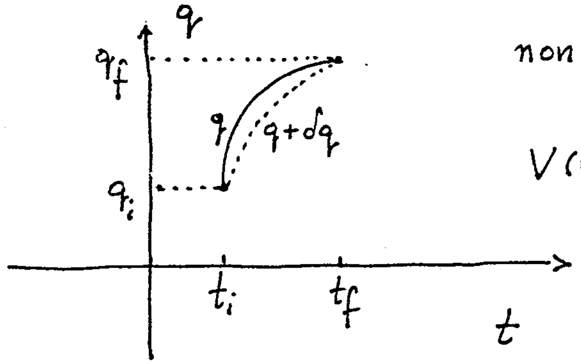
THE UNIFIED ACTION PRINCIPLE

(addendum to PH 141/lecture 2)

Action Principle



nonrelativistic particle in one dimension

$V(q)$ potential energy

$$S = \int_{t_i}^{t_f} L(q, \dot{q})\, dt$$

$\delta S = 0$ for physical orbit

$q(t) \to q(t) + \delta q(t)$

$$L = K - V$$

$$K = \frac{1}{2} m \dot{q}^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$ Euler-Lagrange equation (Newton equation)

$$m\ddot{q} = f \qquad f = -\frac{\partial V}{\partial q}$$

$$p = \frac{\partial L}{\partial \dot{q}}$$ definition of canonical momentum

$$p = m\dot{q}$$

$$H = p\dot{q} - L \quad \text{Hamiltonian}$$

$$H = \frac{p^2}{m} - \frac{p^2}{2m} + V(q) = \frac{p^2}{2m} + V(q) \quad \text{energy}$$

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quantum theory $[q, p] = i\hbar$

oscillator example $V(q) = \frac{1}{2} m\omega^2 q^2$ operators

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(q - i \frac{1}{m\omega} p\right)$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(q + i \frac{1}{m\omega} p\right)$$

creation and annihilation operators

$$[a, a^+] = 1$$

$$H = \hbar\omega \left(a^+ a + \frac{1}{2}\right)$$

zero point energy (can be left out by normalization)

note on gravity and cosmological constant

$a|0\rangle = 0$ ground state

$a^+|0\rangle$ first excited state

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle \qquad H|n\rangle = (n + \tfrac{1}{2})\hbar\omega |n\rangle$$

$$\langle\psi(t)| a |\phi(t)\rangle = \langle\psi(0)| e^{\frac{i}{\hbar}Ht} a \, e^{-\frac{i}{\hbar}Ht} |\phi(0)\rangle$$

$$a(t) = e^{\frac{i}{\hbar}Ht}\, a\, e^{-\frac{i}{\hbar}Ht} \qquad a(0) = a$$

Heisenberg picture

$$\frac{d\,a(t)}{dt} = \frac{i}{\hbar}[H, a] \qquad \dot{a}(t) = -i\omega\, a(t)$$

$$a(t) = e^{-i\omega t} a$$

$$a^{+}(t) = e^{i\omega t} a^{+}$$

$$q(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(a e^{-i\omega t} + a^{+} e^{i\omega t}\right)$$

satisfies the Euler-Lagrange equation $m\ddot{q} = -m\omega^2 q$

$\ddot{q} + \omega^2 q = 0$

classically a, a^* are determined by initial conditions $q(0)$ and $p(0)$

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generalization to N degrees of freedom

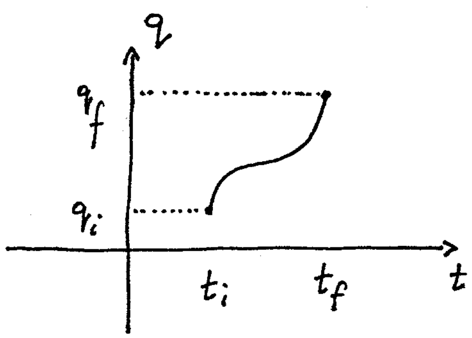
$$S = \int_{t_i}^{t_f} L(q_k, \dot{q}_k)\, dt \qquad L(q_1, \dot{q}_1, q_2, \dot{q}_2, \dots, q_N, \dot{q}_N)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} \qquad k = 1, 2, \dots, N$$

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \qquad k = 1, 2, \dots, N \qquad \text{canonical momenta}$$

$$H = \sum_k p_k \dot{q}_k - L \qquad \text{Hamiltonian}$$

Action Principle and Quantum Theory (digression)



$$S = \int_{t_i}^{t_f} L(q, \dot{q})\, dt$$

$\uparrow$ K − V

$\delta S = 0$ action principle in classical physics selects physical trajectory

$$\delta S = \int_{t_i}^{t_f} \left[L(q + \delta q, \dot{q} + \delta \dot{q}) - L(q, \dot{q}) \right] dt$$

$$\delta S = \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \, dt$$

by partial integration

must vanish for arbitrary δq

Newton equation

in Quantum Mechanics:

$$G(q_f, q_i; t_f, t_i) = \langle q_f | e^{-\frac{i}{\hbar} H (t_f - t_i)} | q_i \rangle$$

fundamental object (propagator)

at $t = t_i$ particle is prepared at $q = q_i$
probability amplitude (complex) that particle will be found at $q = q_f$ at some later time $t = t_f$

$$\langle \psi_f | e^{-\frac{i}{\hbar} H t} | \psi_i \rangle = \int dq' \int dq \; \psi_f^*(q') \, G(q', q; t_f - t_i) \, \psi_i(q)$$

at $t = t_i$ $\quad |\psi_i\rangle$ general initial state

$t = t_f$ $\quad |\psi_f\rangle$ probability amplitude of some final state

$$G(q_f, q_i; t_f - t_i) = \sum_{\text{all paths}} e^{\frac{i}{\hbar} S(\text{path})}$$

Quantum Action Principle

Feynman Path Integral

(contains classical limit)

$\frac{S}{\hbar}$ phase of a path

$$\hbar = \frac{h}{2\pi} = 1.0546 \times 10^{-34} \, \text{Js} = 6.5822 \times 10^{-22} \, \text{MeV s}$$

Planck's constant

microscopic motion $S \lesssim \hbar$ all paths contribute (quantum regime)

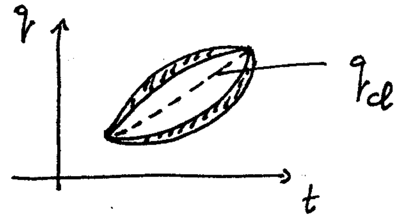
electron moving over distance $q_f - q_i = 0.1 \times 10^{-8}$ cm (size of atom), or smaller

$v = 0.1\,c$ nonrelativistic

$$t_f - t_i = \frac{0.1 \times 10^{-8} \, \text{cm}}{0.3 \times 10^{10} \frac{\text{cm}}{\text{s}}} \approx .3 \times 10^{-18} \, \text{s}$$

$$S = \frac{1}{2} \underbrace{0.5 \, \text{MeV}/c^2}_{m_e} \cdot (0.01\,c^2) \cdot 0.3 \times 10^{-18} \, \text{s} = 7 \times 10^{-22} \, \text{MeVs}$$

$\frac{\delta S}{\hbar}$ phase is slowly changing in quantum motion over macroscopically measurable spread of path bundle, nonclassical trajectories all contribute in quantum situation

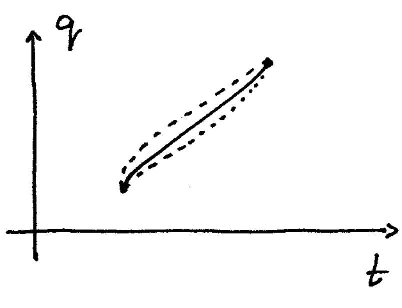


<u>macroscopic body</u> (classical limit)

$m = 1\ gr \quad \left(m_e = 9.1 \times 10^{-28}\ gr\right)$

$v = \frac{1}{10} c$

$l = 1\ cm \qquad S \sim 10^{17}\ MeVs \sim 10^{40}\ \hbar$



only very narrow band around the classical path contributes