Black Holes (Ph 161)

An introduction to General Relativity.

Lecture II

Ph 161 Black Holes

Homework Assignment 4

Due Tuesday, February 28, 2006

This should be your own work; do not copy problem solutions.

(1.) Go over your notes from class and Hartle's book and show that if the metric functions for a given coordinate system do not depend on one of the coordinates then the corresponding *covariant* component of four-momentum is conserved for a freely falling particle. By "conserved" we mean that it is constant along the geodesic, the curve swept out by the freely falling particle. The covariant components of four-momentum are the functions $p_\mu \equiv g_{\mu\nu} p^\nu$. (Hint: we did all of this in class!)

(2.) Discuss why we do not have global "energy" conservation in General Relativity. (Of course, energy and momentum are always *locally* conserved.) How does symmetry replace this idea? (Hint: This is what problem (1.) is all about.)

(3.) Hartle Chapter 9: problem 6.

Read Chapter 9 of Hartle's Book

Ph 161 Black Holes

Homework Assignment 5

Due Tuesday, March 7, 2006

This should be your own work; do not copy problem solutions.

(1.) Why don't orbits close on themselves in General Relativity? Discuss where the precession of, *e.g.*, Mercury's orbit comes from in terms of the conserved quantities along geodesics in Schwarzschild geometry. (Consult Hartle Chapter 9.)

(2.) Discuss why stars supported by pressure coming from particles with relativistic speeds become unstable in General Relativity. Look carefully at Hartle Chapter 12, problem 2 and note that the pressure forces must always balance gravitational forces for stars to be in equilibrium.

(3.) Decide on your paper/talk topic. You may want to give a very rough outline.

Hint for (1.) & (2.): The basic answer for both of these questions is that gravity is nonlinear in General Relativity. Spacetime curvature has mass-energy and so curves spacetime! Therefore, unlike Newtonian gravitation, the "gravitational forces" in General Relativity grow faster than $1/r^2$.

Read Hartle, Chapter 12

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Homework Assignment 6

Due Tuesday, March 14, 2006

This should be your own work; do not copy problem solutions.

(1.) Write down the Kruskal-Szekeres metric in terms of the coordinates (U, V, θ, φ) discussed in class and in Chapter 12 of Hartle's book. How are these coordinates related to the Schwarzschild coordinates (t, r, θ, φ)?

(a.) Draw the Kruskal spacetime diagram (U-V plane) and place on it curves corresponding to Schwarzschild radial coordinate $r = 0$, $2M$, and $4M$. Justify your result.

(b.) On this Kruskal spacetime diagram place curves corresponding to Schwarzschild timelike coordinate $t = 0$, $-\infty$, $+\infty$, $-M$, and $+M$. Justify your result.

(c.) By drawing the world line of a physical observer falling through $r = 2M$ who sends out periodic light beams, argue why the surface $r = 2M$ acts like a causal horizon. (Light lines in the Kruskal diagram are 45 degree straight lines - why?)

Hint: all of these tasks were done explicitly in class (see your notes and course web pages): I want you to work through it again by yourself.

Papers and Talks

Talks: Wednesday, March 15, 6:00 PM - 8:30 PM
104 Peterson Hall
&
Lecture, Thursday, March 16, 9:30 AM-10:50 AM

Papers: due in class, Thursday, March 16

Core Collapse Supernovae (Types II, Ib, Ic)

I. Collapse and Bounce Epoch

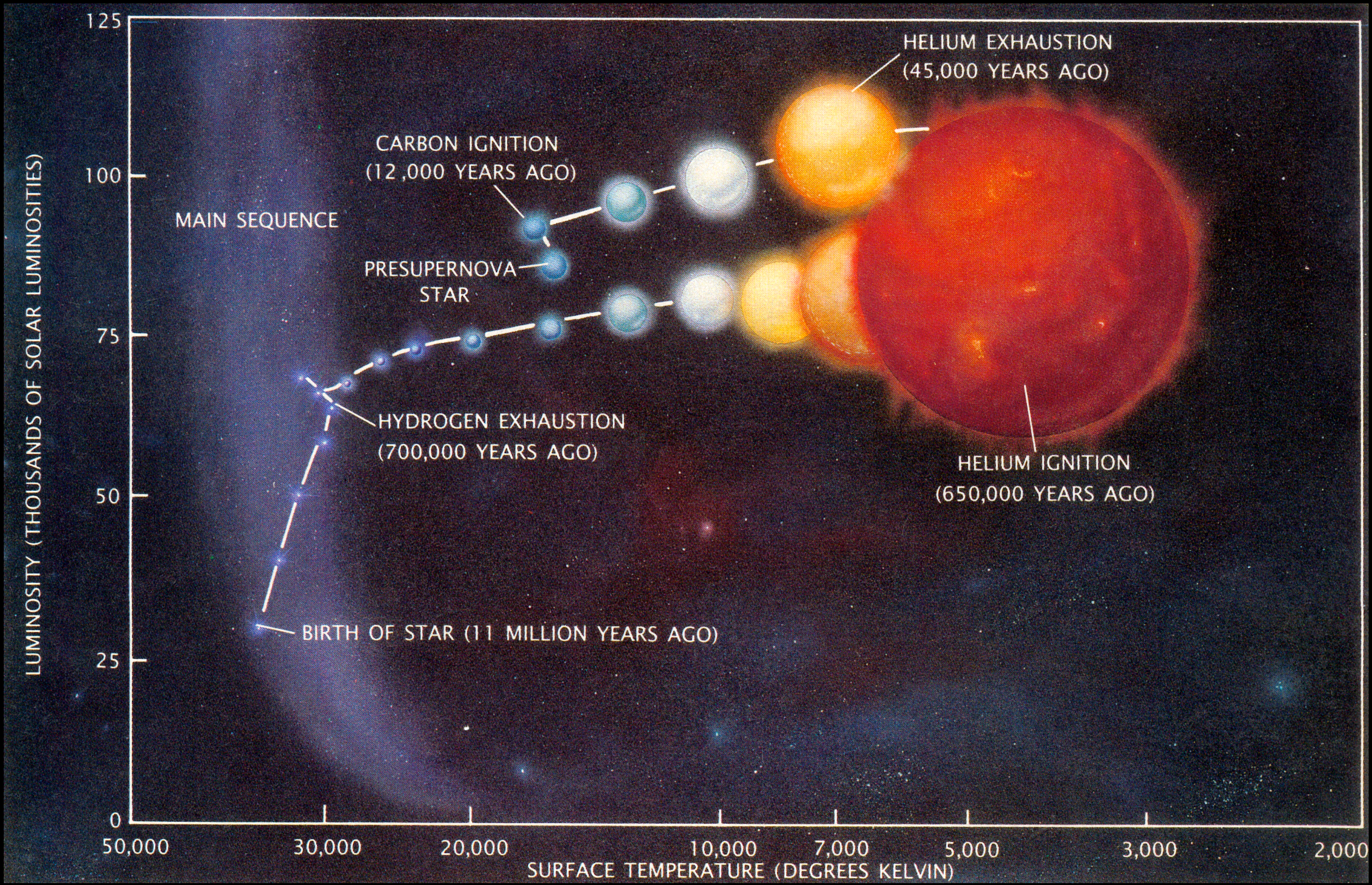
- Massive star (>10 solar masses) evolves in millions of years
- Forms "Fe"-core of 1.4 to 1.6 solar masses
- Core goes dynamically unstable
- Collapse duration of order 1 sec
- Entropy-per-baryon S/k of order 1 (really "Cold")
- Shock generated at core bounce (at edge of homologous core)
- Shock energy subsequently degraded by photo-dissociation of nuclei

II. Shock Re-Heating Epoch

- Time "post-bounce," $t_{pb,}$ from 0.1 s to 0.6 s
- Neutrino processes re-energize shock, drive convection
- Entropy-per-baryon S/k of order 40

III. Hot Bubble/r-Process Epoch

- t_{pb} from 1 s to 20 s
- Entropy-per-baryon S/k of order 70 to 500
- Neutrino-driven "wind"



Weaver & Woosley, *Sci Am*, 1987

Nuclear Burning Stages of a 25 M_{sun} Star

Burning Stage	Temperature	Density	Time Scale
Hydrogen	5 keV	5 g cm^{-3}	7×10^6 years
Helium	20 keV	700 g cm^{-3}	5×10^5 years
Carbon	80 keV	2×10^5 g cm^{-3}	600 years
Neon	150 keV	4×10^6 g cm^{-3}	1 year
Oxygen	200 keV	10^7 g cm^{-3}	6 months
Silicon	350 keV	3×10^7 g cm^{-3}	1 day
Core Collapse	700 keV	4×10^9 g cm^{-3}	~ seconds of order the free fall time
"Bounce"	~ 2 MeV	~10^{15} g cm^{-3}	~milli-seconds
Neutron Star	< 70 MeV initial ~ keV "cold"	~10^{15} g cm^{-3}	initial cooling ~ 15-20 seconds ~ thousands of years

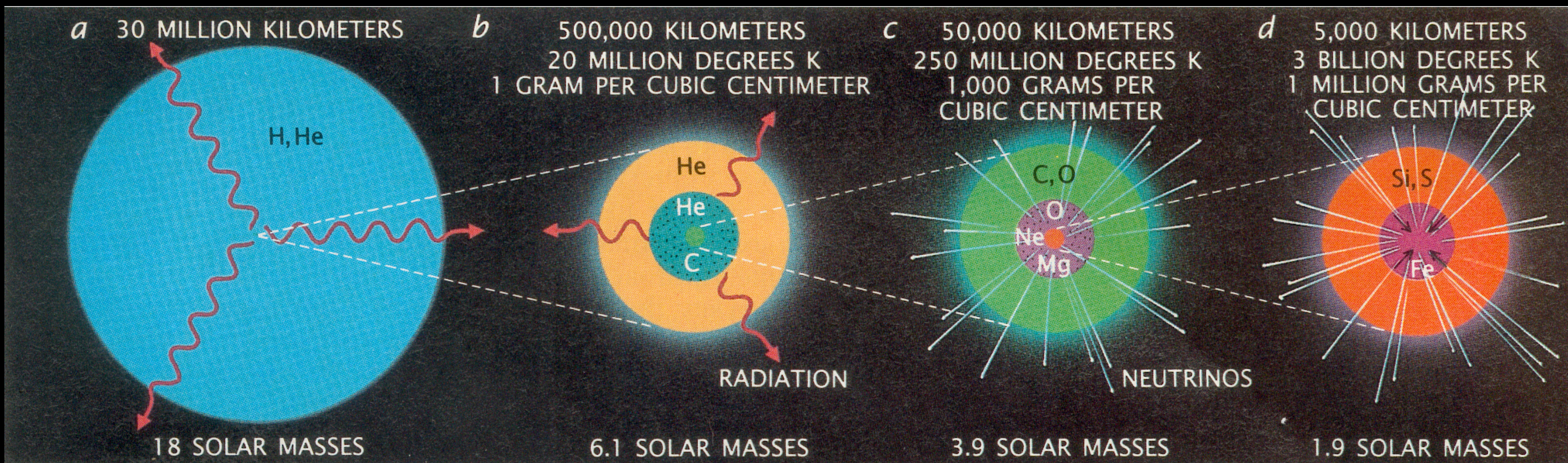
Massive Stars are Giant Refrigerators

From core carbon/oxygen burning onward
the neutrino luminosity exceeds the photon luminosity.

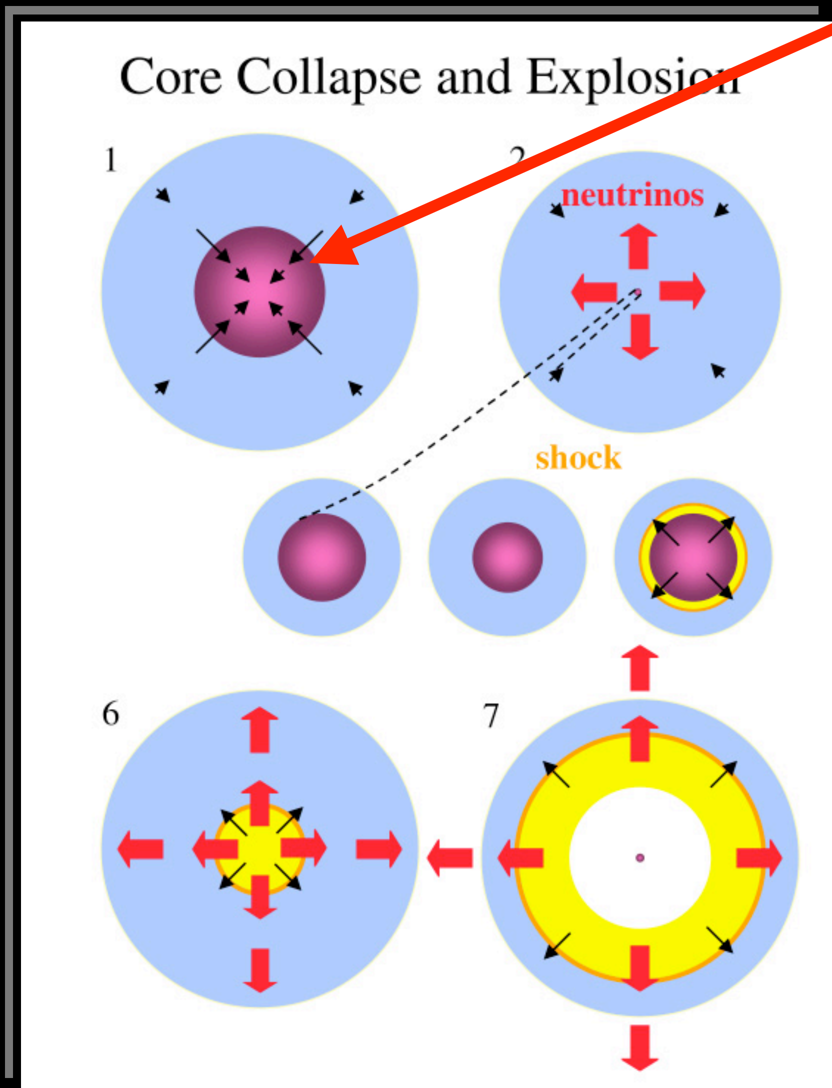
Neutrinos carry energy/entropy away from the core!

Core goes from S/k~10 on the Main Sequence (hydrogen burning)
to a thermodynamically cold S/k ~1 at the onset of collapse!

e.g., the collapsing core of a supernova can be a
frozen (Coulomb) crystalline solid with a
temperature ~1 MeV!



Weaver & Woosley, *Sci Am*, 1987



A. Mezzacappa

Inner core where the infall velocity v is subsonic and proportional to radius $v \sim r$ (homologous) and whose mass is proportional to $Y_e^{\,2}$.

The rest of the original Fe-core, the outer core, falls in supersonically.

Inner core "bounces" as a unit at or above nuclear density (i.e., when nucleons touch). Shock wave generated at this core's edge at bounce.

Shock's energy subsequently is degraded as it plows through outer core material.

Entropy jump is a factor ~10 across shock, so NSE favors "photo-disintegrating" nuclei.

Pulling a nucleon out of nucleus requires ~ 8 MeV. This is 10^{51} ergs for each 0.1 solar masses transited by shock!

Neutrinos Dominate the Energetics of Core Collapse Supernovae

Explosion
only ~1% of neutrino energy

- Total optical + kinetic energy, 10^{51} ergs
- Total energy released in Neutrinos, 10^{53} ergs

10% of star's rest mass!

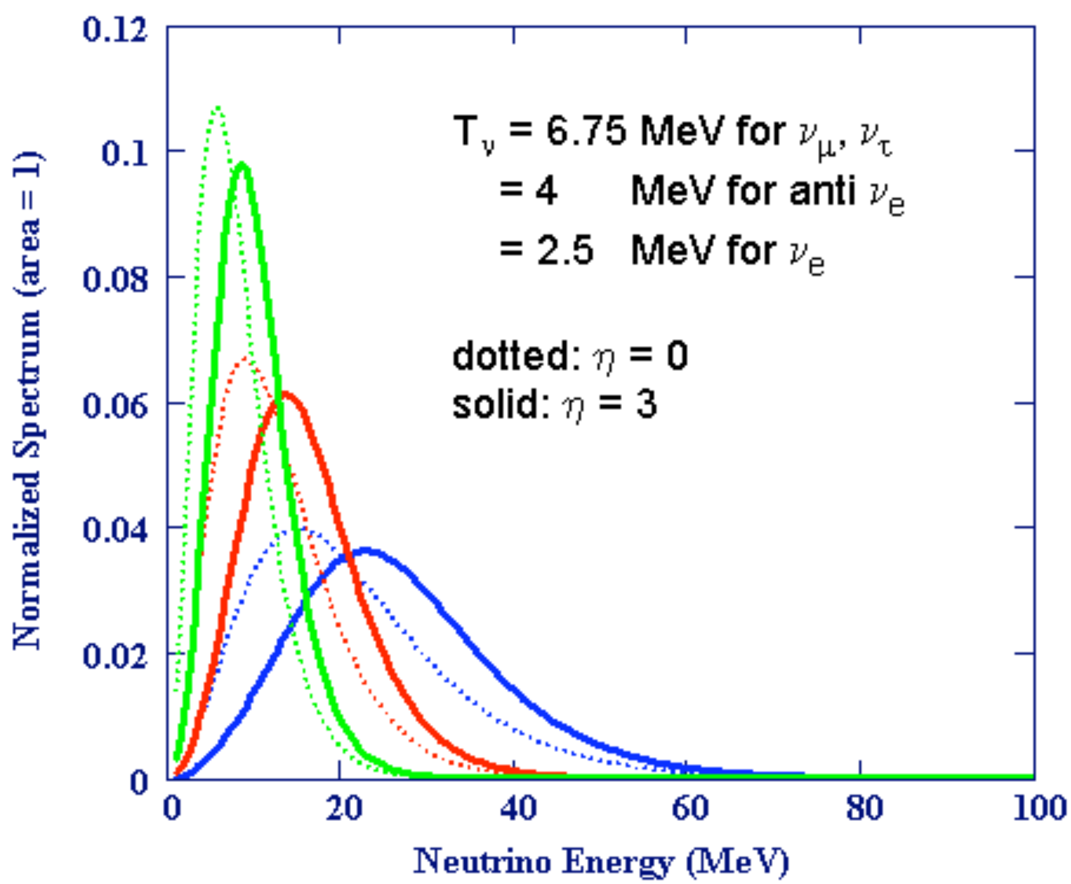
$$\boxed{E_{\mathrm{GRAV}} \approx \frac{3}{5}\frac{\mathrm{G}\,M_{\mathrm{NS}}^2}{R_{\mathrm{NS}}} \approx 3\times 10^{53}\,\mathrm{ergs}\left[\frac{M_{\mathrm{NS}}}{1.4\mathrm{M}_{\mathrm{sun}}}\right]^2\left[\frac{10\ \mathrm{km}}{R_{\mathrm{NS}}}\right]}$$

Neutrino diffusion time, $\boxed{\tau_\nu \approx 2\ \mathrm{s}}$

$$\boxed{L_\nu \approx \frac{1}{6}\frac{GM_{\mathrm{NS}}^2}{R_{\mathrm{NS}}}\frac{1}{\tau_\nu} \approx 4\times 10^{51}\,\mathrm{ergs\ s^{-1}}}$$

Neutrino Energy Spectra

- at "Neutrino Sphere"
- Near Fermi-Dirac energy distribution



The flux of neutrinos in a pencil of directions and energies is

$$d\varphi_\nu \approx \frac{L_\nu}{\pi R_\nu^2}\left(\frac{d\Omega_\nu}{4\pi}\right)\frac{1}{\langle E_\nu \rangle} f(E_\nu)dE_\nu$$

The (black body) neutrino distribution function is

$$f\left(E_\nu\right) = \frac{1}{T_\nu^3 F_3\left(\eta_\nu\right)} \frac{E_\nu^2}{e^{E_\nu/T_\nu - \eta_\nu} + 1}$$

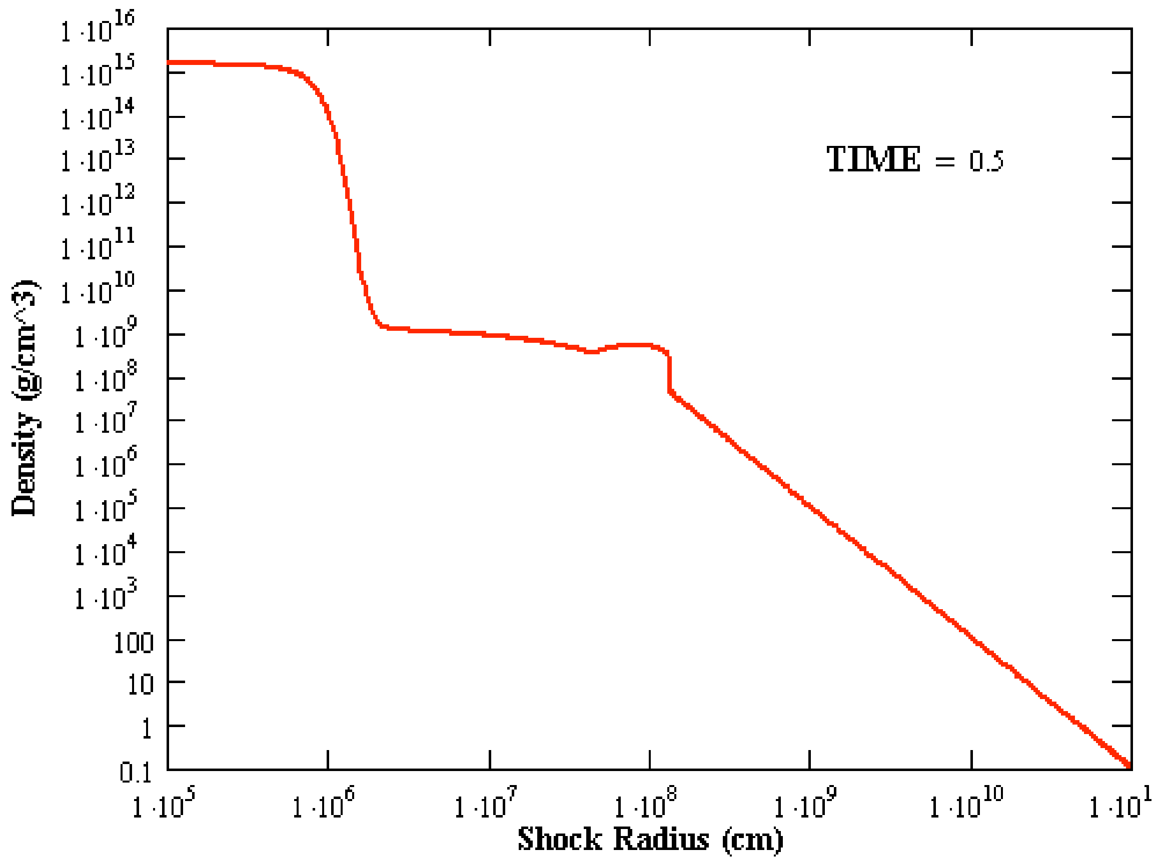
$$F_k\left(\eta_\nu\right) = \int_0^\infty \frac{x^k dx}{e^{x-\eta_\nu} + 1}$$

Neutron-to-proton ratio and energy deposition behind shock are largely determined by these processes:

$$\boxed{\nu_e + n \longrightarrow p + e^-}$$

$$\boxed{\bar{\nu}_e + p \longrightarrow n + e^+}$$

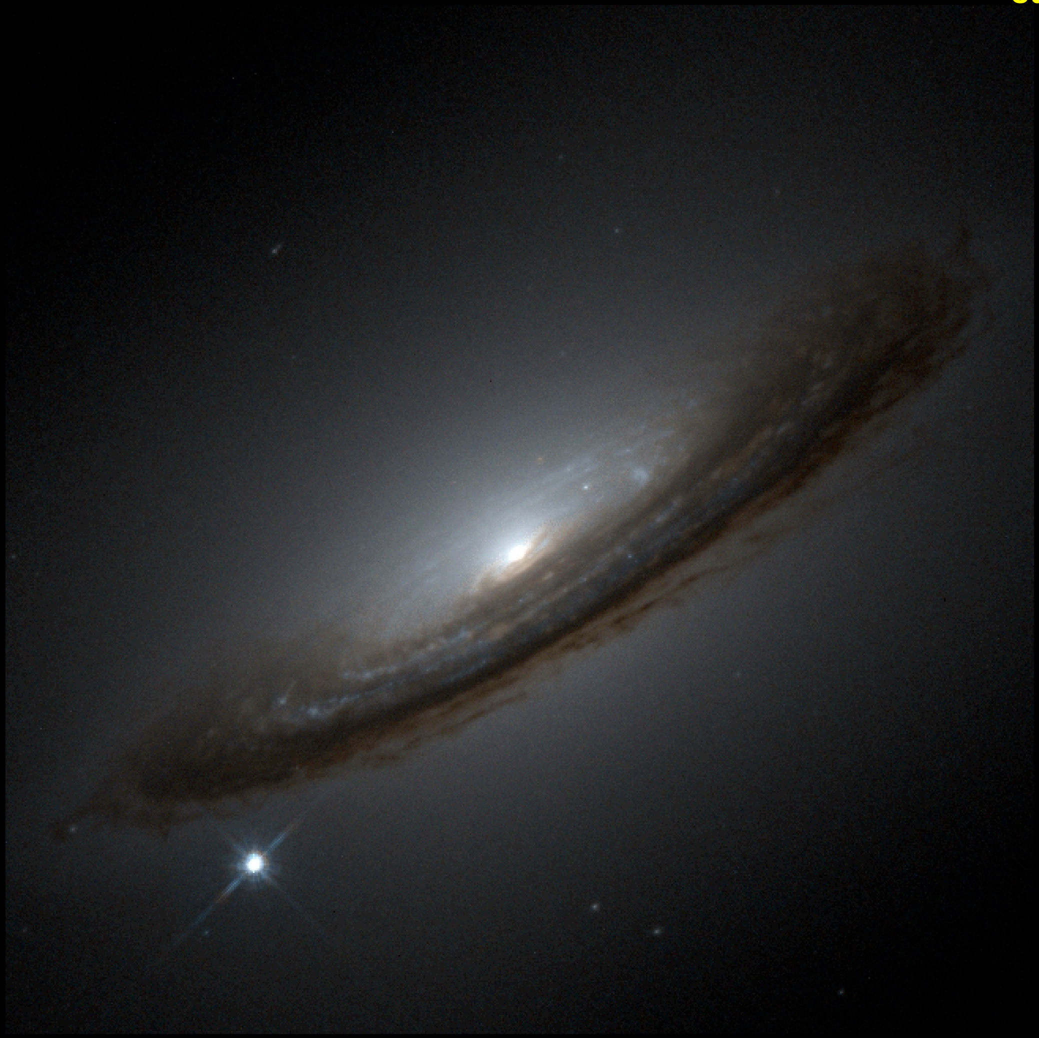
Shock Propagation



R. Schirato & G. Fuller, astro-ph/0205390


The result of coupling in
1% of the neutrino energy?

Photon luminosity of a supernova is huge: $L \sim 10^{10}\ L_{sun}$



cse.ssl.berkeley.edu/

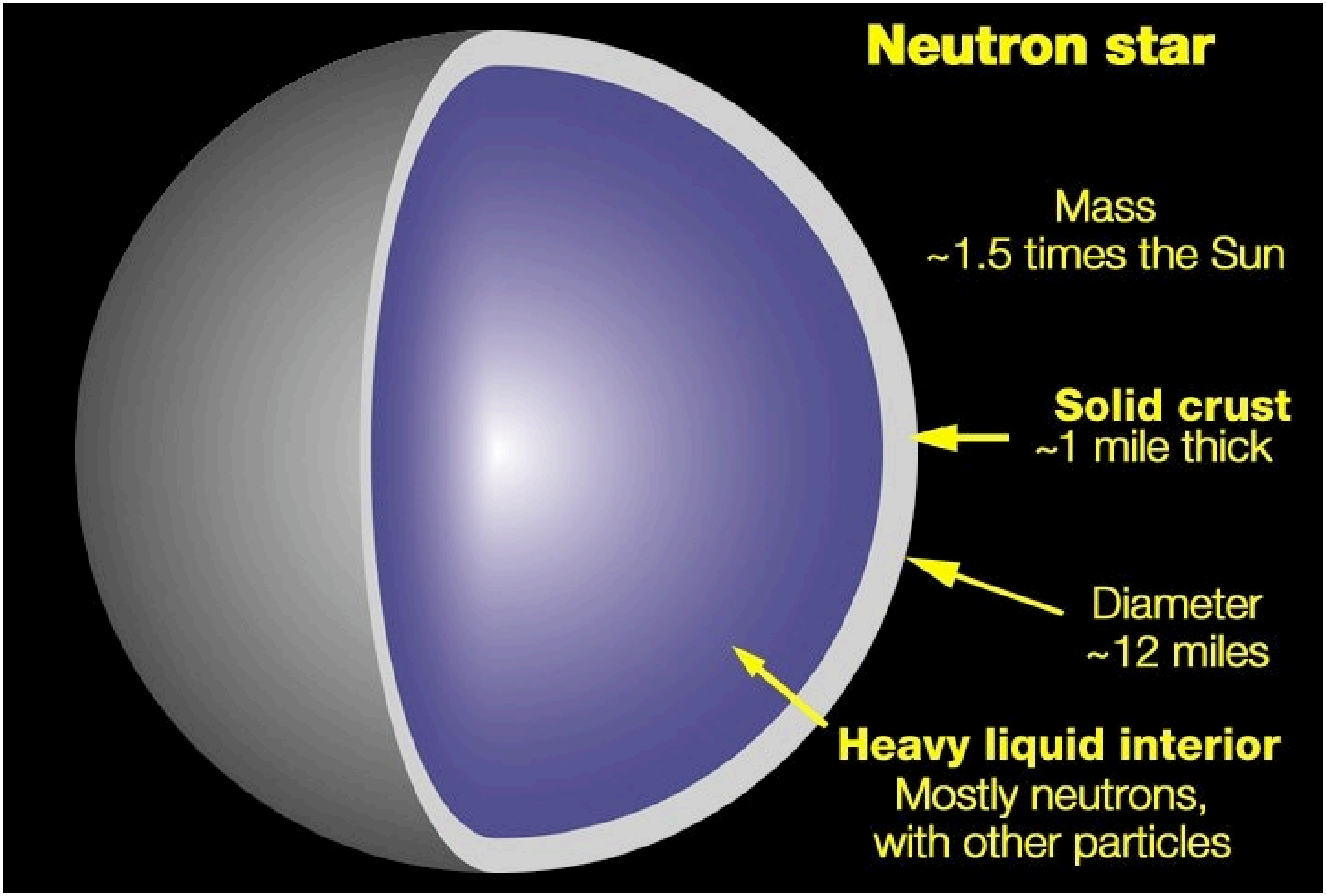
How Frequently Does Stellar Collapse Happen?

Core collapse supernova rate = 1 per galaxy per 30 years
$\sim 10^{-9}$ per galaxy per second

Our galaxy (the milky way) is $\sim 10^{10}$ years old,
so there have been $\sim 10^{8}$ supernovae in the history of the galaxy.

There is ~ 1 galaxy per Mpc^3 and the causal horizon is ~ 3000 Mpc,
so there are (*currently*) $\sim 10^{10}$ galaxies in the "observable" universe,
implying that there are ~ 10 collapse/supernova events every second inside the causal horizon!

"Cold" Neutron Star: interior may consist of exotic phases of nuclear matter.



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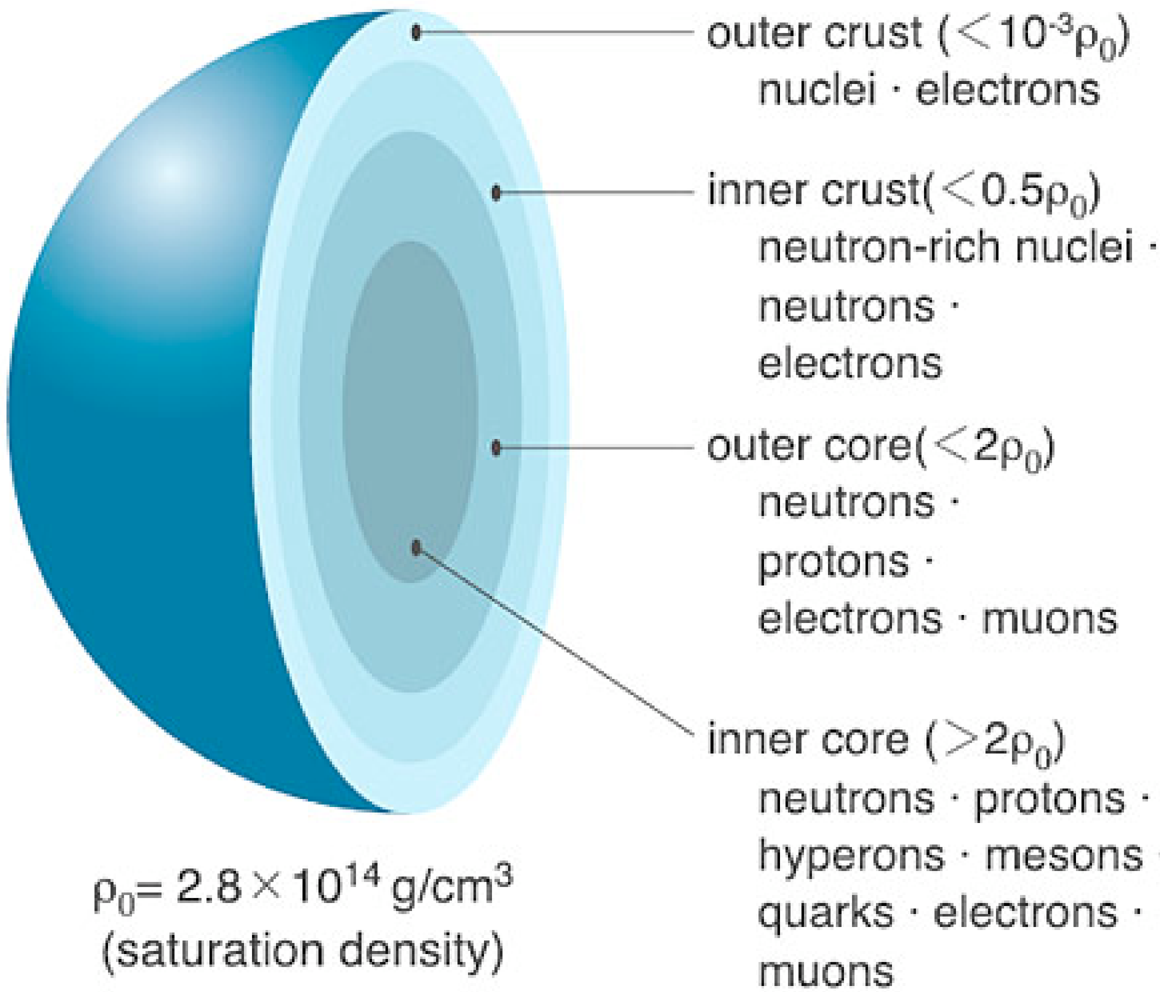


lnisjp.tokai.jaeri.go.jp/

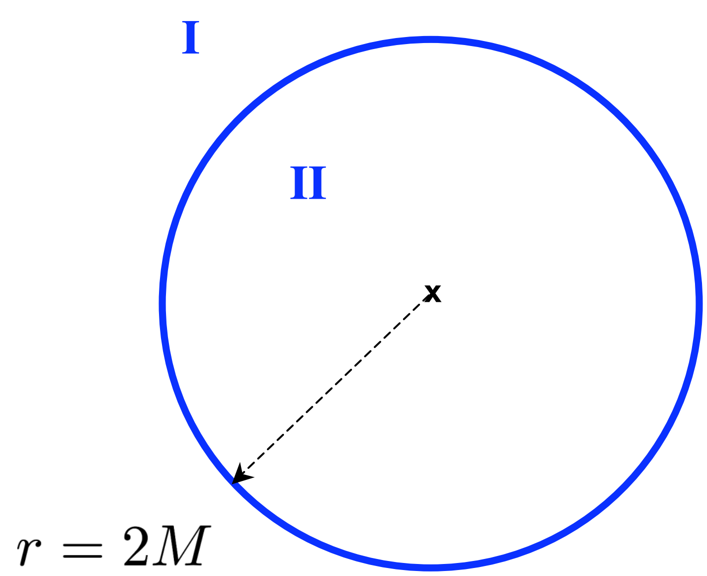
Neutron stars are very close to instability in General Relativity !

Suppose they accrete some matter and go dynamically unstable and begin collapsing. What happens?

There is likely nothing to halt the collapse at this point.
Once the object shrinks inside Schwarzschild radial coordinate $r = 2M$ all of its material is dragged to $r = 0$,
crushed to mathematical point, the singularity.
(Of course, quantum mechanics will intervene once all the material is crushed inside a physical radius $L \sim 10^{-33}$ cm .)

Collapse to a Black Hole in Schwarzschild Geometry

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2}$$



Vacuum everywhere,
in both regions **I** and **II**.

Singularity at $r = 0$ at center.

Coordinate singularity at $r = 2M$.

Transforming the Metric Components

$$g_{\bar{\alpha}\,\bar{\beta}} = \Lambda^{\mu}{}_{\bar{\alpha}}\,\Lambda^{\nu}{}_{\bar{\beta}}\,g_{\mu\,\nu}$$

. . . this is how the "components" of the metric transform under a coordinate transformation, where the transformation matrix elements are (for a so-called "coordinate basis")

$$\Lambda^{\mu}{}_{\bar{\alpha}} = \frac{\partial x^{\mu}}{\partial x^{\bar{\alpha}}}$$

for the two different coordinate systems $\{x^{\bar{\alpha}}\}$ **and** $\{x^{\mu}\}$

If we have one solution to Einstein's Field Equations, *e.g.,* the Schwarzschild metric/coordinates, we can always find another by a coordinate transformation.

Eddington-Finkelstein Coordinates

$$(v, r, \theta, \varphi)$$

Where we define v by this transformation:

$$v = t + r + 2M \ln \left| \frac{r}{2M} - 1 \right|$$

The vacuum Schwarzschild metric can be transformed by this to give a new metric (line element):

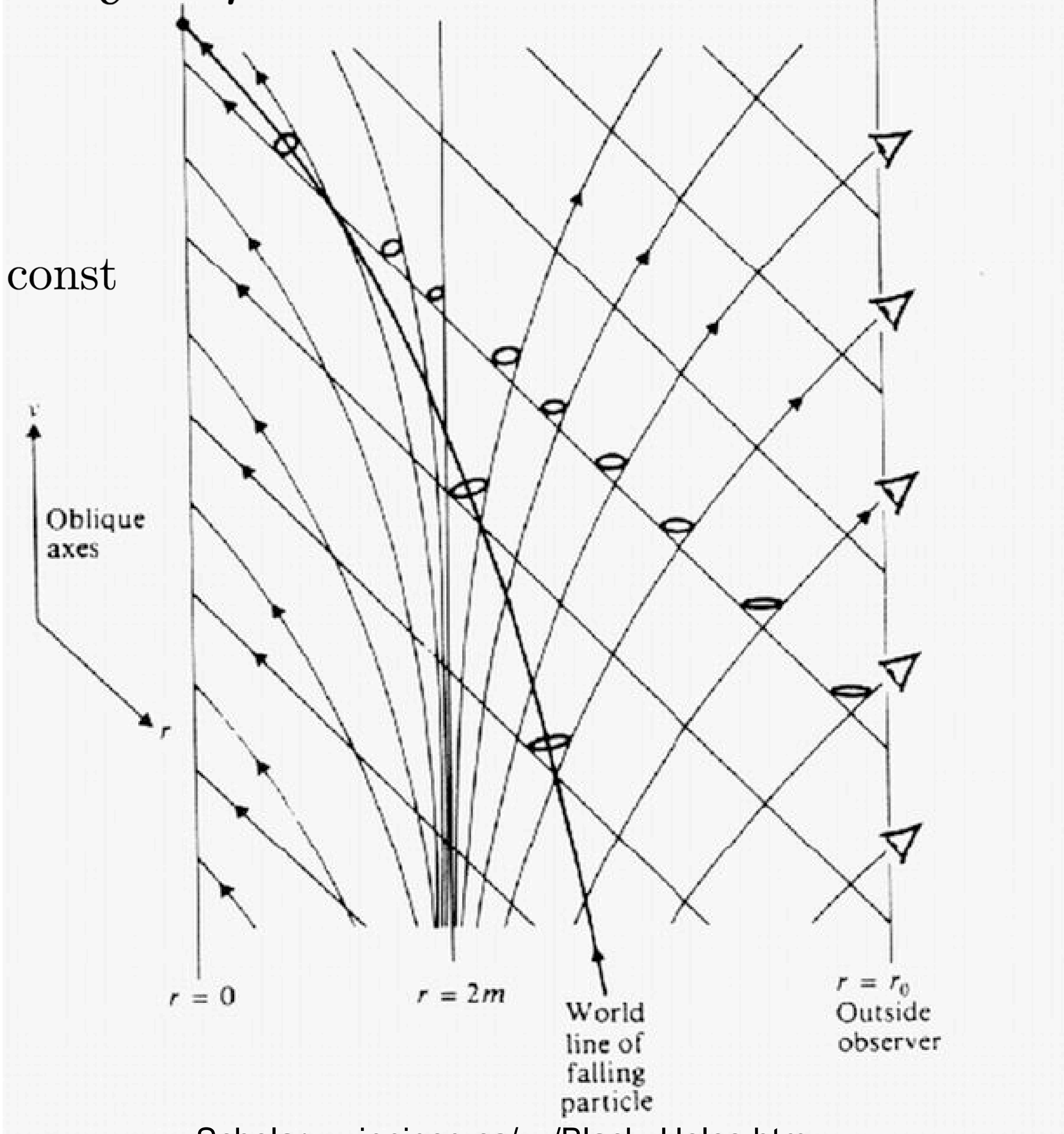
$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2}$$

Radial, null world lines (light lines) with this metric are:

$$v = \text{constant}$$

$$v - 2\left(r + 2M \ln\left|\frac{r}{2M}\right|\right) = \text{const}$$

$v - r$



Scholar.uwinnipeg.ca/…/Black_Holes.htm

Kruskal-Szekeres Coordinates

Transform the metric components from those corresponding to the Schwarzschild coordinates (t, r, θ, ϕ) to those corresponding to a new coordinate system, the Kruskal-Szekeres (K-S) coordinates, (V, U, θ, ϕ) .

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2}$$

Schwarzschild

$$\boxed{ds^2 = \frac{32M^3}{r} e^{-r/2M}\left(-dV^2 + dU^2\right) + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2}$$

Kruskal-Szekeres

where the relation between the coordinates is:

$$r > 2M \quad \begin{cases} U = \left(\dfrac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh\left(\dfrac{t}{4M}\right) \\ V = \left(\dfrac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh\left(\dfrac{t}{4M}\right) \end{cases}$$

$$r < 2M \quad \begin{cases} U = \left(1 - \dfrac{r}{2M}\right)^{1/2} e^{r/4M} \sinh\left(\dfrac{t}{4M}\right) \\ V = \left(1 - \dfrac{r}{2M}\right)^{1/2} e^{r/4M} \cosh\left(\dfrac{t}{4M}\right) \end{cases}$$

$$\boxed{\left(\frac{r}{2M} - 1\right) e^{r/2M} = U^2 - V^2}$$

So we can regard Schwarzschild radial coordinate r as a function of U and V.

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

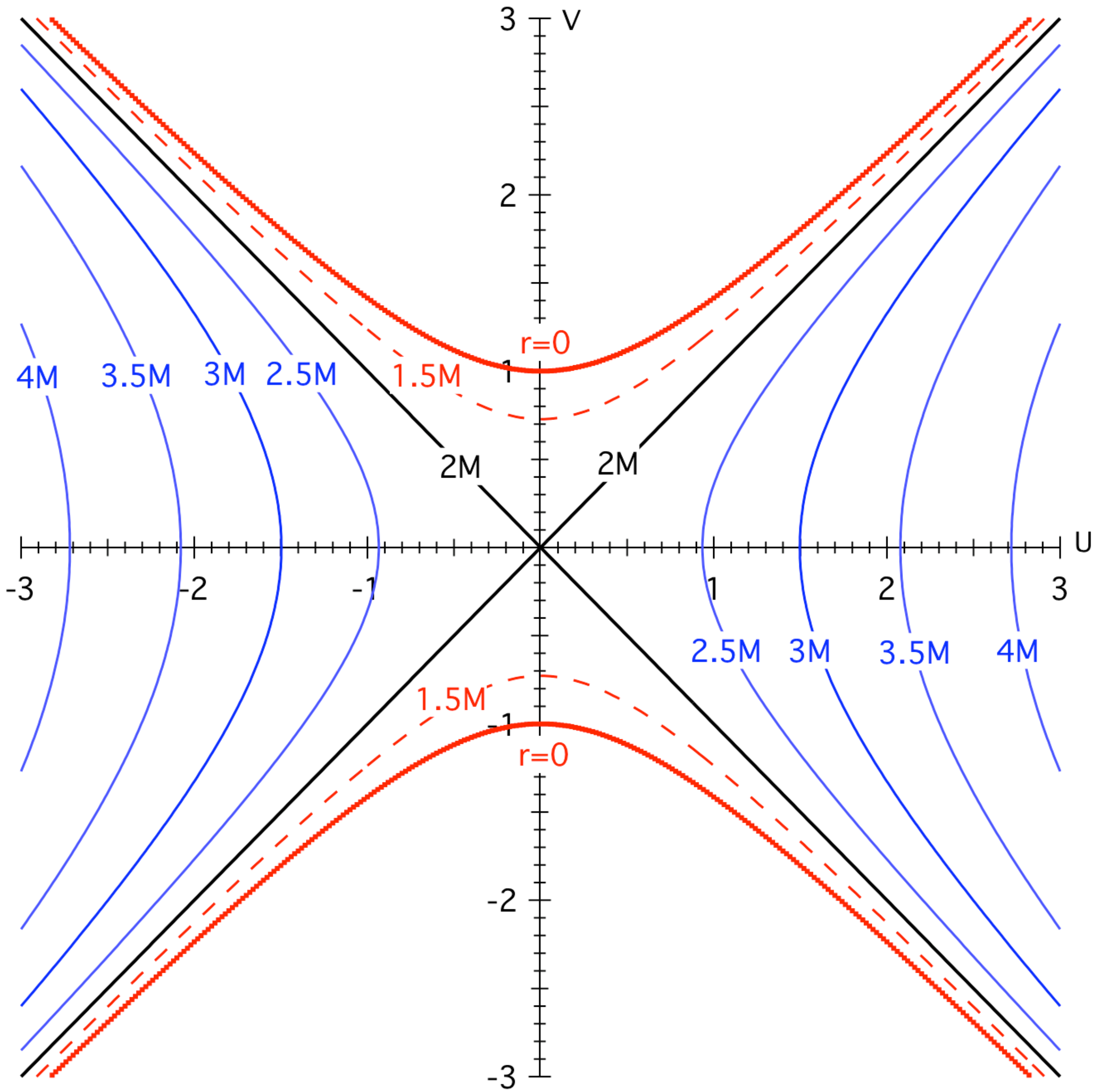
$$\cosh^2 x - \sinh^2 x = 1$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\boxed{\left(\frac{r}{2M}-1\right)e^{r/2M} = U^2 - V^2}$$

Lines of constant Schwarzschild radial coordinate r are curves of constant U^2-V^2, that is, hyperbolae in the U-V plane (Kruskal Diagram).

- **The value $r = 2M$ corresponds to the straight lines** $V = \pm U$**.**
- **Note that $r = 0$, the singularity, corresponds to hyperbolae** $V = \pm\sqrt{U^2 + 1}$
- **The value $r = 4M$ corresponds to hyperbolae** $U = \pm\sqrt{V^2 + e^2}$
- **The value $r = 1.5M$ corresponds to hyperbolae** $V = \pm\sqrt{U^2 + 0.25\, e^{0.75}}$
- **The value $r = 3M$ corresponds to hyperbolae** $U = \pm\sqrt{V^2 + 0.5\, e^{1.5}}$



V
U
3
2
1
-1
-2
-3
r=0
r=0
1.5M
1.5M
2M
2M
2.5M
3M
3.5M
4M
2.5M
3M
3.5M
4M

Constant Schwarzschild timelike coordinate t corresponds to straight lines on the Kruskal diagram. From the definitions of U and V given a few slides back we can form ratios to discover that:

$$\boxed{\tanh\left(\frac{t}{4M}\right) = \frac{V}{U} \text{ for } r > 2M}$$

$$\boxed{\tanh\left(\frac{t}{4M}\right) = \frac{U}{V} \text{ for } r < 2M}$$

$$t = \infty \text{ corresponds to } U = V$$

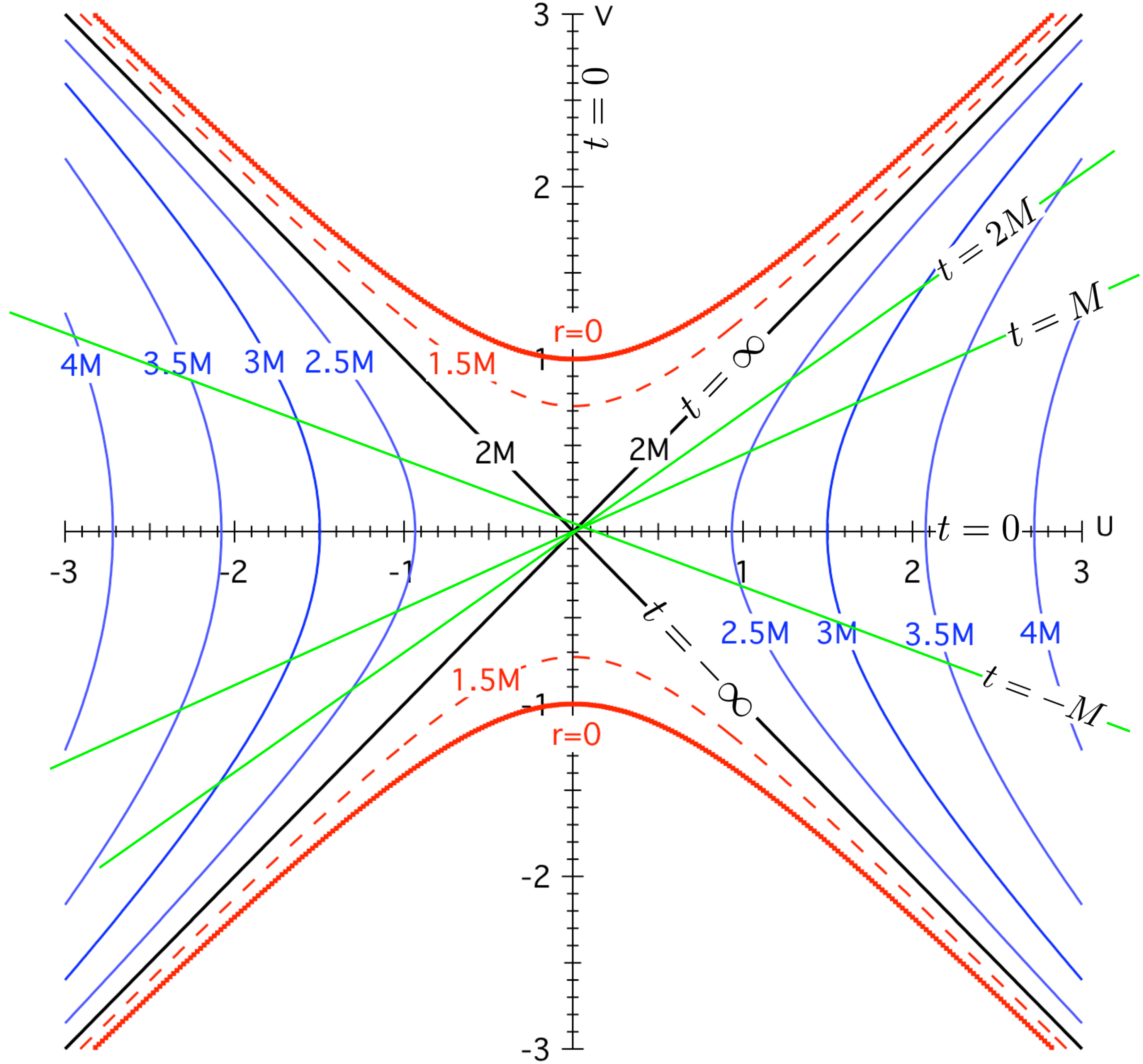
$$t = -\infty \text{ corresponds to } U = -V$$

$$t = 0 \text{ corresponds to } V = 0 \text{ for } r > 2M$$

$$t = 0 \text{ corresponds to } U = 0 \text{ for } r < 2M$$

$$t = -M \text{ corresponds to } V \approx -0.245U$$

$$t = M \text{ corresponds to } V \approx 0.245U$$



V
U
$t=0$
$t=0$
$t=2M$
$t=M$
$t=\infty$
$t=-\infty$
$t=-M$
r=0
r=0
1.5M
1.5M
2M
2M
2.5M
2.5M
3M
3M
3.5M
3.5M
4M
4M

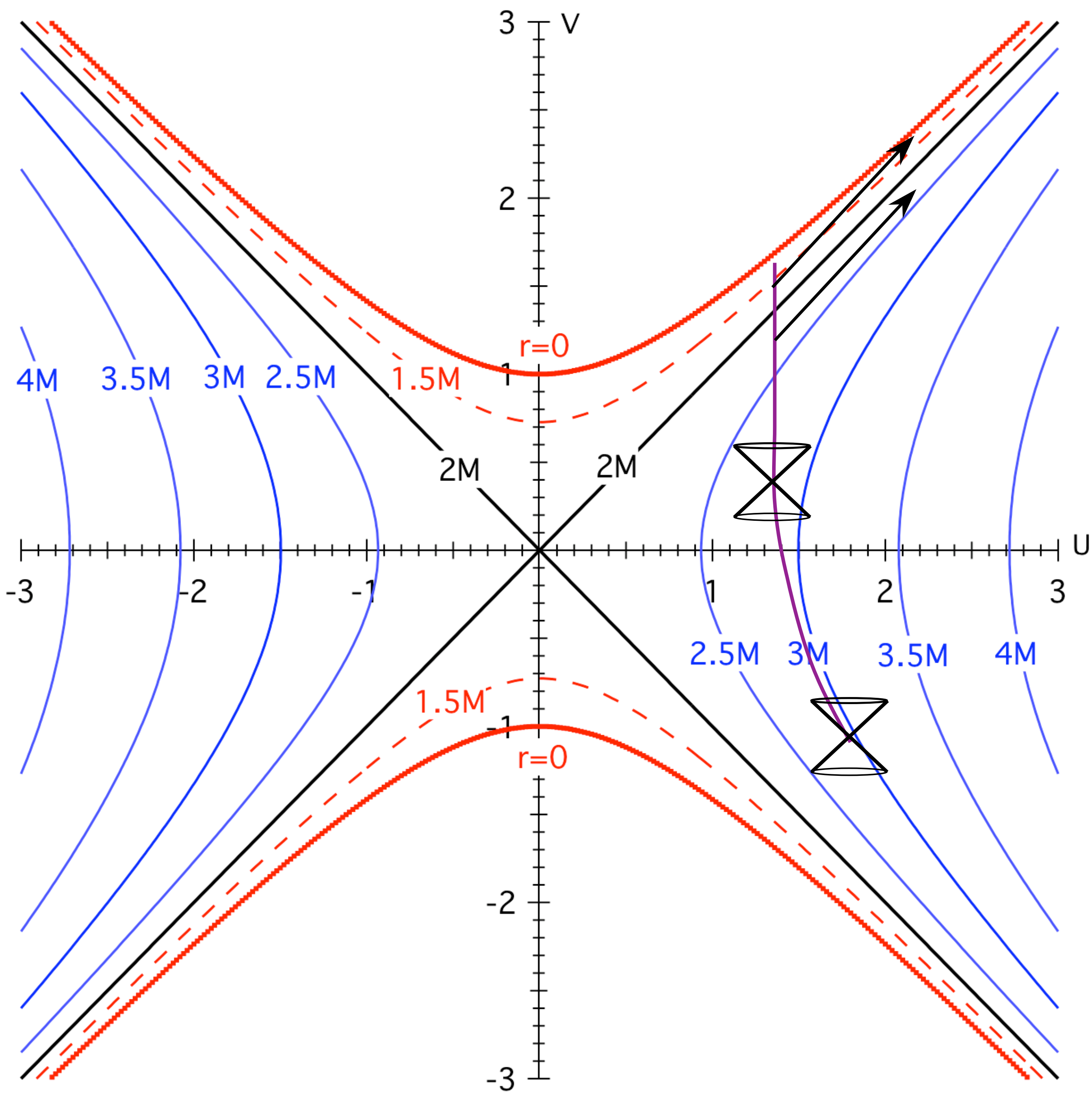
Kruskal-Szekeres Metric

$$\boxed{ds^2 = \frac{32M^3}{r} e^{-r/2M}\left(-dV^2 + dU^2\right) + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2}$$

Note that radial (dθ=dφ=0), **null geodesics (light lines) are 45^o *straight lines* on a Kruskal diagram!**

$$ds^2 = 0 \;\Rightarrow U = \pm V \text{ for radial null worldline}$$

Light cones on a Kruskal diagram are just like in a flat (Minkowski) spacetime diagram: 45^o cones !



V
U
3
2
1
-1
-2
-3
-3
-2
-1
1
2
3
r=0
r=0
1.5M
1.5M
2M
2M
4M
3.5M
3M
2.5M
2.5M
3M
3.5M
4M

Since all physical particles and information must travel on world lines contained in future-directed light cones, we see from the Kruskal diagram that the null surface defined by $r = 2M$ is an *HORIZON*, or *event horizon*, that acts like a one-way membrane.

All particles, including photons, originating inside this surface ($r < 2M$) can never get out and they will always wind up on the singularity at $r = 0$.