Kinetic energy of RB

We had, for the velocity of particle α

$$\vec{v}_\alpha = \vec{V} + \vec{\omega} \times \vec{r}_\alpha$$

It's KE is $T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2$

Total KE is

$$T = \frac{1}{2} \sum_\alpha m_\alpha (\vec{V} + \vec{\omega} \times \vec{r}_\alpha)^2$$

$$= \frac{1}{2} \sum_\alpha m_\alpha V^2 + \sum_\alpha m_\alpha \vec{V} \cdot \vec{\omega} \times \vec{r}_\alpha + \frac{1}{2} \sum_\alpha m_\alpha (\vec{\omega} \times \vec{r}_\alpha)^2$$

$$= \frac{1}{2} M V^2 + \vec{V} \cdot \vec{\omega} \underbrace{\sum_\alpha m_\alpha \vec{r}_\alpha}_{\vec{R} M} + \frac{1}{2} \sum_\alpha m_\alpha (\vec{\omega} \times \vec{r}_\alpha)^2$$

if $\vec{r}_\alpha$ are measured relative to CM, $\vec{R} = 0$.

Hence

$$T = T_{trans} + T_{rot}$$

where

$$T_{trans} = \frac{1}{2} M V^2$$

$$T_{rot} = \frac{1}{2} \sum_\alpha m_\alpha \underbrace{(\vec{\omega} \times \vec{r}_\alpha)}_{\text{rotational velocity about CM.}}{}^2$$

Using vector identity $(\vec{A} \times \vec{B})^2 = A^2 B^2 - (\vec{A} \cdot \vec{B})^2$

$$T_{rot} = \frac{1}{2} \sum_\alpha m_\alpha \left[\omega^2 r_\alpha^2 - (\vec{\omega} \cdot \vec{r}_\alpha)^2 \right]$$

In component form, where $\vec{r}_\alpha = \{x_{\alpha,i}\}$ $i=1,2,3$

$$T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha \left[\left(\sum_i \omega_i^2\right)\left(\sum_k x_{\alpha,k}^2\right) - \left(\sum_i \omega_i x_{\alpha,i}\right)\left(\sum_j \omega_j x_{\alpha,j}\right)\right]$$

using trick $\omega_i = \sum_j \delta_{ij}\omega_j$, above can be rewritten

$$T_{rot} = \frac{1}{2}\sum_{i,j}\omega_i\omega_j \underbrace{\sum_\alpha m_\alpha\left(\delta_{ij}\sum_k x_{\alpha,k}^2 - x_{\alpha,i}x_{\alpha,j}\right)}_{I_{ij}}$$

or

$$\boxed{T_{rot} = \frac{1}{2}\sum_{i,j}\omega_i\omega_j I_{ij}}$$

In compact form

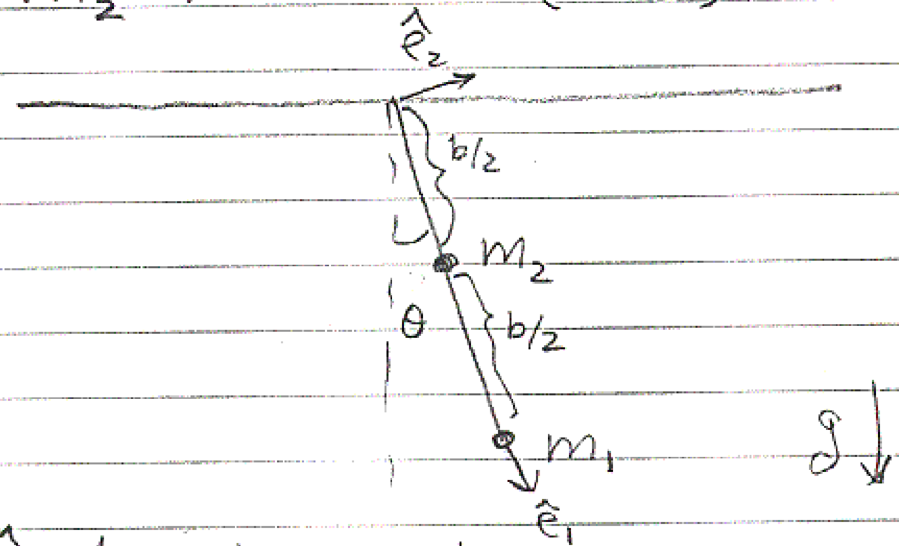
$$T_{rot} = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega} \qquad \underline{\text{scalar}}$$

Since $\vec{L} = \overleftrightarrow{I}\cdot\vec{\omega}$, we have

$$\boxed{T_{rot} = \frac{1}{2}\vec{\omega}\cdot\vec{L}}$$

Application to 2-mass pendulum (Ex 11.4)

Calculate period of oscillation for a rigid pendulum of length b, with m_1 at end and m_2 in middle ($b/2$).



Let $\hat{e}_1$ be along rod
$\hat{e}_2$ be $\perp$ to rod in plane of osc.
$\hat{e}_3$ be out of the " " "

$$\vec{\omega} = \omega_3 \hat{e}_3 = \dot{\theta} \hat{e}_3$$

$$T_{rot} = \frac{1}{2}\omega_3 L_3 = \frac{1}{2}\omega_3^2 I_{33}$$

$$I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)$$

$$\vec{r}_1 = (b, 0, 0) \quad , \quad \vec{r}_2 = (b/2, 0, 0)$$

$$I_{33} = m_1 b^2 + m_2 b^2/4$$

$$\therefore \quad T_{rot} = \frac{1}{2}\left(m_1 b^2 + m_2 b^2/4\right) \dot{\theta}^2$$

Potential energy U. Let $U=0$ at origin

$$U = -m_1 g\, b\cos\theta - m_2 g \frac{b}{2}\cos\theta$$

Lagrangian $\mathcal{L} = T - U$

$$\mathcal{L} = \frac{1}{2}(m_1 b^2 + m_2 b^2/4)\dot\theta^2 + (m_1 g b + m_2 g \tfrac{b}{2})\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -(m_1 g b + m_2 g \tfrac{b}{2})\sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = (m_1 b^2 + m_2 b^2/4)\dot\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} = (m_1 b^2 + m_2 b^2/4)\ddot\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} = 0$$

$$\Rightarrow (m_1 b^2 + m_2 b^2/4)\ddot\theta = -bg\sin\theta\left(m_1 + \frac{m_2}{2}\right)$$

for θ small, $\sin\theta \approx \theta$

therefore, frequency of oscillation

$$\boxed{\omega_0^2 = \frac{g\left(m_1 + \frac{m_2}{2}\right)}{b\left(m_1 + \frac{m_2}{4}\right)}}$$

if $m_2 \ll m_1$, $\omega_0 = \sqrt{g/b}$ as it should

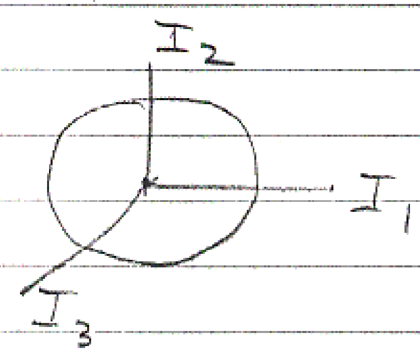
if $m_1 \ll m_2$, $\omega_0 = \sqrt{2g/b}$ as it should

5. Principal axes of inertial

Clearly, the previous analysis would have been more complicated if the BCS was not aligned with the rod. Our goal is to study the dynamics of RB with the minimum of mathematical complexity. This is accomplished by aligning BCS with the principal axes of inertia (PAI)

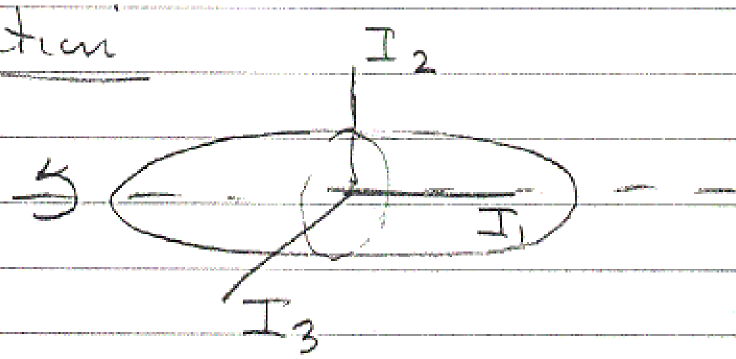
Quite often, the PAI are obvious from inspection

Sphere



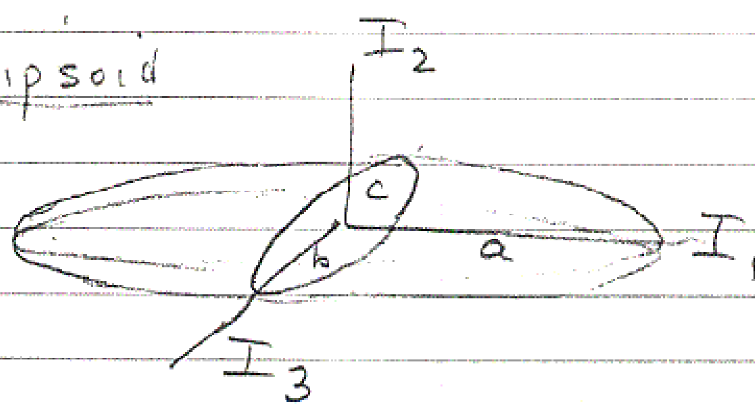
$I_1 = I_2 = I_3$

Figure of revolution



$I_1 \neq I_2 = I_3$

Triaxial ellipsoid



$I_1 \neq I_2 \neq I_3$

How to determine PIA

We want to find a BCS such that $\overleftrightarrow{I}$ is diagonal:

$$I_{ij} = \delta_{ij} I_i$$

$$\overleftrightarrow{I} = \begin{Bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{Bmatrix}$$

In this case

$$L_i = \sum_j I_i \delta_{ij} \omega_j = I_i \omega_i$$

$$T_{rot} = \frac{1}{2} \sum_{ij} I_i \delta_{ij} \omega_i \omega_j = \frac{1}{2} I_i \omega_i^2$$

$\vec{L} \parallel \vec{\omega}$.

Let $\quad \vec{L} = I \vec{\omega}$

From general expression for $\vec{L}$

$$\left.\begin{aligned} L_1 &= I\omega_1 = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ L_2 &= I\omega_2 = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ L_3 &= I\omega_3 = I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} (I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0 \\ I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 &= 0 \\ I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 &= 0 \end{aligned}\right\}$$

We can write this in matrix notation as

$$\overleftrightarrow{M} \cdot \vec{\omega} = 0$$

A nontrivial solution for $\vec{\omega} \neq 0$ is

$$\det(\overleftrightarrow{M}) = 0$$

$$\begin{vmatrix} (I_{11} - I) & I_{12} & I_{13} \\ I_{12} & (I_{22} - I) & I_{23} \\ I_{13} & I_{23} & (I_{33} - I) \end{vmatrix} = 0$$

This is a cubic equation for I. It will have 3 roots: I_1, I_2, I_3.

If $I_1 = I_2 = I_3 \Rightarrow$ spherical top

$I_1 = I_2 \neq I_3 \Rightarrow$ symmetric top

$I_1 \neq I_2 \neq I_3 \Rightarrow$ asymmetric top

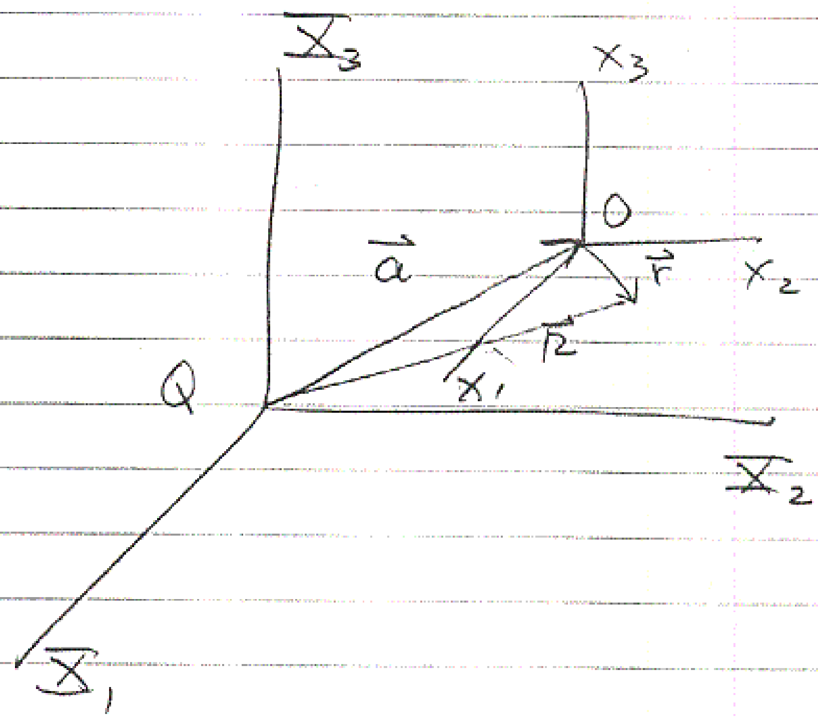
$I_1 = 0;\ I_2 = I_3 \Rightarrow$ rotor

See Example 11.5 for working out PAI for the uniform cube.

6. Moments of Inertia in Different BCS's

Suppose we know I_{ij} about a body's CM, and we want to compute it for a point somewhere else. We shall derive a formula for how $\overleftrightarrow{I}$ changes with translation.

Let $\{x_i\}$ be BCS with origin O at CM
Let $\{X_i\}$ be another BCS with origin Q not at body's CM



Let $\vec{a}$ be vector from Q to O
then

$$\vec{R} = \vec{a} + \vec{r}$$

$\vec{r}$: relative to O, components x_i

$\vec{R}$: relative to Q, components X_i

Formally, we have

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k X_{\alpha,k}^2 - X_{\alpha,i} X_{\alpha,j} \right)$$

Substituting $X_i = a_i + x_i$, we get

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)$$
$$+ \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k (2x_{\alpha,k} a_k + a_k^2) - (a_i x_{\alpha,j} + a_j x_{\alpha,i} + a_i a_j) \right]$$

First term is I_{ij}. Regrouping

$$J_{ij} = I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j \right)$$
$$+ \sum_\alpha m_\alpha \left(2\delta_{ij} \sum_k x_{\alpha,k} a_k - a_i x_{\alpha,i} - a_j x_{\alpha,j} \right)$$

Last term involves $\sum_\alpha m_\alpha x_{\alpha,i} = 0$ since by assumption x_i is relative to CM

$$\therefore J_{ij} = I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j \right)$$

now ( ) doesn't depend on α, $\sum_\alpha m_\alpha \equiv M$

$$\boxed{J_{ij} = I_{ij} + M(\delta_{ij} a^2 - a_i a_j)}$$

2nd term is inertia tensor for a point mass M at position $\vec{X} = \vec{a}$.