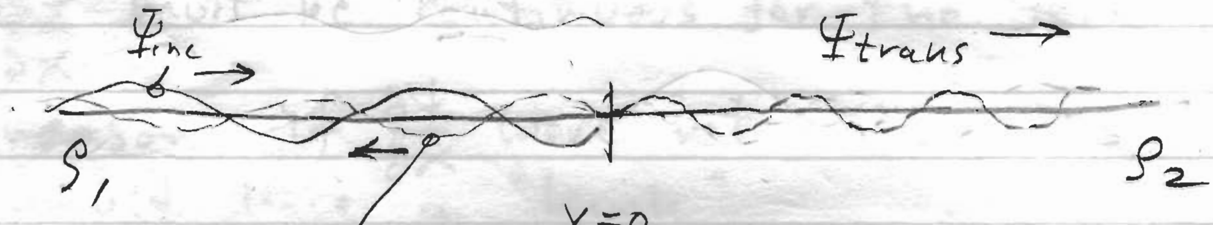
Wave Propagation across a discontinuity: transmission and reflection

Consider a continuous wave train on a string with $\rho = \rho_1$ for $x < 0$, and $\rho = \rho_2$ for $x > 0$. The wave will be partially transmitted and partially reflected when it impinges on the discontinuity. What are the amplitudes of the R and T waves?



region 1 | region 2

$$\Psi_1(x,t) = \Psi_{inc} + \Psi_{refl}$$
$$= Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

$$\Psi_2(x,t) = \Psi_{trans}$$
$$= Ce^{i(\omega t + k_2 x)}$$

N.b.

- waves in both media have same ω
- because ρ is different, but τ same, wave speed is different

$$v_1 = \sqrt{\frac{\tau}{\rho_1}}, \quad v_2 = \sqrt{\frac{\tau}{\rho_2}}$$

- wave numbers are also different

$$k_1 = \frac{\omega}{v_1}, \quad k_2 = \frac{\omega}{v_2}$$

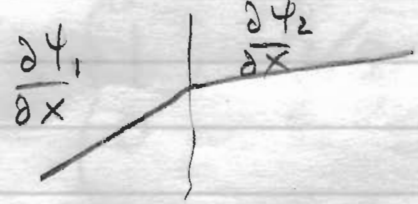
Substituting, $k_2 = k_1 \sqrt{\frac{\rho_2}{\rho_1}}$.

In particular, if $\rho_2 > \rho_1$, $k_2 > k_1$,
$\lambda_2 \equiv \frac{2\pi}{k_2} < \frac{2\pi}{k_1} \equiv \lambda_1$.

Continuity of Ψ and Ψ' allow us to solve for potentially complex B and C given real A.

$\Rightarrow$ Ψ must be continuous because the string is continuous (think $\Psi = \xi$, the displacement)

$\Rightarrow$ $\frac{\partial \Psi}{\partial x}$ must be continuous for the following reason. If $\frac{\partial \psi}{\partial x}$ were not continuous, string would have a "kink"



$x=0$

$$\left.\frac{\partial^2 \psi}{\partial x^2}\right)_{x=0} = \lim_{\Delta x \to 0}\left[\left.\frac{\partial \psi_2}{\partial x}\right)_{x=\frac{\Delta x}{2}} - \left.\frac{\partial \psi_1}{\partial x}\right)_{x=-\frac{\Delta x}{2}} / \Delta x\right] = \infty$$

But by WE, $\ddot{\psi} = -v^2 \frac{\partial^2 \Psi}{\partial x^2} = \infty$ @ $x=0$.

Infinite accelerations are not physical, hence $\frac{\partial \psi}{\partial x}$ is continuous function of x.

Applying $\Psi_1\big|_{x=0} = \Psi_2\big|_{x=0}$

$$\left.\frac{\partial \Psi_1}{\partial x}\right|_{x=0} = \left.\frac{\partial \Psi_2}{\partial x}\right|_{x=0}$$

we have $A + B = C$

$$-k_1 A + k_1 B = -k_2 C$$

Solving for unknown B and C we get

$$B = \frac{k_1 - k_2}{k_1 + k_2} A \qquad \underline{\text{real}}$$

$$C = \frac{2k_1}{k_1 + k_2} A \qquad \underline{\text{real}}, \ \underline{\text{positive}}$$

Define <u>reflection coefficient</u>

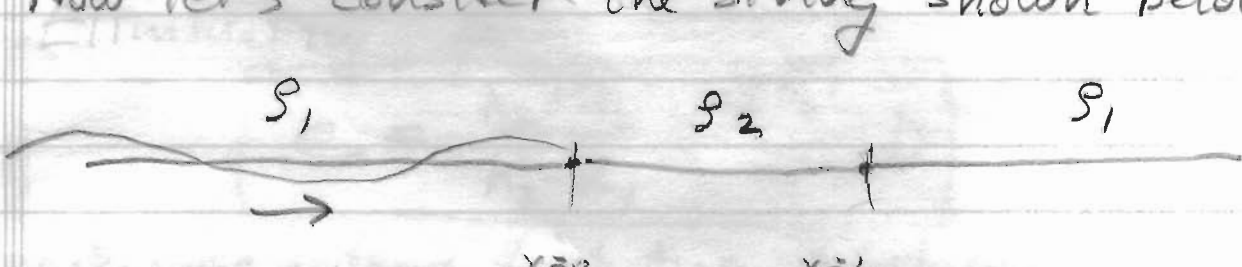
$$R \equiv \frac{|B|^2}{|A|^2} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = \left(\frac{1 - \sqrt{\frac{\rho_2}{\rho_1}}}{1 + \sqrt{\frac{\rho_2}{\rho_1}}}\right)^2$$

$$R \to \begin{cases} 1, & \rho_2/\rho_1 \to \infty \\ 0, & \rho_2/\rho_1 = 1 \end{cases}$$

Define <u>transmission coefft</u>

$$T = 1 - R = \frac{4k_1 k_2}{(k_1 + k_2)^2}$$

Now let's consider the string shown below



with wave moving to the right, Find the value of L which <u>maximizes</u> transmission.

Solution

$$\Psi_1 = A e^{i(\omega t - k_1 x)} + B e^{i(\omega t + k_1 x)}$$

$$\Psi_2 = C e^{i(\omega t - k_2 x)} + D e^{i(\omega t + k_2 x)}$$

$$\Psi_3 = E e^{i(\omega t - k_1 x)}$$

$$k_1 = \frac{\omega}{v_1} = \omega\sqrt{\frac{\rho_1}{\tau}} \; ; \; k_2 = \frac{\omega}{v_2} = \omega\sqrt{\frac{\rho_2}{\tau}}$$

Impose continuity of Ψ, Ψ' @ $x=0$, $x=L$.
At $x=0$, have

$$A + B = C + D \qquad (\Psi \text{ continuous})$$

$$k_1(-A+B) = k_2(-C+D) \qquad (\Psi' \text{ continuous})$$

At $x=L$, have

(*) $$C e^{-ik_2 L} + D e^{ik_2 L} = E e^{-ik_1 L}$$

(**) $$k_2(-C e^{-ik_2 L} + D e^{ik_2 L}) = -k_1 E e^{-ik_1 L}$$

Multiplying (*) by k_2 and adding/subtracting (**), we get

$$C = \frac{1}{2}\left(1 + \frac{k_1}{k_2}\right) E e^{i(k_2 - k_1)L}$$

$$D = \frac{1}{2}\left(1 - \frac{k_1}{k_2}\right) E e^{-i(k_2 + k_1)L}$$

Eliminating E

$$\boxed{C = \frac{k_2 + k_1}{k_2 - k_1} e^{i2k_2 L} D}$$

Likewise, from first two equations

$$A = \frac{1}{2}\left(1 + \frac{k_2}{k_1}\right) C + \frac{1}{2}\left(1 - \frac{k_2}{k_1}\right) D$$

Substituting for C above, we get

$$A = \frac{1}{2k_1}\left[\frac{(k_1+k_2)^2}{(k_2-k_1)}e^{i2k_2L} + (k_2-k_1)\right] D$$

we want a relationship between A and E. Using Eg, relating D and E we obtain, after a little algebra

$$\frac{E}{A} = \frac{4k_1k_2\, e^{i(k_1+k_2)L}}{(k_1+k_2)^2 e^{i2k_2L} - (k_1-k_2)^2}$$

Transmission coefficient is $\dfrac{|E|^2}{|A|^2}$

$$\boxed{T = \frac{8k_1^2k_2^2}{k_1^4+k_2^4+6k_1^2k_2^2-2(k_1^2-k_2^2)^2\cos 2k_2L}}$$

To maximize T, we minimize denominator. This occurs whenever $\cos 2k_2L = 1$

$$\left.\begin{aligned} &\text{or} \quad L = \frac{m\pi}{k_2}, \\ &\text{or} \\ &\quad L = \frac{m\pi}{\omega}\sqrt{\frac{\tau'}{\rho_2}} \end{aligned}\right\} \quad m = 0, 1, 2 \ldots$$

This has optical analogs in the construction of compound lenses, where coatings are applied to minimize reflections.