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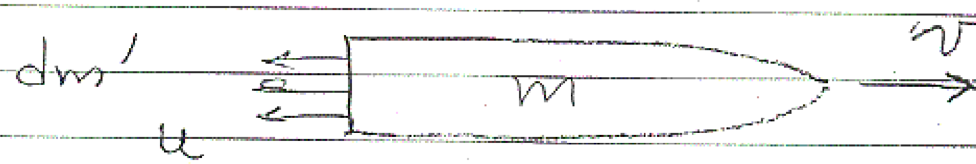
11. Rocket Motion

The motion of a rocket is a useful applicati. of Newton's laws and the conservation of momentum.

Consider a rocket in free space, i.e. $F^{(e)} = 0$. It accelerates by virtue of N3.



- Let rocket have mass m at time t, moving with speed v relative to IRF $\equiv$ inertial reference frame.
- Suppose in time dt an amount dm' of exhaust gas leaves rocket at speed u relative to the rocket, accelerating rocket to speed $v + dv$. Then, by COM for system

$$P_{initial}(t) = P_{final}(t+dt)$$

$$mv = (m - dm')(v + dv) + dm'(v - u)$$

$$mv = mv + m\,dv - dm'v + \cancel{dm'dv} + dm'v - dm'u$$

$$m\,dv = u\,dm'$$

$$\therefore \boxed{dv = u\frac{dm'}{m}}$$

We have ignored ~~second~~ order term $dm'dv$

Change of mass of rocket ship in time dt is

$$dm = -dm'$$

so

$$dv = -u\frac{dm}{m}$$

Integrating from $m(t=0) = m_0$

$v(t=0) = v_0$

$$\boxed{v = v_0 + u \ln\left(\frac{m_0}{m}\right)}$$

Exhaust velocity depends on properties of propellant and nozzle, but can't be varied much because chemical energies are ~ 1 eV/particle. Key to optimizing rockets is quantity m_0/m.

If m_0 is mass of full rocket

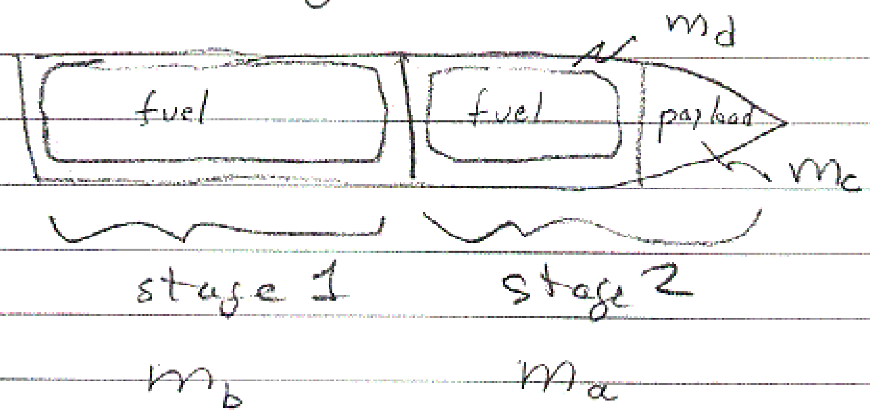
m_1 is mass of empty rocket

$$v_{max} = v_0 + u \ln\left(\frac{m_1}{m_0}\right)$$

Since terminal velocity is limited by m_1/m_0 for a single stage rocket, the concept of <u>multi-stage rockets</u> was developed.

Consider a 2-stage rocket with initial mass m_0 (1st stage rocket + payload + fuel) and final mass m_1

after 1^{st} stage fuel is exhausted



Let m_a be mass of 1^{st} stage payload (2^{nd} stage
m_b be " " 1^{st} stage fuel container

then

$$m_1 = m_a + m_b$$

$$v_1 = v_0 + \ln\left(\frac{m_0}{m_1}\right)$$

Now 2^{nd} stage consists of payload, rocket and fuel

Let m_c be mass of 2^{nd} stage payload
m_d be mass of 2^{nd} stage fuel container

Define

$$m_2 = m_c + m_d$$

$$= \text{mass of } 2^{nd} \text{ stage minus fuel}$$

then

$$v_2 = v_1 + u \ln\left(\frac{m_a}{m_2}\right)$$

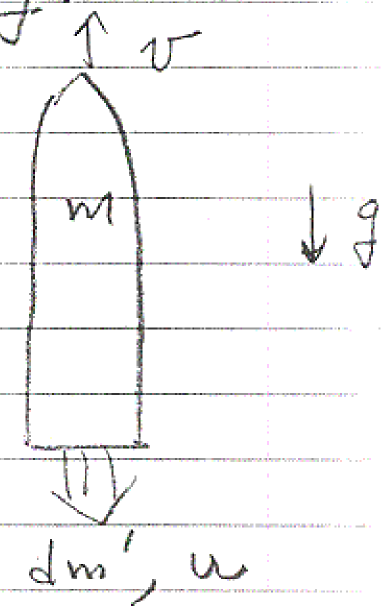
$$= v_0 + u\left[\ln\left(\frac{m_0}{m_1}\right) + \ln\left(\frac{m_a}{m_2}\right)\right]$$

$$\boxed{v_2 = v_0 + u \ln\left(\frac{m_0}{m_1}\frac{m_a}{m_2}\right)}$$

The quantity $M_0 M_a / M_1 M_2$ can be made much larger than M_0/M_1. Principle can be extended to 3, 4 ... stage rockets. Multi-stage rockets are used to achieve escape velocity from a planet's surface (e.g., Earth).

Vertical ascent under gravity.

Now consider situation where rocket is acted upon by external force due to gravity.



N2 tells us

$$\frac{d}{dt}(mv) = F^{(e)} = -mg$$

In time dt

$$d(mv) = dp = p(t+dt) - p(t) = -mg\,dt$$

substituting

$$(m-dm')(v+dv) + dm'(v-u) - mv = -mg\,dt$$

ignoring S.O.T.

$$m\,dv - u\,dm' = -mg\,dt$$

$$m\,dv + u\,dm = -mg\,dt$$

Let $\alpha \equiv -\frac{dm}{dt}$ be constant burn rate (+

$$m\,dv - \alpha u\,dt = -mg\,dt$$

solving for dv

$$dv = \left(-g + \frac{\alpha}{m} u\right) dt$$

but $dt = -dm/\alpha$, so

$$dv = \left(\frac{g}{\alpha} - \frac{u}{m}\right) dm$$

Integrating both sides

$$\int_0^v dv = \int_{m_0}^{m} \left(\frac{g}{\alpha} - \frac{u}{m}\right) dm$$

$$\boxed{v = -\frac{g}{\alpha}(m_0 - m) + u \ln\left(\frac{m_0}{m}\right)}$$

but $m_0 - m = \alpha t$, so

$$\boxed{v = -gt + u \ln\left(\frac{m_0}{m}\right)}$$

This is just the free space result with $v_0 = 0$, and assuming a simple relation between m and t: $m = m_0 - \alpha t$.
Inserting this

$$\boxed{v(t) = -gt + u \ln\left(\frac{m_0}{m_0 - \alpha t}\right)}$$

For a given exhaust velocity u, this equation specifies the minimum burn rate α required to blast off.

Want

$$\frac{dv}{dt}(t=0) > 0$$

$$\frac{dv}{dt} = -g + u(\;)^{-1}\frac{d}{dt}(\;)$$

$$= -g + u\left(\frac{m_0 - \alpha t}{m_0}\right)\frac{m_0(-1)}{(m_0-\alpha t)^2}(-\alpha)$$

$$= -g + \frac{\alpha u}{(m_0 - \alpha t)} > 0 \quad @\ t=0$$

$$\boxed{\alpha > g m_0 / u}$$

Ex) Burn rate of Saturn V

$m_0 = 2.8 \times 10^6$ kg

$u = 2600$ m/s

$$\alpha > \frac{(9.8)(2.8\times 10^6)}{2600} = 10{,}600 \text{ kg/s}$$

How long will first stage fuel last assuming $m_{fuel} = 2.1 \times 10^6$ kg

$$t = \frac{m_{fuel}}{\alpha} = \frac{2.1\times 10^6}{1.06\times 10^4} = 200 \text{ sec.}$$