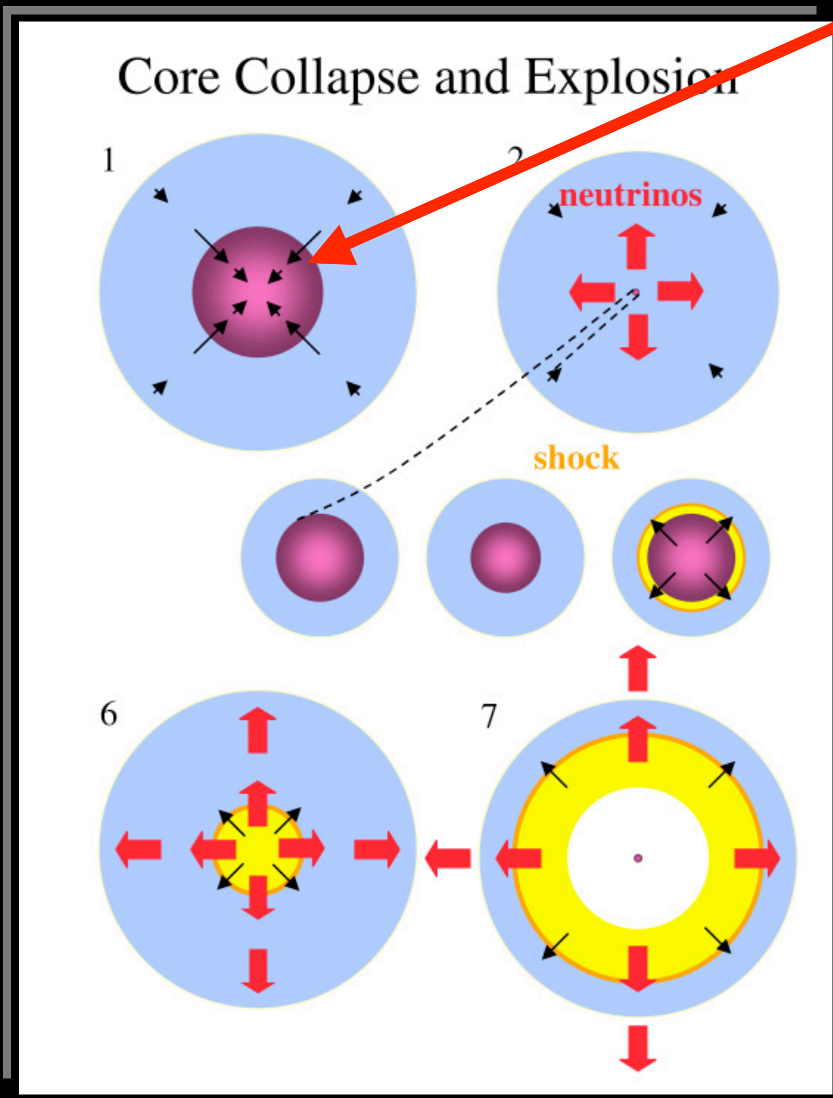


A. Mezzacappa

Inner core where the infall velocity v is subsonic and proportional to radius $v \sim r$ (homologous) and whose mass is proportional to Y_e^2.

The rest of the original Fe-core, the outer core, falls in supersonically.

Inner core "bounces" as a unit at or above nuclear density (i.e., when nucleons touch). Shock wave generated at this core's edge at bounce.

Shock's energy subsequently is degraded as it plows through outer core material.

Entropy jump is a factor ~10 across shock, so NSE favors "photo-disintegrating" nuclei.

Pulling a nucleon out of nucleus requires ~ 8 MeV. This is 10^{51} ergs for each 0.1 solar masses transited by shock!

Neutrinos Dominate the Energetics of Core Collapse Supernovae

Explosion
only ~1% of neutrino energy

- Total optical + kinetic energy, 10^{51} ergs
- Total energy released in Neutrinos, 10^{53} ergs

10% of star's rest mass!

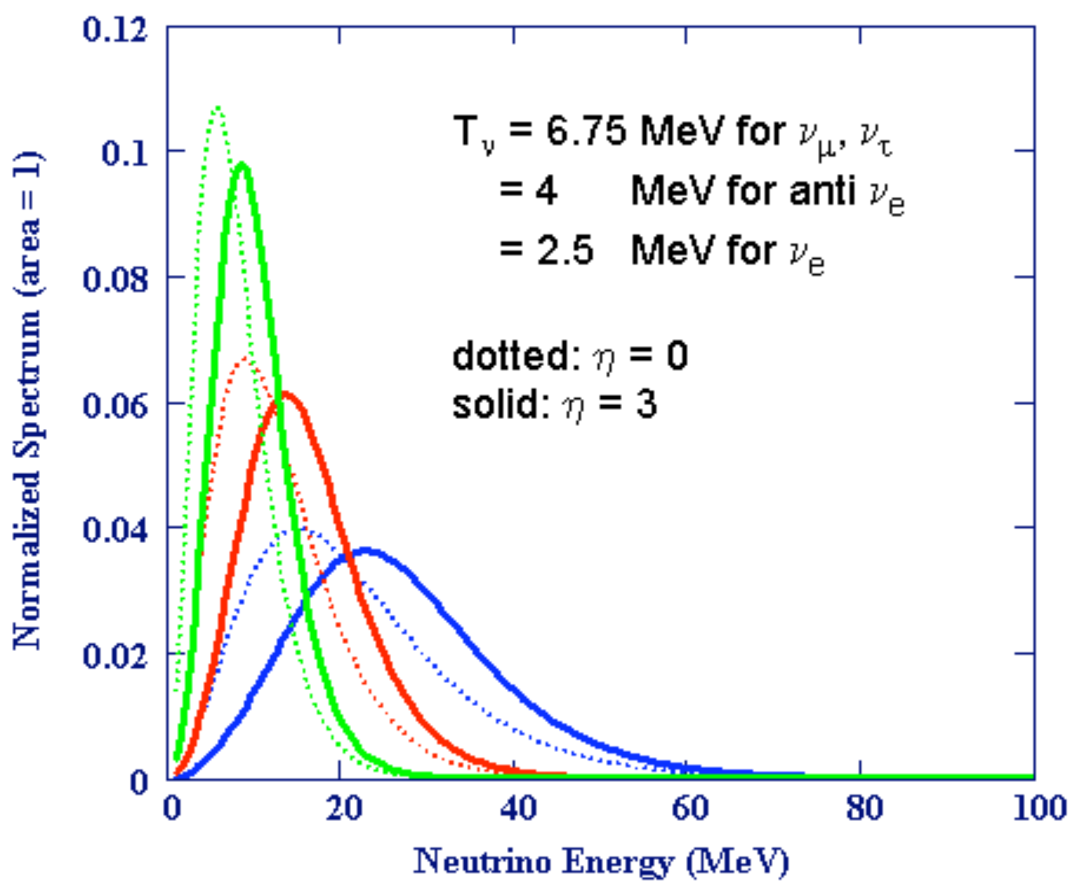
$$E_{\rm GRAV} \approx \frac{3}{5}\frac{G\,M_{\rm NS}^2}{R_{\rm NS}} \approx 3\times10^{53}\,{\rm ergs}\left[\frac{M_{\rm NS}}{1.4{\rm M}_{\rm sun}}\right]^2\left[\frac{10\ {\rm km}}{R_{\rm NS}}\right]$$

Neutrino diffusion time, $\tau_\nu \approx 2$ s

$$L_\nu \approx \frac{1}{6}\frac{GM_{\rm NS}^2}{R_{\rm NS}}\frac{1}{\tau_\nu} \approx 4\times10^{51}\,{\rm ergs\ s^{-1}}$$

Neutrino Energy Spectra

- at "Neutrino Sphere"
- Near Fermi-Dirac energy distribution



The flux of neutrinos in a pencil of directions and energies is

$$d\varphi_\nu \approx \frac{L_\nu}{\pi R_\nu^2}\left(\frac{d\Omega_\nu}{4\pi}\right)\frac{1}{\langle E_\nu \rangle} f(E_\nu)dE_\nu$$

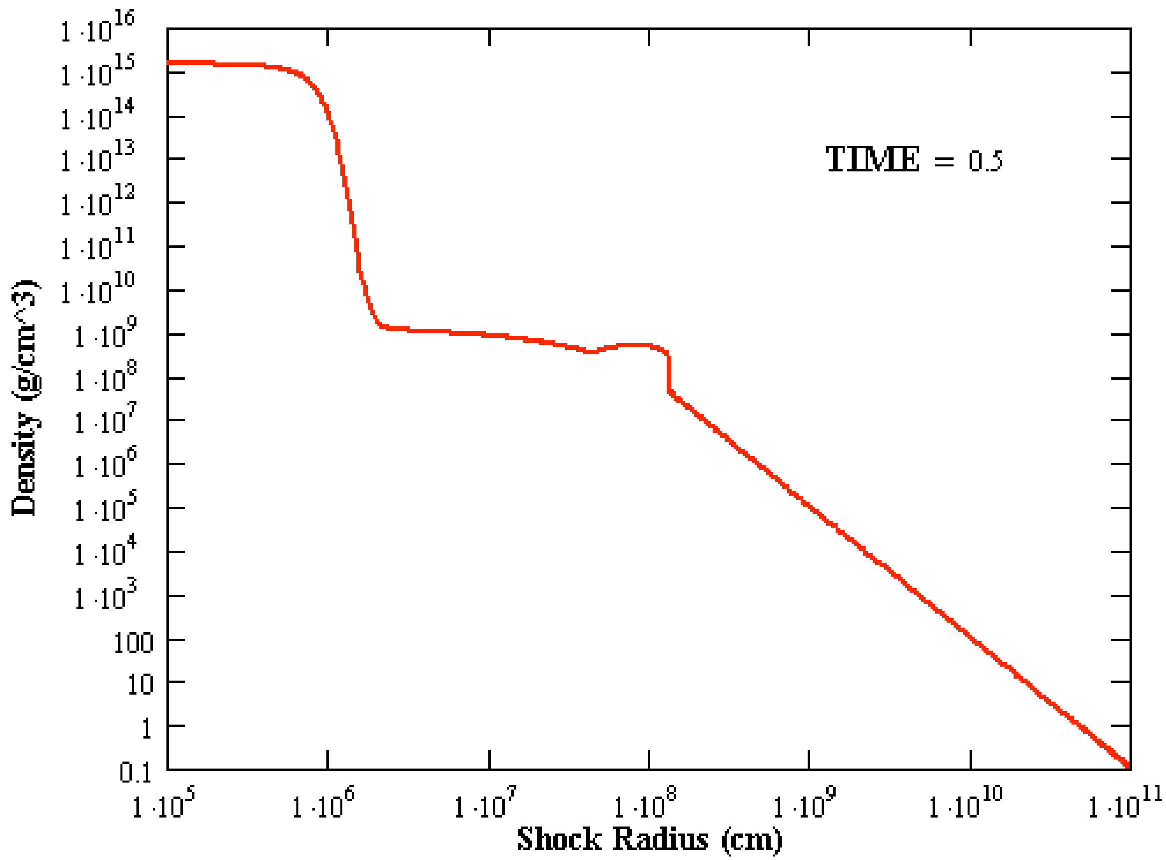
The (black body) neutrino distribution function is

$$f\left(E_\nu\right) = \frac{1}{T_\nu^3 F_3\left(\eta_\nu\right)} \frac{E_\nu^2}{e^{E_\nu/T_\nu - \eta_\nu} + 1}$$

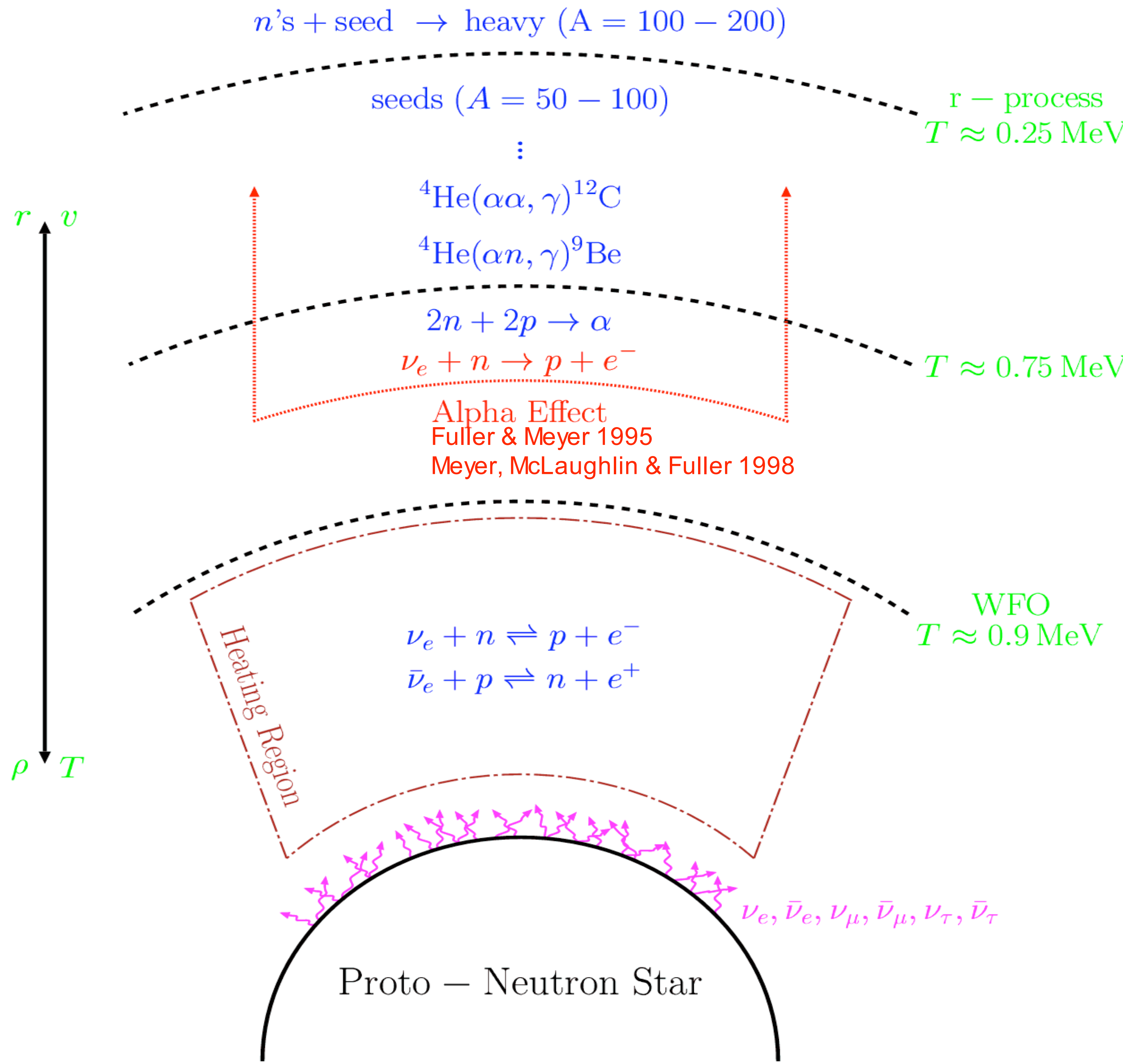
$$F_k\left(\eta_\nu\right) = \int_0^\infty \frac{x^k dx}{e^{x-\eta_\nu} + 1}$$


The result of coupling in
1% of the neutrino energy?

Shock Propagation



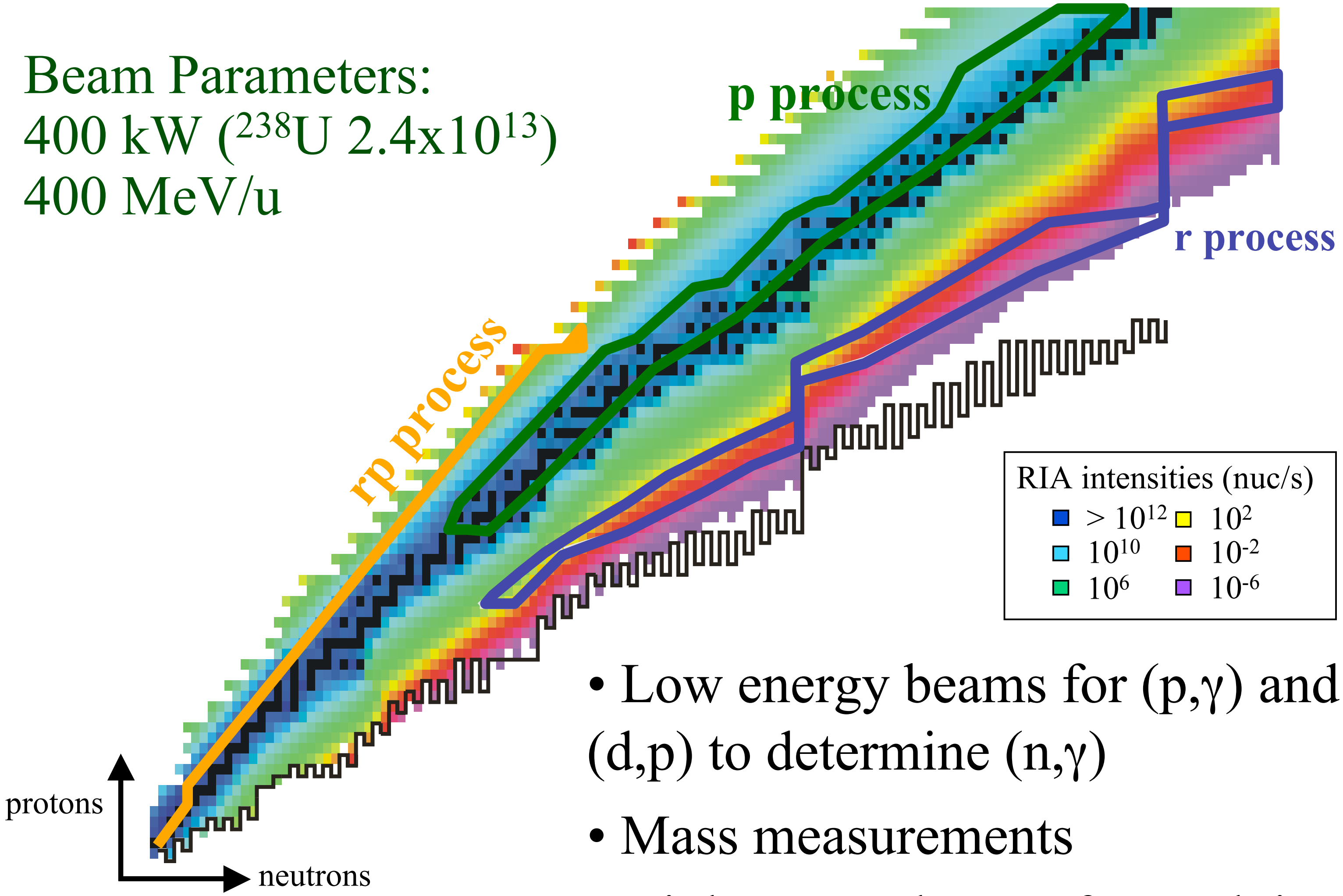
R. Schirato & G. Fuller, astro-ph/0205390



n's + seed $\rightarrow$ heavy (A = 100 − 200)
seeds (A = 50 − 100)
r − process
$T \approx 0.25$ MeV
^{4}He$(\alpha\alpha,\gamma)^{12}$C
^{4}He$(\alpha n,\gamma)^9$Be
r v
$2n + 2p \rightarrow \alpha$
$\nu_e + n \rightarrow p + e^-$
$T \approx 0.75$ MeV
Alpha Effect
Fuller & Meyer 1995
Meyer, McLaughlin & Fuller 1998
WFO
$T \approx 0.9$ MeV
$\nu_e + n \rightleftharpoons p + e^-$
$\bar{\nu}_e + p \rightleftharpoons n + e^+$
Heating Region
ρ T
$\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$
Proto − Neutron Star

RIA will produce nuclei of interest in the r-Process

Beam Parameters:
400 kW (^{238}U 2.4x10^{13})
400 MeV/u



- Low energy beams for (p,γ) and (d,p) to determine (n,γ)
- Mass measurements
- High energy beams for studying Gamow-Teller strength

R-Process Nucleosynthesis

Heavy element abundance determinations in Ultra Metal-Poor halo star CS 22892-052

$$[Fe/H] \approx -3.1$$

Mass A>100 abundance pattern fits that of solar system, lower nuclear mass material has an abundance pattern which does not, in general, fit the solar pattern. This trend is evident in other Ultra Metal-Poor Halo (UMP's) stars as well.

The same pattern is seen in several other UMP's. Qian & Wasserburg believe that these r-process nuclides were deposited on the surfaces of these otherwise quiescent stars by companions that became core collapse supernovae.

universal abundance pattern?

A = 130 & 195 peak have comparable abundances. Why?

Fission cycling in neutron-rich conditions? (McLauglin & Buen 05)