7. How $\overset{\leftrightarrow}{I}$ Transforms under Rotations of BCS

Dynamics of RB is most simply analyzed w.r.t. a BCS in which the inertia tensor is diagonal; i.e., $I_{ij} = I_i \delta_{ij}$. For a general I_{ij}, there always exists a rotation matrix λ_{ij} that aligns the BCS with the principal axes of inertia. We need to know how a 2nd rank tensor transforms under a general rotation. Here, I merely state the result, which is derived in TM, Sec. 11.7.

If $\vec{X}$ is the original BCS
$\vec{X}'$ is the rotated BCS
and λ_{ij} is the rotation matrix s.t.

$$\boxed{x_i' = \sum_j \lambda_{ij} x_j} \quad (1)$$

And, if I_{ij} is the I.T. in $\vec{X}$ BCS
I_{ij}' " " " " $\vec{X}'$ "

Then

$$\boxed{I_{ij}' = \sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl}} \quad (2)$$

Just as ① is the general expression for how vectors transform under rotation, so is ② the general expression for how a 2nd rank tensor transforms.

Using ②, we can ask under what conditions can we diagonalize $\overleftrightarrow{I}$?

We require

$$I'_{ij} = I_i \delta_{ij} = \sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl}$$

Following the steps in TM, Eqs. 11.67–72b, we arrive at the condition

$$\sum_{l} (I_{ml} - I_j \delta_{ml}) \lambda_{jl} = 0$$

This is a set of simultaneous linear alg. eqns; for each value of j, there are 3 equations, one for each value of m.

E.g. $j=1$

$m=1$ $$(I_{11} - I_1)\lambda_{11} + I_{12}\lambda_{12} + I_{13}\lambda_{13} = 0$$

$m=2$ $$I_{21}\lambda_{11} + (I_{22} - I_1)\lambda_{12} + I_{23}\lambda_{13} = 0$$

$m=3$ $$I_{31}\lambda_{11} + I_{32}\lambda_{12} + (I_{33} - I_1)\lambda_{13} = 0$$

and similarly for $j = 2, 3$. In total, we have 9 equations for the 9 unknowns λ_{ij}.

For each j, the Eq. 11.72b has the form

$$\mathbb{M}_j \vec{\lambda}_j = 0$$

For a nontrivial solution to exist for $\vec{\lambda}_j$ we require

$$\det(\mathbb{M}_j) = 0$$

or $\det | I_{m\ell} - I\delta_{m\ell} | = 0$

This is cubic equation for I, whose roots are the PAI. I_1, I_2, I_3.

8. Euler angles

Armed with the above results, we now begin our assault on the dynamics of RB in full 3D. As we recall, a RB has 6 d.o.f. - 3 translational, and 3 rotational. We can always transform the translational d.o.f. away, leaving the 3 rotational. For these we need to define 3 angles to be our dynamical coordinates. How to define?

Answer: Eulerian angles (ϕ, θ, ψ).

Let $\vec{X}'$ be a CS with fixed orientation attached to the body

Let $\vec{X}$ be a CS which rotates with the body

In general, we know a rotation matrix $\tilde{\lambda}_{3D}$ exists s.t.

$$\vec{X} = \tilde{\lambda}_{3D}\vec{X}'$$

The Eulerian angles are generated by a series of 3 1D rotations, which take x_i' to x_i

$$\vec{X}' \xrightarrow[\tilde{\lambda}_\phi]{} \vec{X}'' \xrightarrow[\tilde{\lambda}_\theta]{} \vec{X}''' \xrightarrow[\tilde{\lambda}_\psi]{} \vec{X}$$

$\tilde{\lambda}_\phi, \tilde{\lambda}_\theta, \tilde{\lambda}_\psi$ are each rotation matrices with a simple rotation about ϕ, θ, ψ (See Fig 11-9)

$\boxed{\text{Step 1}}$ $$\vec{X}'' = \tilde{\lambda}_\phi \vec{X}', \quad \tilde{\lambda}_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boxed{\text{step 2}}$ $$\vec{X}''' = \tilde{\lambda}_\theta \vec{X}'', \quad \tilde{\lambda}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$\boxed{\text{step 3}}$ $\vec{x} = \tilde{\lambda}_\psi \vec{x}''' , \quad \tilde{\lambda}_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Putting all this together

$$\vec{x} = \tilde{\lambda}_\psi \tilde{\lambda}_\theta \tilde{\lambda}_\phi \vec{x}'$$

$$\therefore \boxed{\tilde{\lambda}_{3D} = \tilde{\lambda}_\psi \tilde{\lambda}_\theta \tilde{\lambda}_\phi}$$

The components of $\tilde{\lambda}_{3D} = \{\lambda_{ij}\}$ are obtained by matrix multiplication, and are given in Eq. 11.99.

(Show slide)

Kinematics of a RB

As a RB tumbles in space, its Euler angles are continuously changing. We can define Eulerian angular velocities as

$$\left.\begin{aligned} \omega_\phi &= \dot{\phi} \\ \omega_\theta &= \dot{\theta} \\ \omega_\psi &= \dot{\psi} \end{aligned}\right\}$$

simple enough!

Unfortunately, these angular velocities are not what we want, because they are w.r.t. a mixture of fixed and body CS axes:

$\vec{\dot{\phi}}$ along X_3' axis (fixed)

$\vec{\dot{\theta}}$ along line of nodes

$\vec{\dot{\psi}}$ along X_3 axis (body)

We want angular velocities w.r.t. to body CS axes. These are obtained by projecting out components of $\vec{\dot{\phi}}, \vec{\dot{\theta}}, \vec{\dot{\psi}}$ onto $\hat{X}_1, \hat{X}_2, \hat{X}_3$

$$\left.\begin{aligned}\dot{\phi}_1 &= \vec{\dot{\phi}} \cdot \hat{X}_1 = \dot{\phi} \sin\theta \sin\psi \\ \dot{\phi}_2 &= \vec{\dot{\phi}} \cdot \hat{X}_2 = \dot{\phi} \sin\theta \cos\psi \\ \dot{\phi}_3 &= \vec{\dot{\phi}} \cdot \hat{X}_3 = \dot{\phi} \cos\theta\end{aligned}\right\}$$

$$\left.\begin{aligned}\dot{\theta}_1 &= \vec{\dot{\theta}} \cdot \hat{X}_1 = \dot{\theta} \cos\psi \\ \dot{\theta}_2 &= \vec{\dot{\theta}} \cdot \hat{X}_2 = -\dot{\theta} \sin\psi \\ \dot{\theta}_3 &= \vec{\dot{\theta}} \cdot \hat{X}_3 = 0\end{aligned}\right\}$$

$$\left.\begin{aligned}\dot{\psi}_1 &= \vec{\dot{\psi}} \cdot \hat{X}_1 = 0 \\ \dot{\psi}_2 &= \vec{\dot{\psi}} \cdot \hat{X}_2 = 0 \\ \dot{\psi}_3 &= \dot{\psi}\end{aligned}\right\}$$

Collecting the components of $\vec{\omega}$, we have (finally)

$$\boxed{\begin{aligned} \omega_1 &= \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi} \end{aligned}}$$

Using the above, we can relate the Eulerian angles and angular velocity magnitudes $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ to angular velocity in the BCS. We need this because, when we do dynamics, we will be working with $\vec{L}$ and T in BCS:

$$L_i = I_i \omega_i$$

$$T_{rot} = \frac{1}{2}\sum_i I_i \omega_i^2 \quad .$$

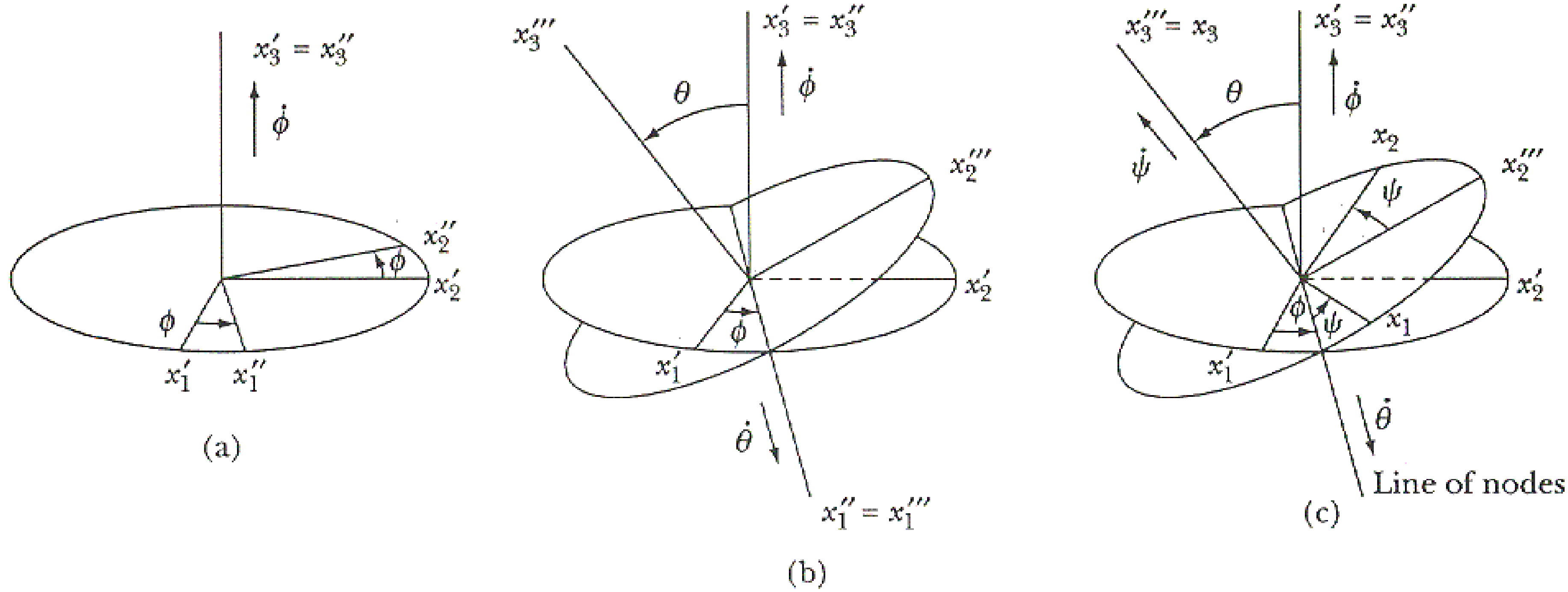

FIGURE 11-9 The Eulerian angles are used to rotate from the x_i' system to the x_i system. (a) First rotation is counterclockwise through an angle ϕ about the x_3'-axis. (b) Second rotation is counterclockwise through an angle θ about the x_1''-axis. (c) Third rotation is counterclockwise through an angle ψ about the x_3'''-axis.

choices for these angles; we find it convenient to use the **Eulerian angles*** ϕ, θ, and ψ.

The Eulerian angles are generated in the following series of rotations, which takes the x_i' system into the x_i system.†

$$\mathbf{x} = \boldsymbol{\lambda}_\psi \mathbf{x}''' \tag{11.}$$

ine common to the planes containing the x_1- and x_2-axes and the es is called the **line of nodes.** The complete transformation from th the x_i system is given by

$$\begin{aligned} \mathbf{x} = \boldsymbol{\lambda}_\psi \mathbf{x}''' &= \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \mathbf{x}'' \\ &= \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \boldsymbol{\lambda}_\phi \mathbf{x}' \end{aligned} \tag{11.}$$

otation matrix $\boldsymbol{\lambda}$ is

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \boldsymbol{\lambda}_\phi \tag{11.}$$

onents of this matrix are

$$\left.\begin{aligned} \lambda_{11} &= \cos\psi \cos\phi - \cos\theta \sin\phi \sin\psi \\ \lambda_{21} &= -\sin\psi \cos\phi - \cos\theta \sin\phi \cos\psi \\ \lambda_{31} &= \sin\theta \sin\phi \\ \lambda_{12} &= \cos\psi \sin\phi + \cos\theta \cos\phi \sin\psi \\ \lambda_{22} &= -\sin\psi \sin\phi + \cos\theta \cos\phi \cos\psi \\ \lambda_{32} &= -\sin\theta \cos\phi \\ \lambda_{13} &= \sin\psi \sin\theta \\ \lambda_{23} &= \cos\psi \sin\theta \\ \lambda_{33} &= \cos\theta \end{aligned}\right\} \tag{11.9}$$

ponents λ_{ij} are offset in the preceding equation to assist in the visu he complete $\boldsymbol{\lambda}$ matrix.)

se we can associate a vector with an infinitesimal rotation, we can as me derivatives of these rotation angles with the components of the a city vector $\boldsymbol{\omega}$. Thus,

$$\omega_\phi = \dot{\phi}$$