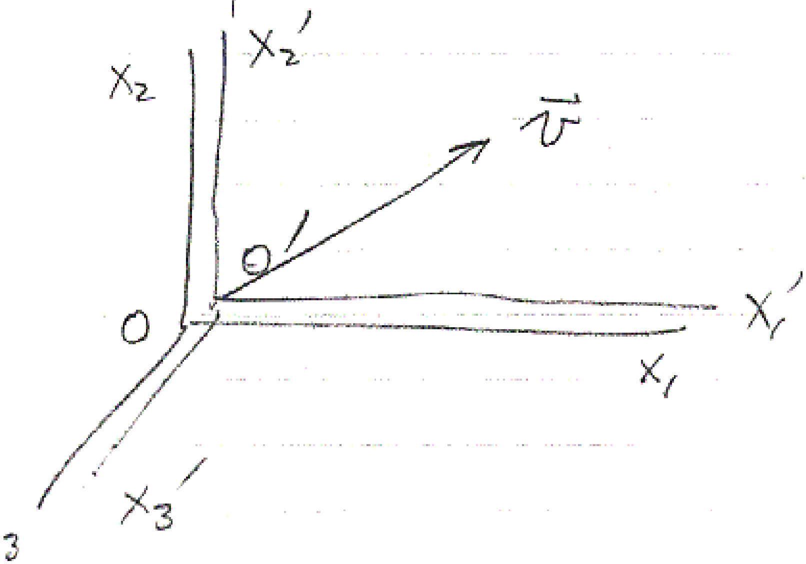
5. Lorentz transformation for general $\vec{v}$

Consider the K' frame moving with velocity $\vec{v}$ relative to the K frame. Let the axes be parallel



The previous results clearly imply that only the component of $\vec{X}$ along $\vec{v}$ is Lorentz contracted, while the components of $\vec{X}$ $\perp$ to $\vec{v}$ are un-affected

$$\vec{X}_{\parallel}' = \gamma(\vec{X}_{\parallel} - \vec{v}t)$$

$$\vec{X}_{\perp}' = \vec{X}_{\perp}$$

$$t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{X}}{c^2}\right)$$

where $\gamma = \dfrac{1}{\sqrt{1 - \frac{\vec{v}\cdot\vec{v}}{c^2}}}$

with the definitions

$$\vec{x}_{\parallel} = [(\vec{v}\cdot\vec{x})\vec{v}]/v^2$$

$$\vec{x}_{\perp} = \vec{x} - \vec{x}_{\parallel}$$

we obtain

$$\boxed{\begin{aligned} \vec{x}' &= \vec{x} + (\gamma-1)\frac{(\vec{x}\cdot\vec{v})}{v^2}\vec{v} - \gamma\vec{v}t \\ t' &= \gamma\left(t - \frac{\vec{x}\cdot\vec{v}}{c^2}\right) \end{aligned}}$$

Proof

$$\vec{x}' = \vec{x}_{\parallel}' + \vec{x}_{\perp}'$$

$$= \gamma(\vec{x}_{\parallel} - \vec{v}t) + \vec{x} - \vec{x}_{\parallel}$$

$$= \vec{x} + (\gamma-1)\vec{x}_{\parallel} - \gamma\vec{v}t$$

$$\boxed{\vec{x}' = \vec{x} + (\gamma-1)[(\vec{v}\cdot\vec{x})\vec{v}/v^2] - \gamma\vec{v}t}$$

Substituting $\vec{v} = v\hat{e}_x$, we recover the "1D" formulae above.

6. Velocity addition formula

Returning to the LT with $\vec{v} \parallel \hat{e}_{x_1}$, we can derive relativistic version of the velocity addition formula. In each frame velocities are defined:

$$\left.\begin{aligned} u_i &\equiv \frac{dx_i}{dt} \\ u_i' &\equiv \frac{dx_i'}{dt'} \end{aligned}\right\}$$

$$u_1' = \frac{dx_1'}{dt'} = \frac{\gamma(dx_1 - v\,dt)}{\gamma\left(dt - \frac{v\,dx_1}{c^2}\right)}$$

$$\boxed{u_1' = \frac{u_1 - v}{1 - \frac{uv}{c^2}}}$$

$$u_2' = \frac{dx_2'}{dt'} = \frac{dx_2}{\gamma\left(dt - \frac{v\,dx_1}{c^2}\right)}$$ x_1

$$u_2' = \frac{u_2}{\gamma\left(1 - \frac{u_1 v}{c^2}\right)}$$ u_1

similarly

$$u_3' = \frac{u_3}{\gamma\left(1 - \frac{u_1 v}{c^2}\right)}$$ u_1

The addition formulae for general $\vec{v}$ can be derived in an analogous way to Sec. 5.

7. Lorentz contraction

This result is worked out in Example 14. Consider a rod of length l at rest in K frame. What is its length in K′ frame. Key is to think of events. By length, we mean distance between ends of rod at the same instant of time in K′ frame. Let ends of rod in K frame have event coords.

$\boxed{K}$ 1 —— 2

$[x_1(1), t(1)]$ $\quad$ $[x_1(2), t(2)]$

$$l \equiv x_1(2) - x_1(1)$$

$\boxed{K'}$ 1 —— 2

$[x_1'(1), t'(1)]$ $\quad$ $[x_1'(2), t'(2)]$

$$l' \equiv x_1'(2) - x_1'(1) \quad \text{at} \quad t'(2) = t'(1)$$

apply LT formula

$$l' = \gamma\{[x_1(2) - x_1(1)] - v[t(2) - t(1)]\}$$
$$= \gamma\{l - v[t(2) - t(1)]\}$$

but from ILT for t

$$t(2) = t'(2) + \frac{v x_1(2)}{c^2}, \quad t(1) = t'(1) + \frac{v x_1(1)}{c^2}$$

$$t(2) - t(1) = [t'(2) - t'(1)] + \frac{v}{c^2}[x_1(2) - x_1(1)]$$

$$t(2) - t(1) = \frac{v}{c^2}[x_1(2) - x_1(1)]$$

since by assumption observation is made when $t'(2) = t'(1)$.

Substituting

$$l' = \gamma\{l - \frac{v^2}{c^2} l\}$$

$$= \gamma l / \gamma^2$$

$$\boxed{l' = l/\gamma}$$

Lengths have contracted by a factor γ. This is the Lorentz-Fitzgerald contraction

Note

This result is obtained much easier by using the ILT for x_1 and setting $t'(2) = t'(1)$:

$$x_1(1) = \gamma(x_1'(1) + v t'(1))$$

$$x_1(2) = \gamma(x_1'(2) + v t'(2))$$

$$x_1(2) - x_1(1) = l = \gamma(\underbrace{x_1'(2) - x_1'(1)}_{l'})$$

$$\boxed{l' = l/\gamma}$$

8. Time dilation

This result is worked out in Example 14.2. Consider a clock fixed in the K frame at some position x_1, which "ticks" with a period $\Delta t \equiv t(2)-t(1)$. In the K' frame, according to LT for t

$$\Delta t' \equiv t'(2) - t'(1)$$
$$= \gamma \left\{\left[t(2) - \frac{v x_1(2)}{c^2}\right] - \left[t(1) - \frac{v x_1(1)}{c^2}\right]\right\}$$

Because $x_1(2) = x_1(1)$, we get

$$\boxed{\Delta t' = \gamma \Delta t}$$

To an observer in K' frame, time interva[l] between ticks is larger by a factor γ.
$\Rightarrow$ "moving clocks run more slowly".
It is easy to show that if the clock is fixed in the K' frame ($x_1'(2)=x_1'(1)$) then as seen in the K frame
$\Delta t = \gamma \Delta t'$.

Proper time is defined as the rate at which time advances in an observer's rest frame. If $\Delta\tau$ is an interval of time measured in one IRF, than to a moving observer

$$\boxed{\Delta t' = \gamma \Delta\tau}$$