$$= \gamma m u^2 + mc^2 \sqrt{1 - u^2/c^2}\,\Big|_0^u$$

$$= \gamma m u^2 + mc^2/\gamma \qquad - mc^2$$

$$\gamma T = \gamma^2 m u^2 + mc^2 - \gamma mc^2$$

$$= \frac{mu^2}{1 - u^2/c^2} + mc^2 - \gamma mc^2$$

$$= \frac{mu^2 + mc^2 - mu^2}{1 - u^2/c^2} - \gamma mc^2$$

$$= \gamma^2 mc^2 - \gamma mc^2$$

$$\therefore \quad T = \gamma mc^2 - mc^2$$

$$\boxed{T = (\gamma - 1) mc^2}$$

For small u, $\left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2}$

and $T \to \frac{1}{2} mu^2$

so, we recover Newtonian formula.

The quantity

$$E_0 \equiv mc^2$$

is the rest energy.

We then have

$$\gamma mc^2 = T + E_0$$

We identify the quantity γmc^2 with the total energy, since it is the sum of kinetic and rest mass.

$$\boxed{E \equiv \gamma mc^2 = T + E_0} \quad \star$$

Note as $v \to 0$, $\gamma \to 1$ and $E \to E_0$. The equation $E_0 = mc^2$ says that mass has an energy equivalent. Thus, rather than thinking of separate conservation laws for mass and energy in relativity we have <u>conservation of mass-energy</u>

<u>Example 14.8</u> Binding energy of D.

(see TM)

We can further manipulate $\star$. We have

$$p = \gamma mu$$

$$p^2c^2 = \gamma^2 m^2 u^2 c^2$$

$$= \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right)$$

but $\frac{u^2}{c^2} = 1 - \frac{1}{\gamma^2}$, so

$$p^2c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right)$$

$$= \gamma^2 m^2 c^4 - m^2 c^4$$

Example : Solar fusion

The Sun shines by virtue of energy released when 4 H nuclei (p) are converted to 1 He nucleus (α) by nuclear fusion. The energy released per reaction is simply calculated if we know the masses:

$$m_p = 1.0072766 \quad u$$
$$m_\alpha = 4.000408 \quad u$$

$$\Delta E = 4 m_p c^2 - m_\alpha c^2$$
$$= (0.028698 \quad u \quad) c^2$$

$$(1 \ u\ ) c^2 = 931.48 \text{ MeV}$$

$$\therefore \Delta E = 26.73 \text{ MeV}$$

This energy comes out in the form of 2 γ's and 2 ν_e's per reaction. Only the γ's effectively heat the Sun.

but $E_0^2 = m^2c^4$, $E^2 = \gamma^2 m^2 c^4$

so

$$p^2c^2 = E^2 - E_0^2$$

or

$$\boxed{E^2 = p^2c^2 + E_0^2}$$

This relates a particle's (body's) momentum, rest energy, and total energy. Since a photon has no mass, $E_0 = 0$, hence

$$E = pc \quad \text{(photon)}.$$

Since $\quad E = h\nu$

we have $\quad \boxed{p = \frac{h\nu}{c}}$

12. Spacetime and Four-Vectors

Because space and time become mixed in a LT, it is convenient to think of a 4-dimensional continuum called spacetime, first introduced by Minkowski. An event in ST has coordinates

$$\text{Event:} \quad (x_1, x_2, x_3, ct)$$

The ST interval between two events is invariant under LT

$$\Delta S_{12} = (\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 - c^2 \Delta t^2)^{1/2}$$

$$\Delta s'_{12} = (\Delta x_1'^2 + \Delta x_2'^2 + \Delta x_3'^2 - c^2 \Delta t'^2)^{1/2}$$

$$\Delta s'_{12} = \Delta s$$

If we introduce the variable $x_4 = ict$ ($i = \sqrt{-1}$), then we can write the ST interval as

$$\Delta s = \left(\sum_{\mu=1}^{4} \Delta x_\mu^2 \right)^{1/2}$$

$$\Delta s' = \left(\sum_{\mu=1}^{4} \Delta x_\mu'^2 \right)^{1/2} = \Delta s$$

In the 4-dimensional space (x_1, x_2, x_3, x_4) Δs is just the distance between 2 points. Since $\Delta s = \Delta s'$, the LT is distance preserving, it corresponds to a rotation in <u>Minkowski space</u>.

$\Rightarrow$ The LT is an orthogonal transform in Minkowski space. In analogy with 3D rotations, we can write

$$x'_\mu = \sum_\nu \lambda_{\mu\nu} x_\nu$$

rotation matrix in 4-D Minkowski space

In the case of K′ moving along X_1 with relative velocity $v = \beta c$, we have from inspection

$$[\lambda] = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

<u>Check</u>

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ ict' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ ict \end{pmatrix}$$

$$x_1' = \gamma x_1 + (i\beta\gamma)(ict)$$
$$= \gamma(x_1 - vt)$$

$$x_2' = x_2$$
$$x_3' = x_3$$

$$ict' = -i\beta\gamma x_1 + \gamma(ict)$$

$$t' = \gamma t - \frac{\beta\gamma x_1}{c}$$

$$= \gamma\left(t - \frac{x_1 v}{c^2}\right)$$

Any quantity that transforms according to

$$A'_\mu = \sum_\nu \lambda_{\mu\nu} A_\nu$$

is called a 4-vector. Obviously, the coordinates of an event in Minkowski space (MS) is a 4-vector

$$\mathbb{X} = (x_1, x_2, x_3, ict)$$
$$= (\vec{x}, ict)$$

Relativistic momentum and energy

Armed with the above, we now show that $\vec{P}_{rel}$, E_{rel} are elements of a 4-vector. First, a little jargon.

Any scalar quantity that is invariant under a LT is called a 4-scalar. Clearly, the ST interval is a 4-scalar.

$$ds = \sqrt{\sum_\mu dx_\mu^2}$$

Now, in a frame comoving with the clock $(dx_1 = dx_2 = dx_3 = 0;\ dt = d\tau)$

$$ds = \sqrt{0+0+0-c^2 d\tau^2}$$
$$d\tau = \frac{i}{c} ds$$

$\therefore$ $d\tau$ is also a 4-scalar.

If $\mathbb{X}$ is a 4-vector, $d\mathbb{X}$ is a 4-vector and $\frac{d\mathbb{X}}{d\tau} \equiv \mathbb{V}$ is a 4-vector; the <u>4-velocity</u>

$$\mathbb{V} = \frac{d\mathbb{X}}{d\tau} = \left(\frac{d\vec{x}}{d\tau}, ic\frac{dt}{d\tau} \right)$$

$$= \left(\frac{d\vec{x}}{dt}, ic \right) \frac{dt}{d\tau}$$

$$= \gamma^{u} (\vec{u}, ic)$$

where $\gamma^{u} = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = |\vec{u}|/c$

Since rest mass is a 4-scalar, the <u>4-momentum</u> $\mathbb{P}$ is also a 4-vector.

$$\mathbb{P} \equiv m\mathbb{V}$$

$$\mathbb{P} = \left(\frac{m\vec{u}}{\sqrt{1-\beta^2}}, iP_4 \right)$$

where $P_4 = \frac{mc}{\sqrt{1-\beta^2}}$

The first 3 components are our relativistic momentum $\vec{P}_{rel}$, while the 4^{th} component is related to E via

$$p_4 = \gamma mc = \frac{E}{c}$$

Therefore, the 4-vector momentum can be written

$$\boxed{\mathbb{P} = \left(\vec{p}_{rel}, i\frac{E}{c}\right)}$$

Since $\mathbb{P}$ is a 4-vector, it transforms according to

$$\mathbb{P}'_\mu = \sum_\nu \lambda_{\mu\nu} \mathbb{P}_\nu$$

Explicity, apply matrix for $K \to K'$

$$\boxed{\begin{aligned} p_1' &= \frac{p_1 - (v/c^2)E}{\sqrt{1-\beta^2}} \\ p_2' &= p_2 \\ p_3' &= p_3 \\ E' &= \frac{E - vp}{\sqrt{1-\beta^2}} \end{aligned}}$$

— — — — — —

FIN