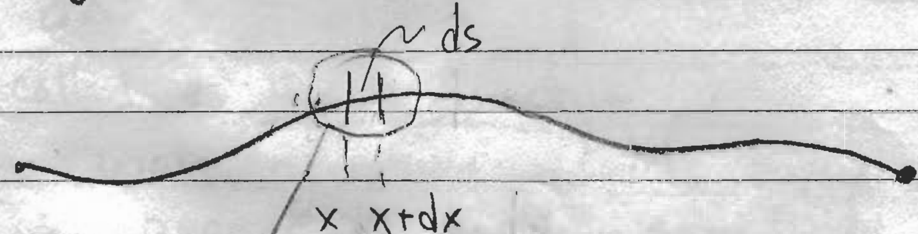
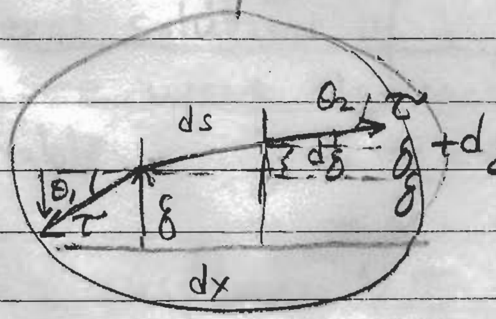
Chapter 13: Continuous Systems; Waves

13.4 We now want to study SOP that are not
13.6-9 rigid. Macroscopic fluids (liquids & gases) and elastic solids can be described as <u>deformable continua</u>, and support wave motions. This chapter is about <u>wave dynamics</u>.

The <u>quintessential</u> wave-supporting continuum is a <u>string under tension</u>. We will derive the <u>wave equation</u> governing transverse motions of the string.

Consider a string of constant <u>mass density</u> ρ under tension τ. Consider a section of length ds displaced from its equilibrium configuration





$$F = ma$$

$$-\tau \sin\theta_1 + \tau \sin\theta_2 = \rho\, ds\, \frac{\partial^2 \delta}{\partial t^2}$$

for small θ,
$\sin\theta \approx \tan\theta$
$ds \approx dx$

$$-\tau \left.\frac{\partial \delta}{\partial x}\right)_x + \tau \left.\frac{\partial \delta}{\partial x}\right)_{x+dx} = \rho\, ds \frac{\partial^2 \delta}{\partial t^2}$$

$$\tau \frac{\partial^2 \delta}{\partial x^2}\, dx = \rho\, ds \frac{\partial^2 \delta}{\partial t^2}$$

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\rho}{\tau}\frac{\partial^2 \xi}{\partial t^2}$$ This is the wave equation for displacement $\xi(x,t)$.

$[\rho] = ML^{-1}$
$[\tau] = MLT^{-2}$

$\therefore [\rho/\tau] = T^2 L^{-2} \propto 1/\text{velocity}^2$

$v = \sqrt{\tau/\rho}$, we have

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2} = 0$$

Any quantity $\psi(x,t)$ which obeys

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0}$$

is called a wave function.

The wave equation admits wave solutions which propagate with speed $\frac{dx}{dt} = \pm v$.

General solution of WE.

To see wave character, introduce 2 new variables $\xi = x + vt$

$\eta = x - vt$

Now transform WE to these new variables

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \eta}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \eta}\right) = \frac{\partial^2 \psi}{\partial \xi^2} + 2\frac{\partial^2 \psi}{\partial \xi \partial \eta} + \frac{\partial^2 \psi}{\partial \eta^2}$$

In a similar way, we find

$$\frac{1}{v}\frac{\partial \psi}{\partial t} = \frac{1}{v}\left(\frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial t}\right) = \frac{\partial \psi}{\partial \xi} - \frac{\partial \psi}{\partial \eta}$$

$$\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{v}\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial t}\right) = \frac{\partial^2 \psi}{\partial \xi^2} - 2\frac{\partial^2 \psi}{\partial \xi \partial \eta} + \frac{\partial^2 \psi}{\partial \eta^2}$$

but, by WE we have

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

this regones $\dfrac{\partial^2 \psi}{\partial \xi \partial \eta} = 0$

General Solution to above is

$$\psi = f(\xi) + g(\eta)$$

or $\boxed{\psi = f(x+vt) + g(x-vt)}$

<u>How to interpret?</u>

ψ is the superposition of two wave forms:

$f(\xi)$ moving to left with speed v

$g(\eta)$ " " right with speed v.

$f(x+vt)$ $g(x-vt)$

Ex) Fig. 13-3

Reflection

If the wave encounters a fixed point, (e.g. end of guitar string), it will be reflected

$$f(x+vt) \rightarrow -f(-x+vt).$$

To see this, realize that a fixed point requires $f(vt)=0$ for all t.

This will be true if we imagine the mirror image of f moving to the right, and of opposite sign

Ex). Fig. 13.4

Periodic solutions of WE

For many problems in mechanics, sol$^{\underline{n}}$ of WE is **harmonic**, i.e., describes **vibrations**.

Consider trial solution of WE

$$\Psi(x,t) = \psi(x)e^{i\omega t}$$

Substituting, we get

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\omega^2}{v^2}\psi = 0$$

Now $\psi = \psi(x)$.

If the string consisted of n discrete masses, there would be n characteristic frequencies (analyzed in detail in TM12.9)

If the string is continuous, there are an ∞ # of frequencies in principle.

If we pick a specific value of $\omega = \omega_r$, then there exists some $\psi_r(x)$ s.t.

$$\frac{\partial^2 \psi_r}{\partial x^2} + \frac{\omega_r^2}{v^2}\psi_r = 0 .$$

ψ_r is called a <u>wave mode</u>
We can write the general solution for $\Psi(x,t)$ as a sum over modes

$$\boxed{\Psi(x,t) = \sum_r \psi_r(x) e^{i\omega_r t}}$$

<u>Separation of variables</u>
A powerful technique for the sol^n of PDE's arising in continuum mechanics is separation of variables. Our choice $\Psi(x,t) = \psi(x) e^{i\omega t}$ is a case in point.
We can be more general:

$$\Psi(x,t) = \psi(x)\chi(t)$$

Substituting into the WE, we obtain

$$\chi \frac{d^2\psi}{dx^2} - \frac{\psi}{v^2}\frac{d^2\chi}{dt^2} = 0$$

By using $\frac{d}{dx}$, $\frac{d}{dt}$, we recognize ψ, χ are functions of a single variable.

Gathering terms, we can rewrite

$$\underbrace{\frac{v^2}{\psi}\frac{d^2\psi}{dx^2}}_{\text{func. of } x} = \underbrace{\frac{1}{\chi}\frac{d^2\chi}{dt^2}}_{\text{func. of } t}$$

Can only be true if ψ, χ are constant (trivial sol$^{\underline{n}}$) or each side = constant,

$$\frac{v^2}{\psi}\frac{d^2\psi}{dx^2} = -\omega^2 = \frac{1}{\chi}\frac{d^2\chi}{dt^2}$$

LHS

$$\frac{d^2\psi}{dx^2} + \frac{\omega^2}{v^2}\psi = 0$$

$$\therefore \quad \psi(x) = A e^{i(\omega/v)t} + B e^{-i(\omega/v)t}$$

RHS

$$\frac{d^2\chi}{dt^2} + \omega^2\chi = 0$$

$$\therefore \quad \chi(t) = C e^{i\omega t} + D e^{-i\omega t}$$

Multiplying together

$$\begin{aligned}\psi(x)\chi(t) = &\ AC \exp[i(\omega/v)(x+vt)] \\ &+ AD \exp[i(\omega/v)(x-vt)] \\ &+ BC \exp[-i(\omega/v)(x-vt)] \\ &+ BD \exp[-i(\omega/v)(x+vt)]\end{aligned}$$

General solution

$$\Psi(x,t) = \sum_r a_r \exp\left[\pm i(\omega_r/v)(x \pm vt)\right]$$

If we define the wave number k

$$k^2 \equiv \frac{\omega^2}{v^2}, \quad k = \pm \frac{\omega}{v}$$

Then the WE becomes

$$\boxed{\frac{d^2\psi}{dx^2} + k^2\psi = 0}$$

This is known as the Helmholtz eq$^{\underline{n}}$

Solutions are

$$\Psi(x,t) = \sum_r a_r e^{i(k_r x \pm \omega_r t)}$$

which we recognize as a superposition of plane waves moving the the L and R with speed $v = \omega/k$.

Standing waves

If we superimpose 2 waves of equal ampl. and frequency moving in opposite directions, then

$$\Psi = \Psi_+ + \Psi_- =$$

$$A e^{ik(x+vt)} + A e^{ik(x-vt)}$$

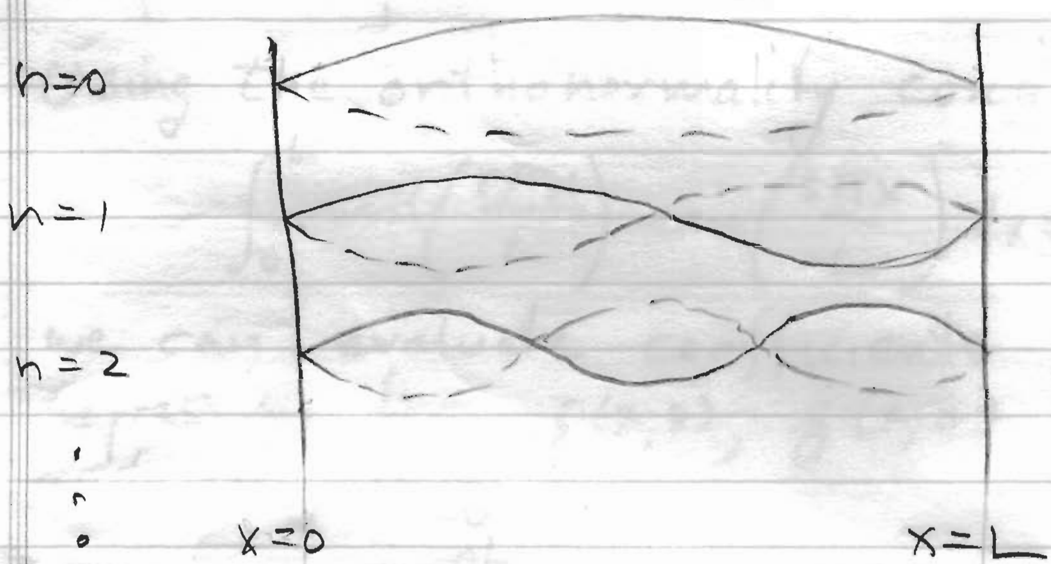
$$= A e^{ikx}(e^{i\omega t} + e^{-i\omega t})$$

$$= 2A e^{ikx} \cos \omega t$$

$$\boxed{Re(\Psi) = 2A \cos kx \cos \omega t}$$ standing waves

$\Psi = 0$ wherever $\cos kx = 0$; i.e
for
$$x = (2n+1)\pi/2k \quad , \quad n = 0, 1, \dots \infty$$

These are called <u>nodes</u>.
Vibrating string has solutions of this form, shifted by $\pi/2k$ to account for fixed ends
$$\Psi(0,t) = \Psi(L,t) = 0$$



These solutions are called <u>normal modes</u>.

$\frac{n\pi}{k}$

<u>General sol^n for vibrating string with fixed ends.</u>
We can write gen. sol^n as the sum of normal modes
$$g(x,t) = \sum_{r=1}^{\infty} \hat{a}_r e^{i\omega_r t} \sin\left(\frac{r\pi x}{L}\right)$$
where $\hat{a}_r$ is complex to allow for random phase.

Let $\hat{a}_r = \mu_r + i\nu_r$

amplitude phase

@ $t=0$ $\quad \xi(x,0) = \sum_r \mu_r \sin\left(\frac{r\pi x}{L}\right)$

$$\dot{\xi}(x,t) = \sum_r (\mu_r + i\nu_r)\, i\omega_r e^{i\omega_r t} \sin\left(\frac{r\pi x}{L}\right)$$

$$Re\left[\dot{\xi}(x,0)\right] = -\sum_r \nu_r \omega_r \sin\left(\frac{r\pi x}{L}\right)$$

Using the orthonormality condition

$$\int_0^L \sin\left(\frac{r\pi x}{L}\right) \sin\left(\frac{s\pi x}{L}\right) dx = \frac{L}{2}\delta_{rs}$$

we can evaluate coefficients by multiplying expressions for $\xi(x,0)$, $\dot{\xi}(x,0)$ by $\sin\left(\frac{s\pi x}{L}\right)$ and integrating

$$\mu_r = \frac{2}{L}\int_0^L \xi(x,0) \sin\left(\frac{r\pi x}{L}\right) dx$$

$$\nu_r = -\frac{2}{\omega_r L}\int_0^L \dot{\xi}(x,0) \sin\left(\frac{r\pi x}{L}\right) dx$$

Given the displacement and velocity of string at $t=0$, subsequent motion is completely determined (see example 13.1).