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47.* Deflection of starlight by the sun

Estimate the deflection of starlight by the sun using an elementary analysis. Discussion: Consider first a simpler example of a similar phenomenon. An elevator car of width L is released from rest near the surface of the earth. At the instant of release a narrow beam of light is fired horizontally from one wall of the car toward the other wall. After release the elevator car is an inertial frame. Therefore the light beam will cross the car in what is a straight line *with respect to the car*. With respect to the *earth*, however, the beam of light is falling—because the elevator is falling. Therefore, in a gravitational field, a beam of light must fall. As another example a ray of starlight in its passage tangentially across the earth's surface will receive a gravitational deflection (over and above any refraction by the earth's atmosphere). However, the time to cross the earth is so short, and in consequence the deflection so slight, that this effect has not yet been detected on the earth. At the surface of the sun, however, the acceleration of gravity has the much greater value of 275 meters per second per second. Moreover, the time of passage across the surface is much increased because the sun has a greater diameter, 1.4×10^9 meters. Determine an "effective time of fall" from this diameter and the speed of light. From this time of fall deduce the net *velocity* of fall toward the sun produced by the end of the whole period of gravitational interaction. (The maximum acceleration acting for this "effective time" produces the same net effect [calculus proof!] produced by the actual acceleration—changing in magnitude and direction along the path—in the entire passage of the ray through the sun's field of force.) Comparing this lateral velocity with the forward velocity of the light deduce the *angle of deflection*. The accurate analysis of special relativity gives the *same result*. However, Einstein's 1915 general relativity predicted a previously neglected effect, associated with the change of *lengths* in a gravitational field that produces something like a supplementary *refraction* of the ray of light and *doubles* the predicted deflection. (Deflection observed in 1947 eclipse of the sun: $(9.8 \pm 1.3) \times 10^{-6}$ radian; in the 1952 eclipse: $(8.2 \pm 0.5) \times 10^{-6}$ radian.)

G. GEOMETRIC INTERPRETATION

48. Geometric interpretation

Develop a geometric interpretation of the Lorentz transformation using the following outline.

(a) Show that in the laboratory spacetime diagram the world line of the origin of the rocket frame will be the line marked t' in Fig. 64. This is the locus of all events that occur at the origin of the rocket frame, that is, *it is the rocket t' axis*. Show that the locus of events that occur at $x' = 1$ meter in the rocket frame is a line that parallels the t' axis in Fig. 64, and similarly for $x' = 2, 3, 4$ meters.

(b) Show that the slope of the t' axis relative to the t axis in Fig. 64 is given by the expression (meters of distance traveled for each)/(meter of light-travel time) $= \beta_r = \tanh \theta_r$. What happens to the slope β_r in the two cases: (1) the rocket travels very slowly and (2) the rocket travels at a speed very close to the speed of light.

(c) Now for the crucial step! Where shall we locate the rocket x' axis in the laboratory spacetime diagram? The principle of relativity says that the measured speed of light must be the same in the two frames. The dotted line in Fig. 65 is the world line of a flash of light. Show that the principle of relativity requires that the rocket x' axis be tilted upward at the same slope as

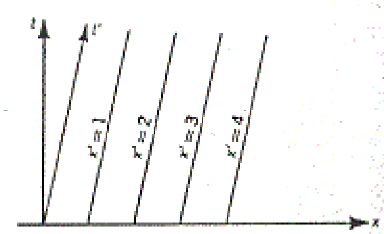
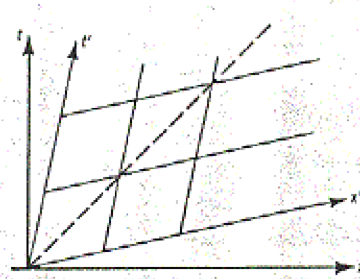


Fig. 64. Location of the rocket time axis in the laboratory spacetime diagram.

Fig. 65. Location of the rocket space axis in the laboratory spacetime diagram.



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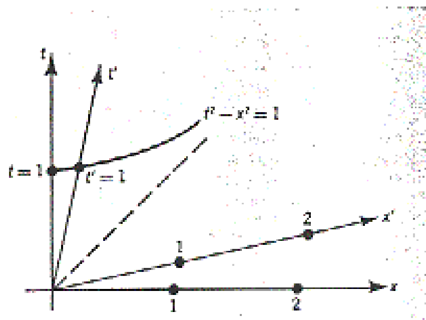

Fig. 66. Calibration of rocket space and time axes.

the rocket t' axis is tilted to the right. Show that the loci of events that occur at rocket times $t' = 1, 2, 3$ meters respectively lie parallel to the rocket x' axis as shown.

(d) Calibrate the rocket axes! Draw the hyperbola $t^2 - x^2 = 1$ (Fig. 66). At the place where the hyperbola crosses the laboratory t axis (where $x = 0$), we have $t = 1$ meter of time. But the interval $t^2 - x^2$ is an invariant so that $(t')^2 - (x')^2 = 1$ also. Therefore at the place where the hyperbola crosses the rocket t' axis (where $x' = 0$), we have $t' = 1$ meter of time. Because of the symmetry and the linearity of the transformation equations, we can use the distance along the rocket t' axis from the origin to $t' = 1$ as a unit distance to lay off along *both* the t' and the x' axes. This completes the derivation of the construction. Next: apply it!

(e) Show that if two events are simultaneous in the laboratory frame they will lie on a line parallel to the laboratory x axis of the spacetime diagram (Fig. 67). Show that if two events are simultaneous in the rocket frame they will lie on a line parallel to the rocket x' axis of the spacetime diagram. Hence the two observers will not necessarily agree on which events are simultaneous. This is the *relative synchronization of clocks*.

(f) Using lines of simultaneity in Fig. 67, show that at rocket time $t' = 1$ meter, the observer in the rocket frame determines that the clock at the laboratory origin has not yet reached one meter of time (i.e., the laboratory clock runs slow), whereas the observer in the laboratory frame observes that the clock at the laboratory origin already reads *more* than one meter of time (i.e., the rocket clock runs slow). This is *time dilation*.

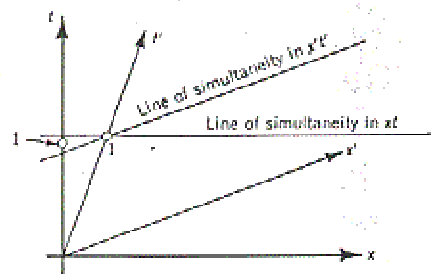

Fig. 67. Illustration of time dilation.

(g) A meter stick lies at rest in the laboratory frame with one end at the origin of that frame (Fig. 68). Measurement of its length in the laboratory frame will give a result like *ab* in Fig. 68. Measurement of its length in the *rocket frame* (i.e., determining the position of the endpoints at the "same time") will give a result like *de* in the figure. Show that this measurement results in an observed *Lorentz contraction* in the rocket frame. Using Fig. 69 show that a meter stick at rest in the rocket frame with one end at the origin of that frame will be Lorentz contracted when observed in the laboratory frame.

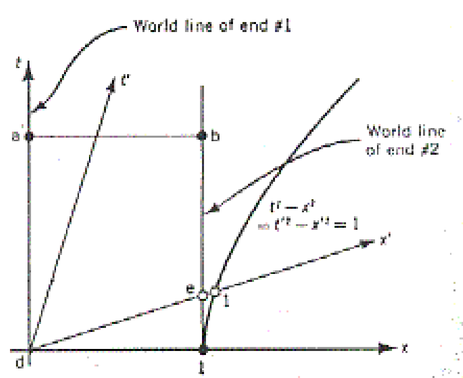

Fig. 68. A meter stick at rest in laboratory frame appears Lorentz-contracted when observed in rocket frame.

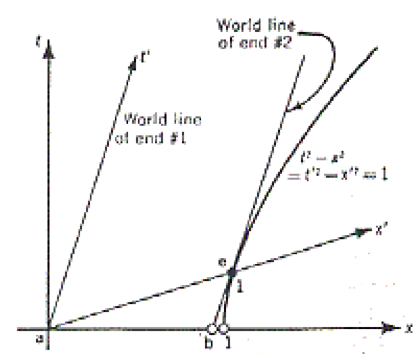

Fig. 69. A meter stick at rest in rocket frame appears Lorentz-contracted when observed in laboratory frame.

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(h) Sketch spacetime diagrams for the relativity of simultaneity, time dilation, and Lorentz contraction in the limiting cases that the relative velocity between laboratory and rocket frames is very small and very large.

(i) Return to the spacetime diagram of Fig. 22 in the chapter, which describes the motion of particles and light flashes in two dimensions. Show that the rocket "plane of simultaneity" is tilted relative to the laboratory plane of simultaneity. Explain the implications of this tilt for the relative simultaneity of events that occur at different positions on the x axis of the laboratory spacetime diagram, and for the relative simultaneity of events that occur at different positions on the y axis of the laboratory spacetime diagram.

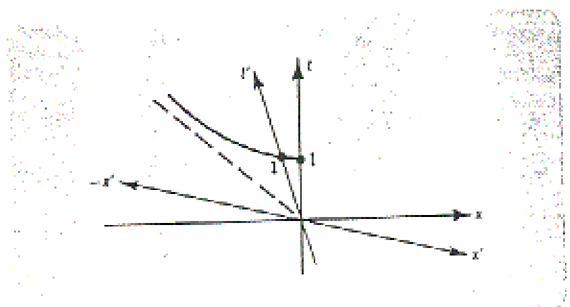

Fig. 70. Location of space and time axes for rocket frame moving in negative laboratory x direction.

(j) Consider a rocket frame moving in the *negative* x direction in the laboratory frame. Verify the features of Fig. 70, in particular the *opposite* sense of the relative synchronization of clocks and the *same* sense of time dilation when compared with the rocket moving in the positive x direction.

49. The clock paradox II—a worked example†

When Peter returned from his fourteen years of traveling (Ex. 27) he was still young enough to learn some relativity. But the more he studied the more puzzled he became. He and his brother Paul, being in relative motion, "each should see the other's clocks running slow." This simple slogan, put in Paul's mouth, made it easy enough to understand why "Peter's clocks—and Peter's aging processes—ran slow," so that Peter was the younger of the two on his return. "But if the slogan is valid," Peter asked, "then would not *I*—if I had investigated—have found *Paul's* clocks running slow? So how did he age more than I?" *Question:* What is the way out of Peter's difficulties?

Solution: As Peter studied more, with this paradox worrying him, he learned that words like "observer" and "observed time" do not have the simple meaning he had at first attributed to them. He should not think of how he might directly have kept day-to-day track of Paul's aging back on earth, either by radio messages or by other methods. That procedure, while conceivable, does not lend itself to the simplest analysis, Peter discovered. The observer in relativity theory, he found, is to be understood as a whole framework of rods and recording clocks moving along with uniform velocity—with the same velocity as Peter himself as he recedes from the earth, $\beta_r = 24/25 = 0.96$. That parade of clocks ("Peter's clocks and Peter's reference frame") zooms by the earth. As each clock passes Paul it punches out (1) the reading of Paul's clock and (2) its own reading and location. The shorthand phrase "Peter observes Paul" means that Peter collects these cards—or the information on them—at some later time.

"So what?" Peter asked himself at this point. "In any case I know that the reading of Paul's clock increases from one punchout to the next only $(1 - \beta_r^2)^{1/2} = 7/25$ as much as the increase in readings of my own clock. So Paul is the man who should have been younger at the end of my journey, not me. But look at his gray hair! Where am I going wrong?"

Running over in his mind once again the events of his journey, Peter could not help but remember the moment when he had stopped his outward trip and started his return to the earth. "*I* stopped and *I* turned back; but," he suddenly asked himself, "what about my inertial reference frame? How can an inertial frame turn back? He looked into this issue more and more carefully. He found himself forced to conclude that the reference frame employed for the first part of his flight—and especially the lattice clock alongside him that had recorded information for the seven outbound years—must have kept on their swift way like a stream of superhighway traffic as one car makes a U-turn into the returning lanes. Another stream of clocks accompanied him home

†See E. Lowry, American Journal of Physics, **31**, 59 (1963).

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—a second inertial reference frame. For all the seven years of return one of these clocks remained faithfully alongside. When it took over escort duty, it adopted the seven-year reading of the outbound clock. It read fourteen years at the time when Peter rejoined Paul.

The inbound parade of clocks was passing the earth all these seven years. One by one as they went by they punched out their readings and the readings of Paul's clock. The punch cards made a growing pile on the ground at Paul's feet. As those seven years went by for Peter's inbound escort, the cards showed that Paul's clocks ran off only 7/25 of this time; that is (7/25) of 7 years or 1.96 years.

"What on earth is the matter with my reasoning?" Peter asked aloud at this point. "Now I find myself concluding that Paul should have aged 1.96 years on my outbound trip, and 1.96 years on my inbound trip, or altogether 3.92 years. Yet I *know* I aged fourteen years, and I *know* he aged more than I did. What have I overlooked?" So saying, he drew a spacetime diagram (Fig. 71), and at least had the answer to his difficulty—the time AB that he had so far left out of account. This time, Peter saw, corrects for the difference between the standards of simultaneity of his outgoing and returning reference frames. A separate calculation, using the results of Ex. 11, gives for this time the value 46.08 years. This supplement has to be added to Paul's aging as measured by Peter's two sets of recording clocks. Peter's final calculation for Paul's age (including his age of 21 years when the trip began) gave

$$21 + 1.96 + 46.08 + 1.96 = 71 \text{ years}$$

He could thankfully rejoice in his own comparative youth of $21 + 14 = 35$ years (uncorrected for the time required to learn spacetime physics!). The present analysis does not purport to be the simplest way to calculate the aging of the twins. For that one goes back to Paul's analysis, outlined in Ex. 27. There one has to consider only a single inertial reference frame, the frame with its origin at Paul. The present analysis illustrates how *any* correct method of analysis leads to the same result as any other correct method of analysis.

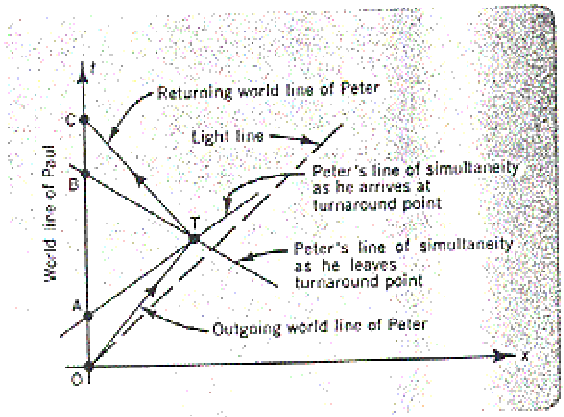


Fig. 71. Peter's bookkeeping on Paul's aging process. During Peter's outbound journey (OT in diagram) his clock flashes a new year seven times. An array of synchronized clocks escort him. Each makes its own seventh year flash somewhere along the "line of simultaneity" AT and punches out a record. The Peter clock which punches out a record at A sees Paul's clock reading only 1.96 years ("slowing of a clock as viewed from a moving reference frame"). On the return journey a different array of synchronized clocks escorts Peter ("second inertial reference frame"). Each of them flashes a seven year sign as it crosses the line of simultaneity BT. The one which travels alongside Peter makes seven more flashes along the world line TC, the last of them signaling fourteen years of travel just as Peter rejoins Paul at C. During the period BC, while the clocks of Peter's inbound reference frame indicate the passage of seven years, Paul has aged only another 1.96 years (again the "slowing of a clock as viewed from a moving reference frame"). But the bookkeeping done so far by Peter's two inertial reference frames is incomplete. Neither one of them does the job of counting the time lapse AB. It is 46.08 years ("correction for change in standard of simultaneity" between Peter's outgoing and incoming inertial reference frames). Thus the slowing of Paul's clocks as observed by Peter's two sets of recording clocks in no way keeps Peter from ending up younger than Paul.

11. Relativistic momentum

The relativistic velocity addition formula requires that we modify how we define momentum if it is to be conserved in all inertial frames. To see this, imagine the perfectly elastic collision of two balls as seen in the K and K′ frames.

⇒ in K frame, let ball A have purely vertical velocity before collision

$$\left.\begin{aligned} u_{A1} &= 0 \\ u_{A2} &= u_0 \end{aligned}\right\}$$

⇒ in K′ frame, let ball B have purely vertical velocity before collision

$$\left.\begin{aligned} u'_{B1} &= 0 \\ u'_{B2} &= -u_0 \end{aligned}\right\}$$

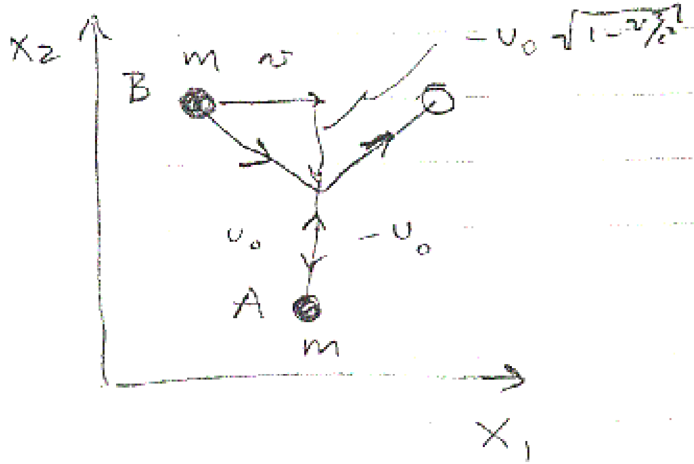
⇒ Ball B in K frame has velocity components

$$u_{B1} = v$$

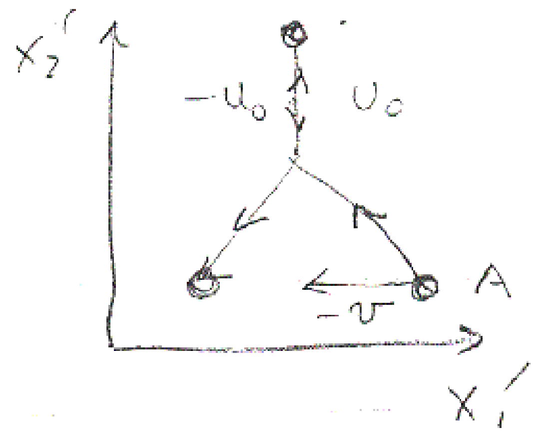
$$u_{B2} = -u_0\sqrt{1-v^2/c^2}$$

$$\left(u_2 = \frac{u_2'}{\gamma\left(1+u_1'\frac{v}{c^2}\right)} \quad ; \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}} \right)$$

This is what collision looks like in K frame



This is what collision looks like in K' frame



In K frame, change in A's x_2 momentum is

$$\Delta P_{A2} = -2mu_0$$

but change in B's x_2 momentum is

$$\Delta P_{B2} = 2mu_0\sqrt{1-v^2/c^2}$$

$$\Delta P_{A2} + \Delta P_{B2} \neq 0 \quad !!$$

It appears vertical momentum is not conserved. Moreover, if viewed in K' frame, we would conclude

$$\Delta P'_{A2} = -2mU_0\sqrt{1-v^2/c^2}$$

$$\Delta P'_{B2} = 2mU_0$$

$$\Delta P'_{A2} + \Delta P'_{B2} \neq 0$$

Not only not conserved, but a different number. Clearly, definition of momentum needs modification.

The relativistic momentum is defined

$$\boxed{\vec{P} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} = \gamma m\vec{u}}$$

Note the Lorentz factor γ depends on $\vec{u}$, the particle speed, and not v, the relative velocity between K and K′.

Note also that

$$\lim_{u\to 0} \vec{P} = m\vec{u}$$

i.e., the Newtonian result.

Example 14.6 in text shows that this definition, momentum is conserved in our toy example.

12. Relativistic energy

With new definition for $\vec{P}$, how do concepts of energy, work, and force change?

If we retain our definition of Work:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = T_2 - T_1$$

and modify N2 to read

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(\gamma m \vec{u})$$

then the work required to accelerate a body of mass m from rest ($T_1 = 0$) is

$$W = T = \int \frac{d}{dt}(\gamma m \vec{u}) \cdot \vec{u}\, dt$$

where we have used $d\vec{r} = \vec{u}\, dt$

$$T = m \int u \, d(\gamma u)$$

Integrating by parts

$$T = m\left[\gamma u^2 \Big|_0^u - \int_0^u \gamma u \, du \right]$$

$$= m\left[\gamma u^2 - \int_0^u \frac{u\, du}{\sqrt{1 - u^2/c^2}} \right]$$

$$= \gamma m u^2 + mc^2\sqrt{1-u^2/c^2}\Big|_0^u$$

$$= \gamma m u^2 + mc^2/\gamma \qquad - mc^2$$

$$= \gamma m u^2 + \frac{\gamma mc^2}{\gamma^2}$$

$$= \gamma mc^2\left(\frac{u^2}{c^2} + \frac{1}{1-u^2/c^2}\right) - mc^2$$

$$= \gamma mc^2\left(\frac{u^2 - u^4/c^2 + c^2}{c^2-u^2}\right)$$

$$\vdots$$

$$= \gamma mc^2 - mc^2$$

$$\boxed{T = (\gamma-1)mc^2}$$

For small u, $\left(1-\frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2}$

and $T \to \frac{1}{2}mu^2$

so, we recover Newtonian formula.