Beat Waves

The linearity of the WE allows us to build up solutions by the principle of linear superposition

$$\Psi(x,t) = \sum_{r=1}^{n} A_r e^{i(\omega_r t - k_r x)}$$

where n is the number of wave modes.
Let us consider $n=2$, with $A_1 = A_2 = A$ and the two modes whose frequencies and wavenumbers differ by a small amount

$$\Psi_1(x,t) = A e^{i(\omega t - kx)}$$

$$\Psi_2(x,t) = A e^{i(\Omega t - K x)}$$

$$\Omega = \omega + \Delta\omega$$

$$K = k + \Delta k$$

then

$$\Psi(x,t) = \Psi_1 + \Psi_2 = A\left[e^{i\omega t} e^{-ikx} + e^{i(\omega+\Delta\omega)t} e^{-i(k+\Delta k)x} \right]$$

writing

$$\omega = \omega + \frac{\Delta\omega}{2} - \frac{\Delta\omega}{2}, \quad k = k + \frac{\Delta k}{2} - \frac{\Delta k}{2}$$

$$\omega + \Delta\omega = \omega + \frac{\Delta\omega}{2} + \frac{\Delta\omega}{2}, \quad k + \Delta k = k + \frac{\Delta k}{2} + \frac{\Delta k}{2}$$

then we can write

$$\Psi(x,t) = A\left[e^{i(\omega + \frac{\Delta\omega}{2})t} e^{-i(k+\frac{\Delta k}{2})x} \right]$$

$$\times \left[e^{-i\left(\frac{(\Delta\omega)t - (\Delta k)x}{2}\right)} + e^{i\left(\frac{(\Delta\omega)t - (\Delta k)x}{2}\right)} \right]$$

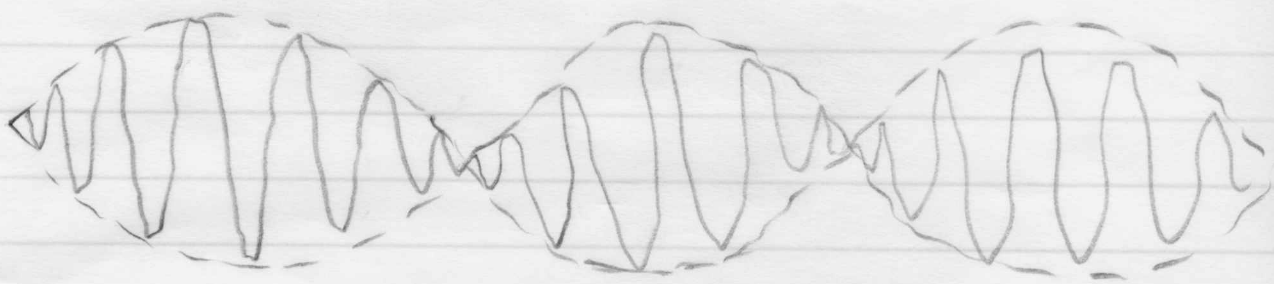
The second factor in [ ] is $2\cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)$

Thus $\Psi(x,t)$ is a plane wave moving to the right with

$$\omega' \equiv \omega + \Delta\omega/2$$
$$k' \equiv k + \Delta k/2$$

Whose amplitude as modulated by

$$2A\cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)$$



This phenomenon is known as <u>beats</u>, and is the principal behind FM (frequency modulated) radio transmissions.

<u>Phase velocity and group velocity</u>

Now let's consider the propagation of this compound waveform in space. The real part of the wave function Ψ is

$$Re(\Psi) = 2A\cos\left[\frac{(\Delta\omega)t - (\Delta k)x}{2}\right]\cos(\omega' t - k' x)$$

The <u>high frequency</u> waves have constant phase for values of x and t such that

$$\omega' t - k' x = \phi_0 \leftarrow \text{constant}$$

i.e., $\frac{dx}{dt} = \frac{\omega'}{k'} = \frac{\omega + \frac{1}{2}\Delta\omega}{k + \frac{1}{2}\Delta k} \equiv v_{phase}$

The low frequency modulation has constant phase for values of x and t such that

$$(\Delta\omega)t - (\Delta k)x = \phi_1 \leftarrow \text{constant}$$

i.e.
$$\frac{dx}{dt} = \frac{\Delta\omega}{\Delta k} \equiv v_{group}$$

In the limit $\Delta k, \Delta\omega \to 0$

$$\boxed{\begin{aligned} v_{phase} &= \frac{\omega}{k} \\ v_{group} &= \frac{d\omega}{dk} \end{aligned}}$$

Consider the string, with $k = \frac{\omega}{v} = \omega\sqrt{\frac{\rho}{\tau}}$

Then

$$v_{phase} = \frac{\omega}{k} = v$$

$$v_{group} = \frac{d\omega}{dk} = \sqrt{\frac{\tau}{\rho}} = v.$$

i.e., $v_{phase} = v_{group}$

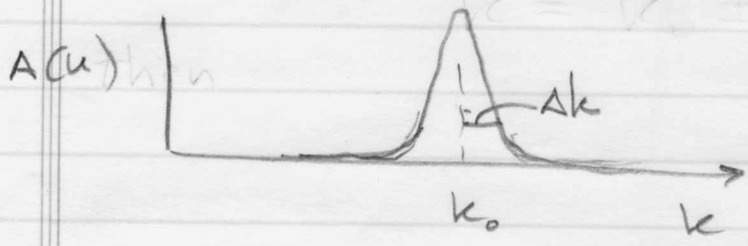
A system for which ω is a linear function of k is called a <u>non-dispersive medium</u>: all wave modes propagate with the same speed.

<u>Wave Packets</u>

In the limit of an ∞ number of wave modes

$$\Psi(x,t) = \sum_{r=1}^{n} A_r e^{i(\omega_r t - k_r x)} \Longrightarrow \int_{-\infty}^{\infty} dk\, A(k) e^{i(\omega t - kx)}$$

$A(k)$ is called the spectral distribution function, and encodes the relative amplitudes of wave modes of different k. If $A(k)$ is sharply peaked about $k = k_0$, and becomes vanishingly small for



$k = k_0 \pm \Delta k$

then $\Psi(x,t) \approx \int_{k_0-\Delta k}^{k_0+\Delta k} dk \, A(k) \, e^{i(\omega t - kx)}$

This type of wave function is called a wave packet, a term introduced by Schrödinger.
For general $\omega(k)$, we can write

$$\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right)_{k=k_0} (k-k_0) + \cdots$$

$$= \underset{\uparrow\,\omega(k_0)}{\omega_0} + \underset{\uparrow\,\frac{d\omega}{dk}(k_0)}{\omega_0'} (k-k_0)$$

We can write argument of exponential

$$\omega t - kx = \omega_0 t - k_0 x + (k-k_0)(\omega_0' t - x)$$

by adding and subtracting $k_0 x$.

$\Psi(x,t)$ becomes

$$\Psi(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} \underbrace{A(k)\, e^{i(k-k_0)(\omega' t - x)}}_{\text{effective amplitude}} \cdot \underbrace{e^{i(\omega_0 t - k_0 x)}}_{\text{wave mode}} \, dk$$

The effective amplitude is

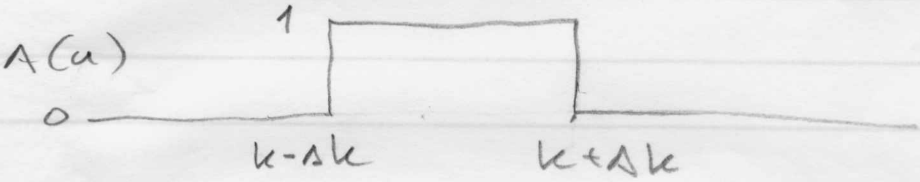
$$A(k) \cos\left[(k-k_0)(\omega' t - x)\right]$$

which for small $k-k_0$ is slowly varying in time.

Example

Consider a wave packet with

$$A(k) = \begin{cases} 1, & |k-k_0| < \Delta k \\ 0, & \text{otherwise} \end{cases}$$



Solve for $\Psi(x,t)$

Solution

$$\Psi(x,t) = \int_{k-\Delta k}^{k+\Delta k} e^{i(\omega t - kx)} \, dk$$

substitute $\omega = \omega_0 + \omega_0'(k-k_0)$, then

$$\Psi(x,t) = e^{i(\omega_0 - \omega_0' k_0)t} \int_{k-\Delta k}^{k+\Delta k} e^{i(\omega_0' t - x)k} \, dk$$

Do the definite integral :

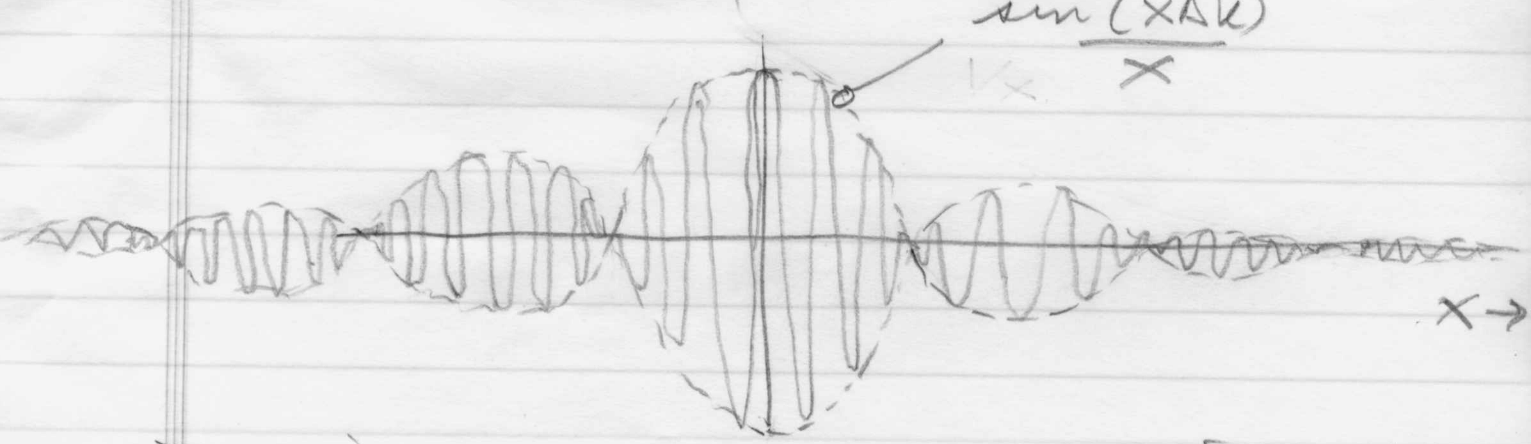
$$\Psi(x,t) = e^{i(\omega_0 - \omega_0' k_0)t} \times \left[\frac{e^{i(\omega_0' t - x)(k_0 + \Delta k)} - e^{i(\omega_0' t - x)(k_0 - \Delta k)}}{i(\omega_0' t - x)} \right]$$

$$= 2\, e^{i(\omega_0 - \omega_0' k_0)t}\, e^{i(\omega_0' t - x)k_0} \times \left[\frac{e^{i(\omega_0' t - x)\Delta k} - e^{-i(\omega_0' t - x)\Delta k}}{(2i)(\omega_0' t - x)} \right]$$

$$= 2\, \frac{\sin\left[(\omega_0' t - x)\Delta k\right]}{(\omega_0' t - x)}\, e^{i(\omega_0 t - k_0 x)}$$

Plot $\mathrm{Re}\left(\Psi(x,0)\right)$

$$\mathrm{Re}\left(\Psi(x,0)\right) = 2\, \frac{\sin(x\Delta k)}{x} \cos k_0 x$$

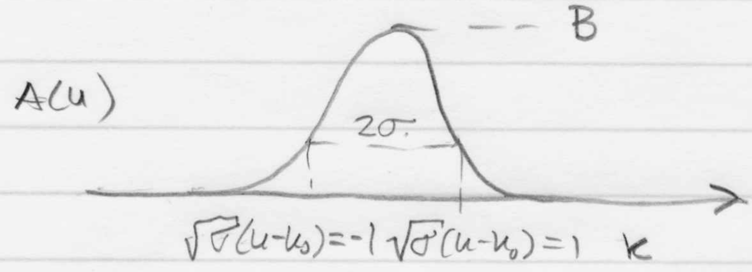


In time, the packet moves to the right with speed $\frac{dx}{dt} = \omega_0'$.

Gaussian Wave Packet

An interesting and important special case is when $A(k)$ is a Gaussian

$$A(k) = B e^{-\sigma(k-k_0)^2}$$



We will see that this spectral distribution leads to a wave packet which is localized in space.

To see this, evaluate $\Psi(x,0)$

$$\Psi(x,0) = B\int_{-\infty}^{\infty} e^{-\sigma(k-k_0)^2} e^{-ikx}\, dk$$

$$= B e^{-ik_0x} \int_{-\infty}^{\infty} e^{-\sigma(k-k_0)^2} e^{-i(k-k_0)x}\, d(k-k_0)$$

change of variables $u = k - k_0$

$$= B e^{-ik_0x} \int_{-\infty}^{\infty} e^{-\sigma u^2} e^{-iux}\, du$$

Integral is of the form

$$\int_{-\infty}^{\infty} e^{-ax^2} e^{bx}\, dx = \int_{-\infty}^{\infty} e^{-a(x^2 - \frac{b}{a}x)}\, dx$$

This can be converted to standard form by completing the square in the exponent

$$-a\left(x^2-\frac{b}{a}x\right)=-a\left(x^2-\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)+\frac{b^2}{4a}$$

$$=-a\left(x-\frac{b}{2a}\right)^2+\frac{b^2}{4a}$$

with another change of variables

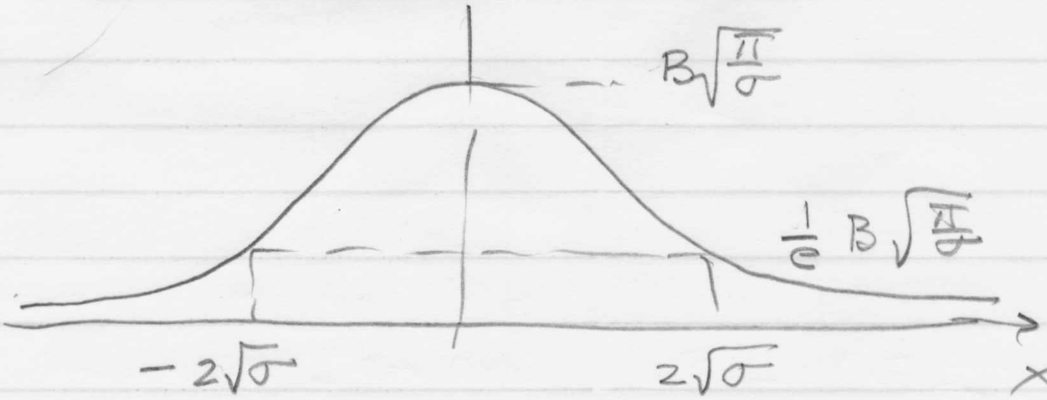
$$y=x-\frac{b}{2a}$$

our integral becomes

$$\int_{-\infty}^{\infty} e^{-ax^2}\, e^{bx}\, dx = e^{\frac{b^2}{4a}} \underbrace{\int_{-\infty}^{\infty} e^{-ay^2}\, dy}_{\sqrt{\frac{\pi}{a}}}$$

Therefore, letting $a=\sigma$, $b=-ix$

$$\boxed{\Psi(x,0)=B\sqrt{\frac{\pi}{\sigma}}\; e^{-ik_0x}\; e^{-x^2/4\sigma}}$$



Notice reciprical relation of width in k-space and x-space; narrow k, wide x and vice versa.