Relativistic Doppler Shift

$$\rightsquigarrow \quad E = h\nu_0 \quad , \quad p = h\nu_0/c$$

In a frame approaching with speed v
From LT

$$p' = \frac{p + (v/c^2)E}{\sqrt{1-\beta^2}}$$

$$= \frac{\frac{h\nu_0}{c} + \frac{v}{c^2}h\nu_0}{\sqrt{1-\beta^2}}$$

$$\frac{h\nu'}{c} = \frac{\frac{h\nu_0}{c}(1+\beta)}{\sqrt{1-\beta^2}}$$

$$\therefore \quad \nu' = \nu_0 \frac{(1+\beta)}{\sqrt{1-\beta^2}} = \nu_0 \sqrt{\frac{(1+\beta)^2}{1-\beta^2}}$$

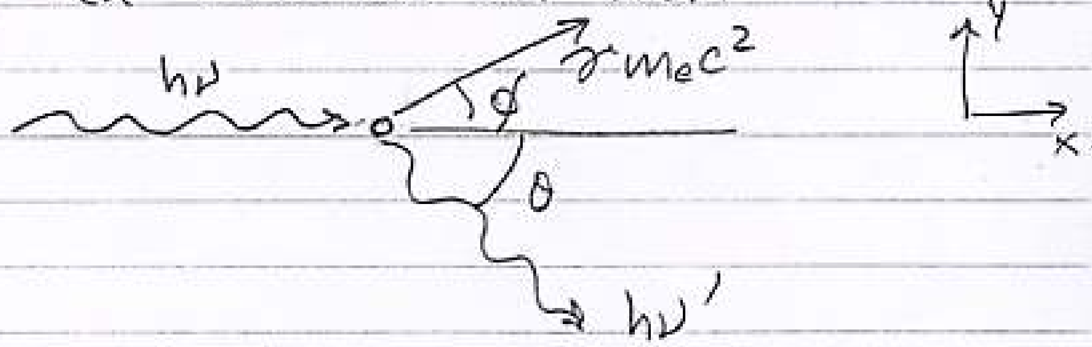
$$\boxed{\nu' = \nu_0 \sqrt{\frac{1+\beta}{1-\beta}} > \nu_0 \quad \text{blueshift}}$$

In a frame receding with speed v

$$\nu' = \nu_0 \sqrt{\frac{1-\beta}{1+\beta}} < \nu_0 \quad \text{redshift}$$

Compton scattering

Consider a photon of $E = h\nu$ scattering off of an electron at rest.



From cons. of. En.

$$h\nu + m_e c^2 = \gamma m_e c^2 + h\nu' \qquad (1)$$

Mom. cons. in X direction gives

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + \gamma m_e v \cos\phi \qquad (2)$$

Mom cons. in Y direction gives

$$\gamma m_e v \sin\phi = \frac{h\nu'}{c}\sin\theta \qquad (3)$$

Eliminate ϕ using $\sin^2\phi + \cos^2\phi = 1$

From (2)

$$\cos\phi = \frac{1}{\gamma m_e v}\left[\frac{h\nu}{c} - \frac{h\nu'}{c}\cos\theta\right] \qquad (4)$$

$$\sin\phi = \frac{h\nu'}{\gamma m_e v}\sin\theta \qquad (5)$$

Therefore

$$1 = \frac{1}{\gamma^2 m_e^2 v^2}\left[\left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 - 2\left(\frac{h\nu}{c}\right)\left(\frac{h\nu'}{c}\right)\cos\theta\right] \qquad (6)$$

It is easy to show

$$\gamma^2 v^2 = c^2(\gamma^2 - 1) \qquad (7)$$

From (1), solving for γ and subst. in (7), we have

$$\gamma = \frac{h(\nu-\nu')}{m_e c^2} + 1$$

$$\gamma^2 v^2 = c^2\left[\frac{h^2(\nu-\nu')^2}{m_e^2 c^4} + \frac{2h(\nu-\nu')}{m_e c^2}\right]$$

$$= \left[\frac{h^2(\nu-\nu')^2}{m_e^2 c^2} + \frac{2h(\nu-\nu')}{m_e}\right] \qquad (8)$$

From (6) and (8) (equating $\gamma^2 v^2$)

$$\left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 - 2\left(\frac{h\nu}{c}\right)\left(\frac{h\nu'}{c}\right)\cos\theta =$$

$$2hm_e(\nu-\nu') + \frac{h^2}{c^2}(\nu-\nu')^2$$

after some algebra, collecting terms in ν and ν'

$$\left[\frac{2m_e c^2}{h} + 2\nu(1-\cos\theta)\right]\nu' = \frac{2m_e c^2}{h}\nu$$

Hence

$$\nu' = \left[\frac{1}{1 + \frac{h\nu}{m_e c^2}(1-\cos\theta)}\right]\nu$$

Since $E' = h\nu'$, Cons. En. gives energy of

Electron

$$T = \gamma m_e c^2 - m_e c^2 = h\nu - h\nu'$$

$$= E - E'$$

It is easy to show

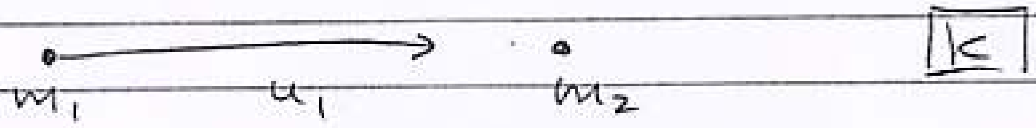
$$T = \frac{E}{m_e c^2}\left[1 - \frac{1}{1 + \frac{E}{m_e c^2}(1-\cos\theta)}\right]$$

$$\boxed{T = \frac{E}{m_e c^2}\left[\frac{1-\cos\theta}{1 + \frac{E}{m_e c^2}(1-\cos\theta)}\right]}$$

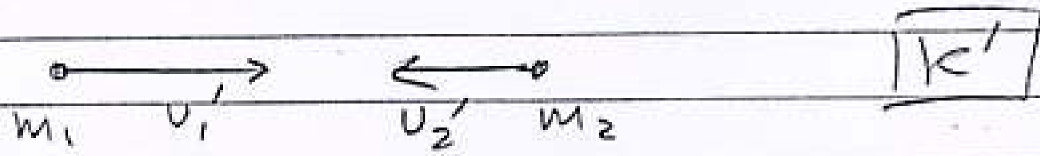
Relativistic collisions

If the velocities of particles are relativistic ($v/c \to 1$), then analysis in Ch. 9 needs to be modified.

Consider particle of mass m_1 and speed u_1 colliding with a stationary particle of mass m_2



Let us call this IRF the lab frame, and associate it with the K frame from before. In the center of momentum frame, which we will associate with the K′ frame, we have $p_1' = p_2'$ since $p_1' - p_2' = 0$



Speed of CM frame

$$\gamma_1' m_1 u_1' = \gamma_2' m_2 v_2'$$

$$\gamma = 1/\sqrt{1-\beta^2}, \quad \beta = u/c$$

$\gamma^2 = \frac{1}{1-\beta^2}$

$1-\beta^2 = \frac{1}{\gamma^2}$

now

$$\beta\gamma = \beta/\sqrt{1-\beta^2}$$

$$= \sqrt{\frac{\beta^2}{1-\beta^2}} = \sqrt{\frac{1-1/\gamma^2}{1/\gamma^2}}$$

$$= \sqrt{\gamma^2 - 1},$$

So $p_1' = \gamma_1' m_1 u_1' = \gamma_1' m_1 \beta_1' c = m_1 c\sqrt{\gamma_1'^2 - 1}$ ④

$= p_2' = m_2 c\sqrt{\gamma_2'^2 - 1}$.

Now, by the LT of the 4-momentum we have

$$p_1' = \left(p_1 - \frac{u_2'}{c^2} E_1\right)\gamma_2' \quad ④④$$

where

$$\left.\begin{aligned} p_1 &= \gamma_1 m_1 u_1 \\ E_1 &= \gamma_1' m_1 c^2 \end{aligned}\right\}$$

Substituting these values into ④④, and using ④, after some algebra, get

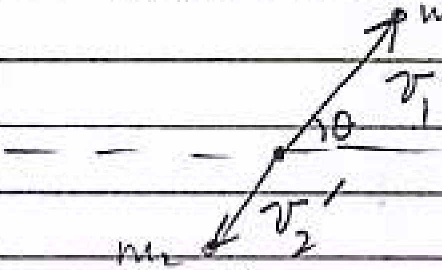
$$\gamma_1' = \frac{\gamma_1 + \frac{m_1}{m_2}}{\sqrt{1 + 2\gamma_1\left(\frac{m_1}{m_2}\right) + \left(\frac{m_1}{m_2}\right)^2}}$$

$$\gamma_2' = \frac{\gamma_1 + \frac{m_2}{m_1}}{\sqrt{}}$$

$$u_2' = \left(1 - \frac{1}{\gamma_2'^2}\right)^{1/2} c = V$$

Elastic Scattering

In the CM frame, the scattered particles look like this



$v_1' = \sqrt{1-}$ $\boxed{K'}$

In the LAB frame, they look like this