Relativistic Rocket

Consider a RR whose instantaneous velocity WRT an IRF is v and whose exhaust gases are ejected with constant velocity V wrt to the rocket. Derive its equation of motion.

Solution

- Let $m(t)$ be the instantaneous mass of the rocket in its rest frame (proper frame of the rocket)
- in the Lab frame, conserving relativistic momentum $p = \gamma m v$ we have

① $$p = \underbrace{\gamma m v}_{\text{before}} = \underbrace{(\gamma + d\gamma)(m+dm)(v+dv)}_{\text{rocket}} + \underbrace{\gamma_w \, dm' \, w}_{\text{exhaust}}$$

where w is velocity of exhaust in IRF

$$\gamma_w = \frac{1}{\sqrt{1 - w^2/c^2}},$$

dm' is mass of ejected fuel

Unlike nonrelativistic case, we cannot assume $dm' = -dm$, which is conservation mass. In SR, mass-energy is conserved:

② $$E = \gamma m c^2 = (\gamma + d\gamma)(m+dm)c^2 + \gamma_w \, dm' c^2$$

Using velocity addition formula,

$$w = \frac{v - V}{(1 - vV/c^2)}$$

From (1), retaining infinitesimals to first order

$$\cancel{\gamma m v} = \cancel{\gamma m v} + \gamma m\, dv + d\gamma\, m v + \gamma\, dm\, v + \gamma_w\, w\, dm'$$

$$\gamma m\, dv + m v\, d\gamma + \gamma v\, dm + \gamma_w\, w\, dm' = 0$$

Now, $d\gamma = \gamma^3 \beta\, d\beta = \gamma^3 \frac{v}{c^2} dv$

So $\gamma m \underbrace{\left(1 + \frac{\gamma^2 v^2}{c^2}\right)} dv + \gamma v\, dm + \gamma_w\, w\, dm = 0$

$$1 + \frac{v^2}{c^2 - v^2}$$

$$\frac{c^2}{c^2 - v^2} = \frac{1}{1 - v^2/c^2} = \gamma^2$$

$$\therefore \quad \gamma^3 m\, dv + \gamma v\, dm + \gamma_w\, w\, dm' = 0$$

(3) $\Rightarrow \quad \gamma^2 m\, dv + v\, dm + \gamma_w \frac{w\, dm'}{\gamma} = 0$

From (2), again retaining only FOT

$$\cancel{\gamma m c^2} = \cancel{\gamma m c^2} + \gamma\, dm\, c^2 + m\, d\gamma\, c^2 + \gamma_w\, dm' c^2$$

$$\gamma_w\, dm' = -(\gamma\, dm + m\, d\gamma)$$

substituting for $\gamma_w' dm'$ in ③

$$\gamma^2 m\,dv + v\,dm - \frac{w}{\gamma}\left(\gamma\,dm + m\,d\gamma\right) = 0$$

$$d\gamma = \gamma^3 \frac{v\,dv}{c^2}$$

$$\gamma^2 m\,dv\left(1 - \frac{vw}{c^2}\right) + dm\,(v - w) = 0$$

dividing by $\left(1 - \frac{vw}{c^2}\right)$ we get

$$\gamma^2 m\,dv + dm\,\underbrace{\frac{(v-w)}{\left(1 - \frac{vw}{c^2}\right)}}_{\bar{V}\text{ from velocity addition}} = 0$$

Finally, dividing by dt

$$\boxed{m\frac{dv}{dt} + \bar{V}\frac{dm}{dt}(1-\beta^2) = 0}$$

In the limit $\beta \to 0$, this reduces to our NR result. As $\beta \to 1$, $\frac{dv}{dt} \to 0$, implying rocket cannot accelerate to faster than c.

~~The EOM admits a simpler interpretation~~
~~Recall that v and t are lab frame quantities, while m is measured in the rocket frame~~