(11/28)

10. Force-free motion of symmetric top

Let's apply Euler's eq^{ns} to a symmetric top in force-free motion (torques $=0$)

Symmetric top:

$$I_1 = I_2 \neq I_3 \quad \text{(ex: football)}$$

Then,

$$\left.\begin{aligned} (I_1 - I_3)\,\omega_2\omega_3 - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\,\omega_3\omega_1 - I_1\dot{\omega}_2 &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}\right\}$$

Since motion is force-free, CM is at rest or in uniform motion. Can transform to a frame where body is at rest.

For last equation, $\dot{\omega}_3 = 0$, or

$$\omega_3(t) = \text{constant}.$$

Isolating $\dot{\omega}_1$, $\dot{\omega}_2$, we have

$$\dot{\omega}_1 = \left(\frac{I_1 - I_3}{I_1}\right)\omega_2\omega_3$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\right)\omega_1\omega_3$$

defining $\quad \Omega \equiv \dfrac{I_3 - I_1}{I_1}\,\omega_3$

$$\dot{\omega}_1 + \Omega\,\omega_2 = 0 \quad , \quad \dot{\omega}_2 - \Omega\,\omega_1 = 0$$

Multiplying 2^{nd} equation by i and adding to the first, we have

$$(\dot{\omega}_1 + i\dot{\omega}_2) - i\Omega(\omega_1 + i\omega_2) = 0$$

Defining complex variable

$$\eta \equiv \omega_1 + i\omega_2$$

we have

$$\dot{\eta} - i\Omega\eta = 0$$

This has solution

$$\eta(t) = Ae^{i\Omega t}$$

or $\omega_1 + i\omega_2 = A\cos\Omega t + iA\sin\Omega t$

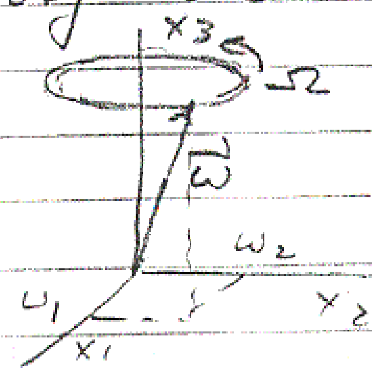
$$\therefore \quad \omega_1 = A\cos\Omega t$$
$$\omega_2 = A\sin\Omega t$$

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = A^2 + \omega_3^2 = \text{constant}$$

Interpretation

⇒ The angular velocity magnitude $|\vec{\omega}|$ is a constant

⇒ The angular velocity vector $\vec{\omega}$ precesses about x_3 body axis with frequency Ω.



⇒ Cone swept out by $\vec{\omega}$ called body cone

Now, since motion is force-free, torques are 0, therefore

$$\frac{dL}{dt} = 0$$

or

$$\vec{L} = \text{constant}.$$

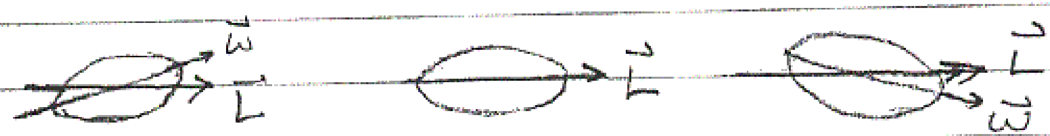
Since energy is conserved, $T_{rot} = \text{const}$.
But

$$T_{rot} = \frac{1}{2}\,\vec{\omega}\cdot\vec{L} = \text{constant}$$

i.e., projection of $\vec{\omega}$ vector onto fixed $\vec{L}$ vector is a constant.

$\Rightarrow$ $\vec{\omega}$ precesses about fixed $\vec{L}$ with frequency Ω

Example: mis-thrown football



Notice that $\dfrac{\Omega}{\omega_3} = \dfrac{I_3 - I_1}{I_1} < 0$

since $I_3 < I_1$. The football precesses in the opposite direction to spin imparted by quarterback.

11. Motion of symmetric top with one point fixed

Motion of a top in a gravitational field supported by a pivot point is more complex, and has no analytic solution except for a few special cases. We will derive Lagrangian, EOM, and describe motion qualitatively.

We establish coord. systems (body and fixed) with common origins at the tip of the top. (Fig. 11-15). We adopt the Eulerian angles (ϕ, θ, ψ) as the generalized coordinates for the top.

Qualitatively, we know from experience the top spins about X_3 and precesses about X_3'.

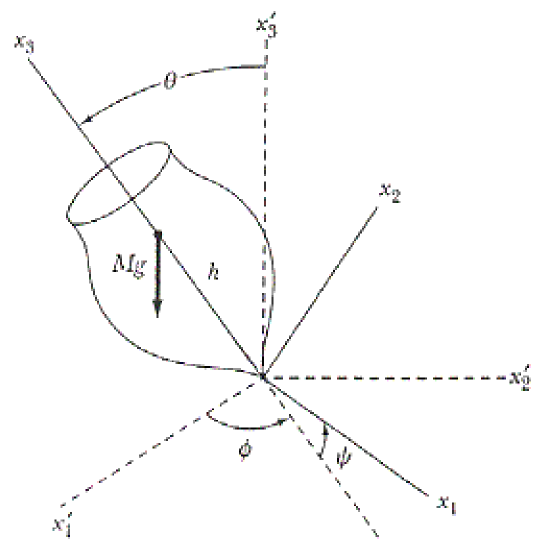


FIGURE 11-15 A symmetric top with its bottom tip fixed rotates in a gravitational field. The Euler angles relate the x_i'- (fixed) axes with the x_i- (body) axes. The angle ψ represents the rotation around the x_3 symmetry axis.

Lagrangian

$$\mathcal{L} = T - U$$

$$T = \frac{1}{2}\sum_i I_i \omega_i^2 = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$$

$$U = Mgh \cos\theta$$

h: dist. from tip to CM

M: mass of top

From Eq. 11.102, it is easy to show

$$\omega_1^2 + \omega_2^2 = \dot\phi^2 \sin^2\theta + \dot\theta^2$$

and

$$\omega_3^2 = (\dot\phi \cos\theta + \dot\psi)^2$$

$$\boxed{\mathcal{L} = \frac{1}{2} I_1 (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi\cos\theta + \dot\psi)^2 - Mgh\cos\theta}$$

EOM

Since $\mathcal{L}$ does not depend on ϕ and ψ, but only on their time derivatives, they are cyclic (cf. TM, Sec. 7.10). Consequently, their conjugate momenta are conserved

$$P_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = (I_1 \sin^2\theta + I_3 \cos^2\theta)\,\dot{\phi} + I_3 \dot{\psi} \cos\theta$$
$$= \text{constant}$$

$$P_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = I_3 \dot{\psi} + I_3 \dot{\phi} \cos\theta = \text{constant}$$

P_ϕ, P_ψ have dimensions $I\omega$, and are therefore angular momenta.
The above says that the angular momenta about the ϕ and ψ axes are conserved.
$\Rightarrow$ P_ϕ, P_ψ are constants of the motion and depend on how the top was set spinning.

above can be solved for $\dot{\phi}$, $\dot{\psi}$ by algebraic manipulation to yield

$$\dot{\psi} = \frac{P_\psi - I_3 \dot{\phi} \cos\theta}{I_3} \qquad (1)$$

and

$$\boxed{\dot{\phi} = \frac{P_\phi - P_\psi \cos\theta}{I_1 \sin^2\theta}} \qquad (2)$$

Substituting for $\dot{\phi}$ in (1), we get

$$\boxed{\dot{\psi} = \frac{P_\psi}{I_3} - \frac{(P_\phi - P_\psi \cos\theta)\cos\theta}{I_1 \sin^2\theta}} \qquad (3)$$

The equation for $\dot{\theta}$ can be derived by invoking energy conservation

$$E = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 \omega_3^2 + Mgh\cos\theta$$
$$= \text{Constant}$$

But we had $P_\psi = I_3(\dot{\psi} + \dot{\phi}\cos\theta)$
$$= I_3\omega_3 \quad = \text{constant}$$

so $\frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} P_\psi^2 / I_3 = \text{constant}$

and therefore $E' = E - \frac{1}{2} I_3 \omega_3^2 = \text{constant}$

$$E' = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + Mgh\cos\theta = \text{constant}$$

Substituting for $\dot{\phi}$, we get

$$\boxed{E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta)}$$

where $V(\theta)$, the <u>effective potential</u>, is

$$V(\theta) \equiv \frac{(P_\phi - P_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + Mgh\cos\theta$$

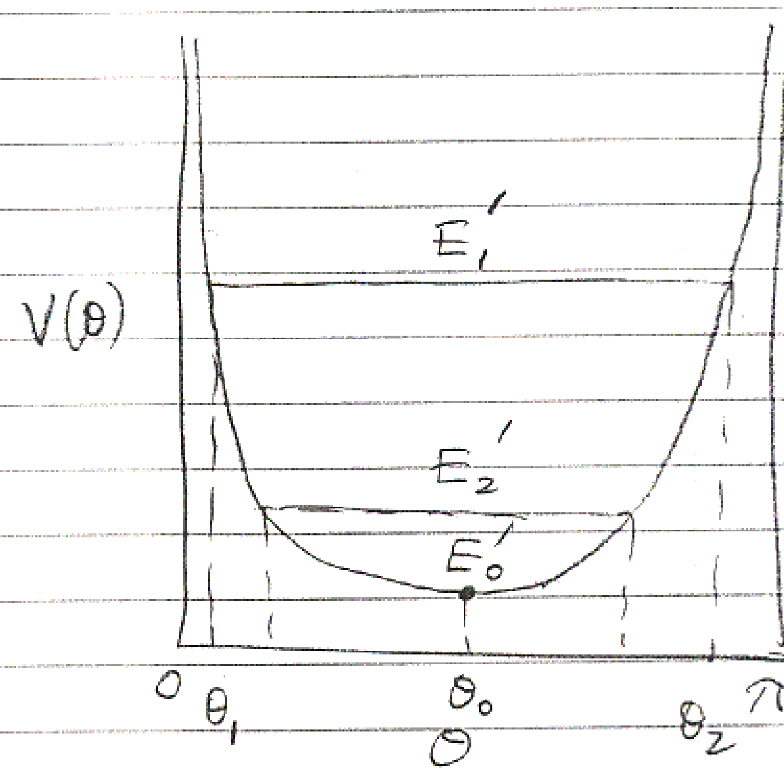
$V(\theta)$ depends on properties of top M, I_1, h conserved momenta P_ϕ, P_ψ, and dynamical variable θ.

$\dot{\theta}$ EOM

$$\boxed{\dot{\theta} = \sqrt{2(E' - V(\theta))/I_1}}$$

E' is a constant of the motion.
depending on value of E', different "orbits" are possible.

Graph of $V(\theta)$



In general, top will "nutate" between $\theta_1 \leq \theta \leq \theta_2$ as it precesses, unless it has $E' = E_0'$, the minimum energy, in which case it precesses without nutating. Depending on values of P_ψ, P_ϕ and E', different motions are possible (see Fig 11.17).

velocity must have opposite signs at $\theta = \theta_1$ and $\theta = \theta_2$. Thus, the nutational-

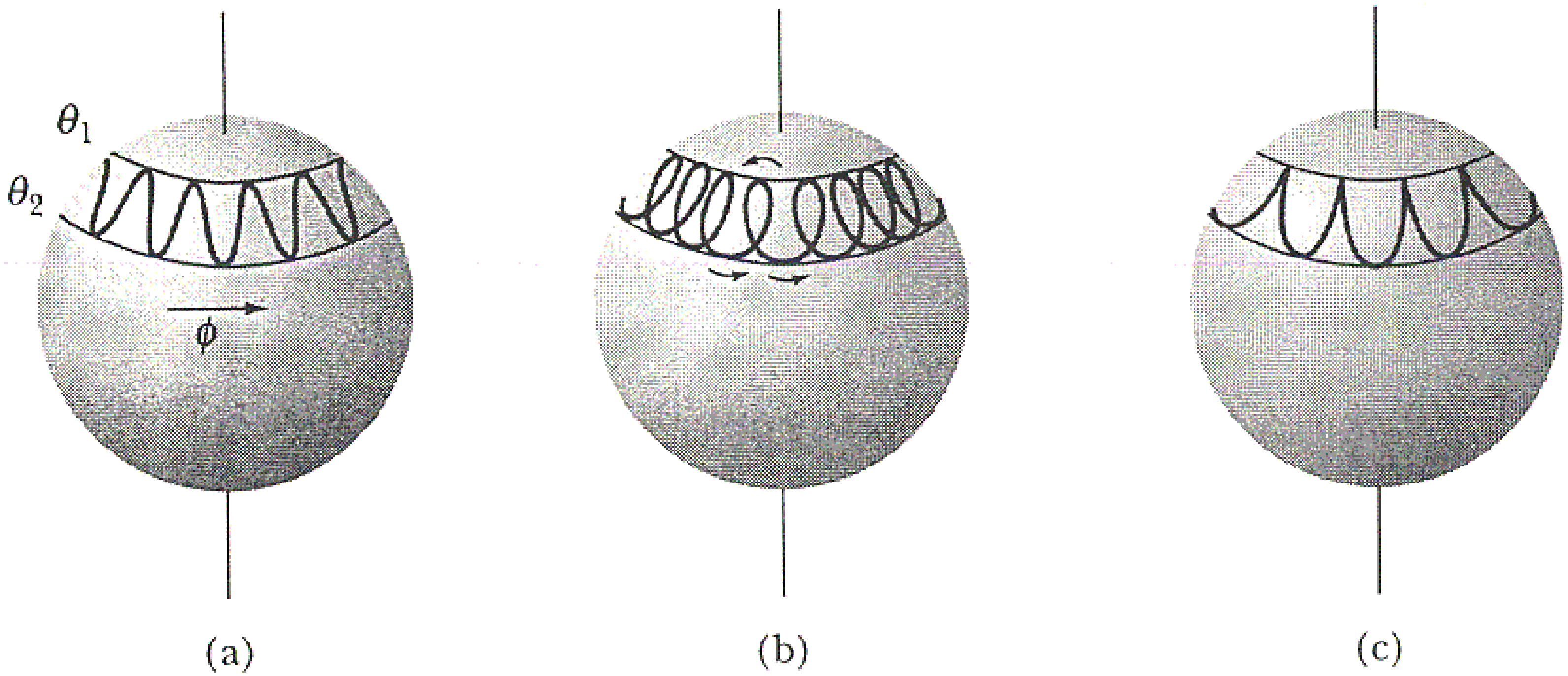


FIGURE 11-17 The rotating top also nutates between the limit angles θ_1 and θ_2. In (a) $\dot{\phi}$ does not change sign. In (b) $\dot{\phi}$ does change sign, and we see looping motion. In (c) the initial conditions include $\dot{\theta} = \dot{\phi} = 0$; this is the normal cusp-like motion when we spin a top and release it.

$\theta_0 > \pi/2$, the fixed tip of the top is at a position *above* the center of mass. Such motion is possible, example, with a gyroscopic top whose tip is actually a ball and rests in a cup that is fixed atop a destal.