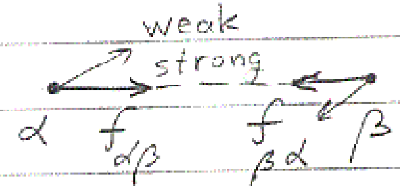
Chapter 9

Dynamics of a System of Particles

Examples
- star cluster
- explosion/impact debris
- atoms/molecules in volume of gas
- chains and meshes

1. Importance of Newton's 3RD Law

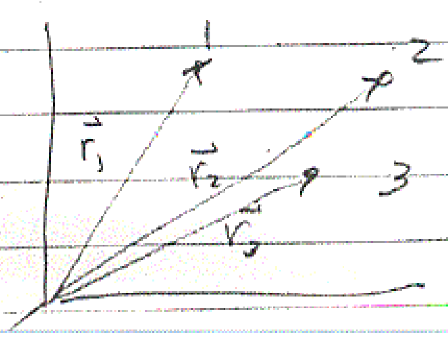


Consider 2 particles in a system of particles (SOP). Let $\vec f_{\alpha\beta}$ be force α exerts on β, and $f_{\beta\alpha}$ vice versa. By N3 $\vec f_{\alpha\beta} = -\vec f_{\alpha\beta}$. This is true regardless of force law, and is called the weak form of N3. If the force is central, i.e. along line connecting α and β, this is called strong form of N3.

The importance of N3 to a system of particles is that only external forces can change the momentum of the system; internal forces cancel in pairs. To show this, we need to build up several important properties of a SOP.

2. Center of Mass (CM)

Consider a cloud of particles with masses m_α and coordinates $\vec r_\alpha$ in some coordinate system.



total mass: $M = \sum_\alpha m_\alpha$

center of mass: $\vec{R} = \frac{1}{M} \sum_\alpha m_\alpha \vec{r}_\alpha$

Note $\vec{R}$ depends on the coordinate system used. (i.e., not translationally or rotationally invariant)
For a continuous mass density distribution $\rho(\vec{r})$,

$$\vec{R} = \frac{\int d^3\vec{r}\; \vec{r}\, \rho(\vec{r})}{\int d^3\vec{r}\; \rho(\vec{r})}.$$

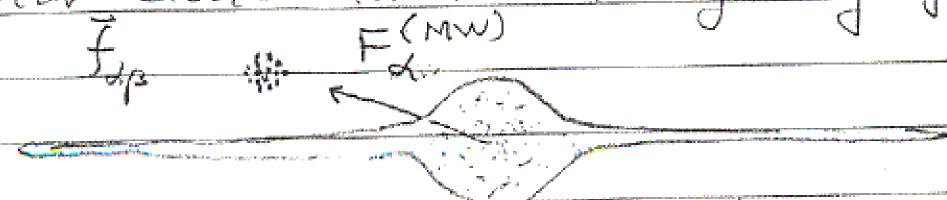
3. Linear Momentum

Force on particle α in a SOP is composed of external and internal forces

$$\vec{F}_\alpha = \vec{F}_\alpha^{(e)} + \vec{F}_\alpha^{(i)}$$
$$= \vec{F}_\alpha^{(e)} + \sum_\beta \vec{f}_{\alpha\beta}.$$

Internal forces arise from self-interactions among SOP (e.g., electrostatic), while external forces are due fields (electric, magnetic, gravity) generated by mass and charges outside the SOP in question.

Example) Star cluster in the Milky Way galaxy



$$\vec{f}_{\alpha\beta} = -\frac{G m_\alpha m_\beta (\vec{r}_\alpha - \vec{r}_\beta)}{|\vec{r}_\alpha - \vec{r}_\beta|^3}; \quad \vec{F}_\alpha^{(MW)} = -m_\alpha \vec{\nabla} \phi_{MW}(\vec{r})$$

By N2, particle α momentum changes as

$$\dot{p}_\alpha \equiv m_\alpha \ddot{\vec{r}}_\alpha = \vec{F}_\alpha^{(e)} + \vec{F}_\alpha^{(i)}$$

or

$$\frac{d^2}{dt^2}(m_\alpha \vec{r}_\alpha) = \vec{F}_\alpha^{(e)} + \sum_\beta \vec{f}_{\alpha\beta}$$

Summing over α, we have

$$\frac{d^2}{dt^2}\underbrace{\sum_\alpha m_\alpha \vec{r}_\alpha}_{M\vec{R}} = \underbrace{\sum_\alpha \vec{F}_\alpha^{(e)}}_{\vec{F}} + \sum_\alpha \sum_{\substack{\beta \\ \alpha\neq\beta}} \vec{f}_{\alpha\beta}$$

The last term vanishes identically since

$$\sum_\alpha \sum_{\substack{\beta \\ \alpha\neq\beta}} \vec{f}_{\alpha\beta} \equiv \sum_{\alpha,\beta\neq\alpha} \vec{f}_{\alpha\beta} = \sum_{\alpha<\beta} (\vec{f}_{\alpha\beta} + \vec{f}_{\beta\alpha}) \doteq 0$$

<u>Proof for n=3 particles</u>

$$\sum_{\alpha,\beta\neq\alpha} \vec{f}_{\alpha\beta} = \vec{f}_{12} + \vec{f}_{13} + \vec{f}_{21} + \vec{f}_{23} + \vec{f}_{31} + \vec{f}_{32} = 0$$

We therefore have

$$\boxed{M\ddot{\vec{R}} = \vec{F}}$$

i.e. CM of SOP obeys N2, where M is total mass, and $\vec{F}$ is total external force.

⇒ CM moves as if it were a single particle of mass M acted on by the total external force.

⇒ This result is independent of the nature of the internal force.

Total linear momentum of system

$$\vec{P} = \sum_\alpha m_\alpha \dot{r}_\alpha = \frac{d}{dt}\sum_\alpha m_\alpha \vec{r}_\alpha = \frac{d}{dt}(M\vec{R}) = M\dot{\vec{R}}$$

and $\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{F}$.
Thus, total linear momentum of SOP is conserved if $\vec{F} = 0$.
N.B. The above is only true in an inertial reference frame. Non-inertial reference frames are discussed in TM Chapter 10.

The above results carry over to the continuum limit. Consider a gas with mass density $\rho(\vec{r})$ and velocity $\vec{v}(\vec{r})$.
Then

$$\vec{P} = \int d^3\vec{r}\; \rho(\vec{r})\, \vec{v}(\vec{r})$$

We have

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{F}$$

where

$$M = \int d^3\vec{r}\; \rho(\vec{r})$$

and

$$\vec{F} = \int d^3\vec{r}\; \rho(\vec{r})\, \vec{a}^{(e)}(\vec{r})$$

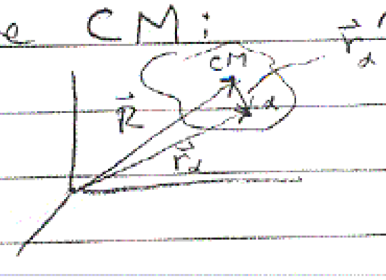
$$= -\int d^3\vec{r}\; \rho(\vec{r})\, \nabla \phi^{(e)}(\vec{r})$$

$\vec{F}$ is simply the external acceleration integrated over the mass distribution. Internal forces in the gas (pressure, viscosity) integrate to 0 by N3.

4. Angular Momentum (AM)

We discussing AM of a SOP, it is convenient to compute it relative to the CM:

$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}'$$



AM of particle α relative to origin of (inertial) coord. sys. is

$$\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$$

Summing over α:

$$\vec{L} = \sum_\alpha \vec{L}_\alpha = \sum_\alpha (\vec{r}_\alpha \times \vec{p}_\alpha) = \sum_\alpha (\vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha)$$

$$= \sum_\alpha (\vec{r}_\alpha{}' + \vec{R}) \times m_\alpha (\dot{\vec{r}}_\alpha{}' + \dot{\vec{R}})$$

$$= \sum_\alpha m_\alpha \left[(\vec{r}_\alpha{}' \times \dot{\vec{r}}_\alpha{}') + (\vec{r}_\alpha{}' \times \dot{\vec{R}}) + (\vec{R} \times \dot{\vec{r}}_\alpha{}') + (\vec{R} \times \dot{\vec{R}}) \right]$$

Middle two terms cancel. To see this, rewrite as

$$\left(\sum_\alpha m_\alpha \vec{r}_\alpha{}' \right) \times \dot{\vec{R}} + \vec{R} \times \frac{d}{dt} \left(\sum_\alpha m_\alpha \vec{r}_\alpha{}' \right)$$

but

$$\sum_\alpha m_\alpha \vec{r}_\alpha{}' = \sum_\alpha m_\alpha (\vec{r}_\alpha - \vec{R})$$

$$= \sum_\alpha m_\alpha \vec{r}_\alpha - \vec{R} \sum_\alpha m_\alpha$$

$$= M\vec{R} - \vec{R}M = 0.$$

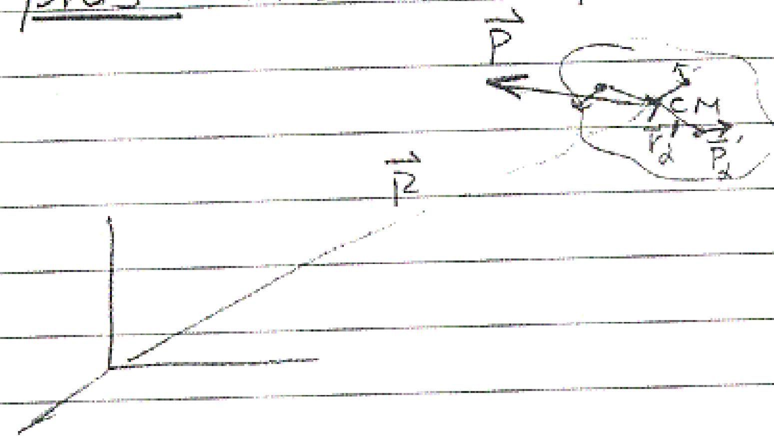
Therefore

$$\vec{L} = \sum_\alpha m_\alpha \left[(\vec{r}_\alpha{}' \times \dot{\vec{r}}_\alpha{}') + (\vec{R} \times \dot{\vec{R}}) \right]$$

$$= \sum_\alpha \vec{r}_\alpha{}' \times \vec{p}_\alpha{}' + M \vec{R} \times \dot{\vec{R}}$$

$$\boxed{\vec{L} = \sum_\alpha \vec{r}_\alpha{}' \times \vec{p}_\alpha{}' + \vec{R} \times \vec{P}}$$

⇒ The total AM of a SOP about an origin is the sum of AM of the CM about the origin **plus** the AM of the SOP about its CM.



Now let's look at the time derivative of $\vec{L}$, and relate it to internal and external torques.

$$\dot{\vec{L}}_\alpha = \vec{r}_\alpha \times \dot{\vec{p}}_\alpha$$
$$= \vec{r}_\alpha \times \left(\vec{F}_\alpha^{(e)} + \sum_\beta \vec{f}_{\alpha\beta}\right)$$

Summing over α:

$$\dot{\vec{L}} = \sum_\alpha \dot{\vec{L}}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{(e)} + \sum_{\alpha,\beta\neq\alpha} (\vec{r}_\alpha \times \vec{f}_{\alpha\beta})$$

Last term can be written

$$\sum_{\alpha,\beta\neq\alpha} (\vec{r}_\alpha \times \vec{f}_{\alpha\beta}) = \sum_{\alpha<\beta} \left[(\vec{r}_\alpha \times \vec{f}_{\alpha\beta}) + (\vec{r}_\beta \times \vec{f}_{\beta\alpha})\right]$$

Define $\vec{r}_{\alpha\beta} \equiv \vec{r}_\alpha - \vec{r}_\beta$, we have (using $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$):

$$\sum_{\alpha,\beta\neq\alpha} (\vec{r}_\alpha \times \vec{f}_{\alpha\beta}) = \sum_{\alpha<\beta} (\vec{r}_{\alpha\beta} \times \vec{f}_{\alpha\beta})$$

Consider the case of central forces (gravity, electrostatics)

$$\vec{r}_{\alpha\beta} \times \vec{f}_{\alpha\beta} = 0.$$

In this case,

$$\dot{\vec{L}} = \sum_{\alpha} [\vec{r}_{\alpha} \times \vec{F}_{\alpha}^{(e)}]$$

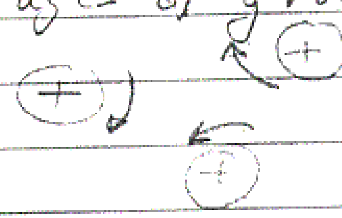
$$= \sum_{\alpha} \vec{N}_{\alpha}^{(e)} = \vec{N}^{(e)}$$

where $\vec{N}^{(e)}$ is the sum of external torques.

$\Longrightarrow$ In the absense of external forces (torques), the AM of a SOP is conserved.

Application:

Disk galaxies have large angular momentum. The above result tells you that if gravity were the only force responsible for their assembly, they got their AM via external torques from other galaxies. A detailed analysis shows that this occurs in the early stages of growth of cosmological density perturbations.



5. Energy

The final conservation law for a SOP we wish to derive is for energy. Consider a SOP in two configirations $r_{\alpha}^{(1)}$ and $r_{\alpha}^{(2)}$. The work done from $1 \rightarrow 2$ is

$$W_{12} = \sum_{\alpha} \int_{1}^{2} \vec{F}_{\alpha} \cdot d\vec{r}_{\alpha} \tag{9.34}$$

Now $\vec{F}_\alpha \cdot d\vec{r}_\alpha = m_\alpha \frac{d\vec{v}_\alpha}{dt} \cdot \frac{d\vec{r}_\alpha}{dt} dt$

$$= m_\alpha \frac{d\vec{v}_\alpha}{dt} \cdot \vec{v}_\alpha \, dt$$

$$= \frac{1}{2} m_\alpha \frac{d}{dt} (\vec{v}_\alpha \cdot \vec{v}_\alpha) \, dt$$

$$= d\left(\frac{1}{2} m_\alpha v_\alpha^2\right)$$

$$= d T_\alpha \leftarrow \text{K.E. of particle } \alpha$$

So

$$W_{12} = \sum_\alpha \int_1^2 dT_\alpha = T_2 - T_1$$

where $T = \sum_\alpha T_\alpha = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2$.

$\Rightarrow$ The work expended changing the SOP config. is equal to the difference of the K.E.s of the system, before and after. Note W_{12} can be $+$, $-$, or 0.

Decomposition of T

Using $\dot{r}_\alpha = \dot{r}'_\alpha + \dot{R}$

we have $\dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha = v_\alpha^2 = (\dot{\vec{r}}'_\alpha + \dot{\vec{R}}) \cdot (\dot{\vec{r}}'_\alpha + \dot{\vec{R}})$

$$= \dot{\vec{r}}'_\alpha \cdot \dot{\vec{r}}'_\alpha + 2\dot{\vec{r}}'_\alpha \cdot \dot{\vec{R}} + \dot{\vec{R}} \cdot \dot{\vec{R}}$$

$$= v_\alpha'^2 + 2(\dot{\vec{r}}' \cdot \dot{\vec{R}}) + V^2$$

then

$$T = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha'^2 + \sum_\alpha \frac{1}{2} m_\alpha V^2 + \dot{\vec{R}} \cdot \frac{d}{dt} \underbrace{\sum_\alpha m_\alpha \vec{r}'_\alpha}_{0}$$

Thus $$\boxed{T = \sum \tfrac{1}{2} m_\alpha v_\alpha^2 + \tfrac{1}{2} M V^2}$$

$\Rightarrow$ Total KE of a SOP is the sum of the KE of particle of mass M moving with CM velocity V, and the KE of the individual particles' motion relative to the CM.

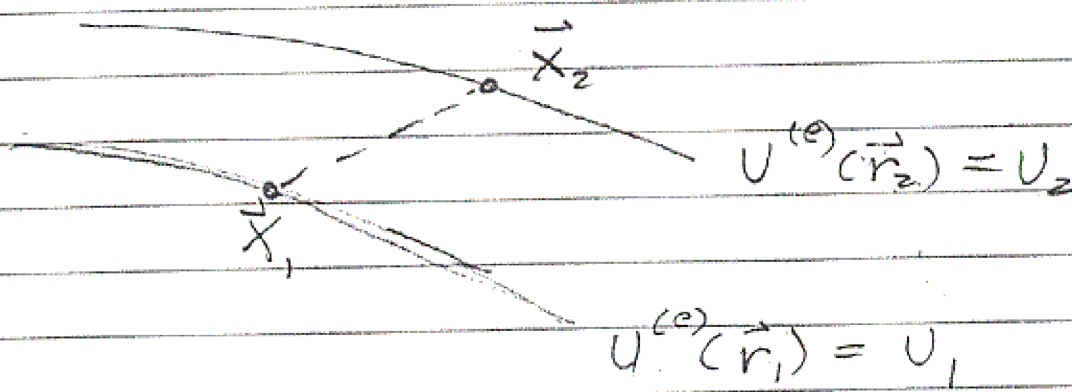
Potential Energy

Substituting $\vec{F}_\alpha = \vec{F}_\alpha^{(e)} + \sum_\beta \vec{f}_{\alpha\beta}$ into Eq. 9.34, we have

$$W_{12} = \sum_\alpha \int_1^2 F_\alpha^{(e)} \cdot d\vec{r}_\alpha + \sum_{\alpha, \beta\neq\alpha} \int_1^2 \vec{f}_{\alpha\beta} \cdot dr_\alpha$$

Let's assume $\vec{F}_\alpha^{(e)}$ and $\vec{f}_{\alpha\beta}$ can be derived from the gradient of potentials

$$\vec{F}_\alpha^{(e)} = -\vec{\nabla}_\alpha U_\alpha^{(e)} \equiv -\frac{\partial}{\partial \vec{r}_\alpha} U^{(e)}(\vec{r}_\alpha)$$



First term of W_{12} is

$$\sum_\alpha \int_1^2 \vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha = -\sum_\alpha \int_1^2 \left(\frac{\partial}{\partial \vec{r}_\alpha} U_\alpha\right) \cdot d\vec{r}_\alpha$$

$$= -\sum_\alpha U(\vec{r}_\alpha) \Big|_{\vec{r}_\alpha(1)}^{\vec{r}_\alpha(2)}$$

This is just the work to move all particles in an external potential $U(\vec{r})$.

The second term of W_{12} is the work required to move all particles against their internal forces; and involves a sum over all pairs, assuming self-energy $= 0$. Let's suppose $\vec{f}_{\alpha\beta}$ depends only on position of particles α & β

$$\vec{f}_{\alpha\beta} \equiv \vec{f}(\vec{r}_\alpha, \vec{r}_\beta)$$

Then, $\vec{f}$ can be expressed as the gradient of a potential

$$\vec{f}_{\alpha\beta} = \vec{f}(\vec{r}_\alpha, \vec{r}_\beta) = -\vec{\nabla}_\alpha U^{(i)}_{\alpha\beta}$$

$$= -\frac{\partial}{\partial \vec{r}_\alpha} U^{(i)}(\vec{r}_\alpha, \vec{r}_\beta)$$

If we hold all particles fixed except for particle α, and move α, then this is identical to the case above where

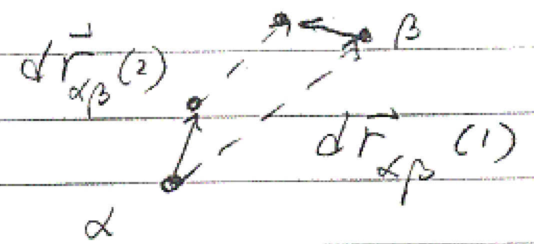
$$U^e(\vec{r}_\alpha) = \sum_{\beta \neq \alpha} U^i(\vec{r}_\alpha, \vec{r}_\beta)$$

But, we want to consider case where all particles move; The second term in W_{12} is

$$\sum_{\alpha, \beta \neq \alpha} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha = \sum_{\alpha < \beta} \int_1^2 \left(\vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha + \vec{f}_{\beta\alpha} \cdot d\vec{r}_\beta \right)$$

$$= \sum_{\alpha<\beta} \int_1^2 \vec{f}_{\alpha\beta} \cdot (d\vec{r}_\alpha - d\vec{r}_\beta)$$

$$= \sum_{\alpha<\beta} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta} \quad , \text{ where } d\vec{r}_{\alpha\beta} \equiv d\vec{r}_\alpha - d\vec{r}_\beta$$



The total derivative of $U_{\alpha\beta}^{(i)}$ is

$$dU_{\alpha\beta}^{(i)} = \sum_j \left(\frac{\partial U^{(i)}}{\partial r_{\alpha,j}} dr_{\alpha,j} + \frac{\partial U^{(i)}}{\partial r_{\beta,j}} dr_{\beta,j} \right)$$

3 coord indices

hold β fixed move α

hold α fixed move β

or, in compact notation

$$dU_{\alpha\beta}^{(i)} = \vec{\nabla}_\alpha U_{\alpha\beta}^{(i)} \cdot d\vec{r}_\alpha + \vec{\nabla}_\beta U_{\alpha\beta}^{(i)} \cdot d\vec{r}_\beta$$

By assumption $\vec{\nabla}_\alpha U_{\alpha\beta}^{(i)} = -\vec{f}_{\alpha\beta}$

but $U_{\alpha\beta}^{(i)} = U_{\beta\alpha}^{(i)}$, so

$$\vec{\nabla}_\beta U_{\alpha\beta}^{(i)} = \vec{\nabla}_\beta U_{\beta\alpha}^{(i)} = -\vec{f}_{\beta\alpha} = \vec{f}_{\alpha\beta}$$

ansatz

N3

Therefore

$$dU_{\alpha\beta}^{(i)} = -\vec{f}_{\alpha\beta} \cdot (d\vec{r}_\alpha - d\vec{r}_\beta)$$

$$= -\vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta}$$

Thus, finally

$$\sum_{\alpha,\beta\neq\alpha} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha = -\sum_{\alpha<\beta} \int_1^2 dU_{\alpha\beta}^{(i)}$$

$$= -\sum_{\alpha<\beta} U_{\alpha\beta} \Big|_{(\vec{r}_\alpha, \vec{r}_\beta)(1)}^{(\vec{r}_\alpha, \vec{r}_\beta)(2)}$$

The total potential U is

$$U = \sum_\alpha U_\alpha^{(e)} + \sum_{\alpha<\beta} U_{\alpha\beta}^{(i)}$$

Then $\quad W_{12} = -U\Big|_1^2 = U_1 - U_2$

Earlier, we found $\quad W_{12} = T_2 - T_1$.

Therefore

$$T_2 - T_1 = U_1 - U_2$$

or

$$T_1 + U_1 = T_2 + U_2$$

so that

$$\boxed{E_1 = E_2}$$

Total energy is conserved. For this reason, any system where both external and internal forces are derivable from potentials are called conservative systems.